Certain submanifolds of trans-Sasakian manifolds

A. Sarkar and N. Biswas

Abstract. In the present paper we have deduced some necessary and sufficient conditions for invariant submanifolds of trans-Sasakian manifolds to be totally geodesic. Characterizations of totally umbilical submanifolds of trans-Sasakian manifolds have also been given.

M.S.C. 2010: 53C40, 53C25.
Key words: trans-Sasakian manifold; invariant submanifold; totally geodesic submanifold; totally umbilical submanifold.

1 Introduction

The notion of trans-Sasakian manifolds was introduced by Blair and Oubina [2],[14]. Three dimensional trans-Sasakian manifolds have also been studied in the paper[6]. Trans-Sasakian manifolds of type (α, β) are generalizations of α-Sasakian and β-Kenmotsu manifolds. It is known that a proper trans-Sasakian manifold exists only for dimension three. In higher dimension it is either α-Sasakian or β-Kenmotsu. In geometry of almost contact manifolds, submanifold theory has become a topic of growing research. There are several works on invariant submanifolds. In [5] the authors studied invariant submanifolds of trans-Sasakian manifolds. In the paper [16], invariant submanifolds of LP-Sasakian manifolds have been studied. In that paper, it was attempted to establish a relation between invariant and totally geodesic submanifolds of LP-Sasakian manifolds. Following this paper, in the present paper we would like to establish relation between invariant submanifolds and totally geodesic submanifolds of trans-Sasakian manifolds. In the paper [8], totally umbilical submanifolds of Sasakian manifolds have been studied. In the paper [13] totally umbilical submanifolds of Kaehlerian manifolds have been considered. In the same line of these papers, in the present paper we have studied totally umbilical submanifolds of trans-Sasakian manifolds. A differentiable manifold can be characterized as a domain of a function satisfying suitable differential equations. Obatta [15] first characterized some Riemannian manifolds as a domain of a function satisfying certain differential equations. It is known that if a function f is defined on a differentiable manifold and f satisfies $\Delta f = -kf$, $k > 0$, then the manifolds is isometric to a sphere. For details see [9],[13],[15].

Differential Geometry - Dynamical Systems, Vol.24, 2022, pp. 177-190.

The present paper is organized as follows: In Section 2, we give necessary preliminaries. Section 3 contains the study of invariant submanifolds of trans-Sasakian manifolds with an example. Section 4 is devoted to study totally umbilical submanifolds of trans-Sasakian manifolds of dimension greater or equal to five. Totally umbilical submanifolds of three-dimensional trans-Saskian manifolds have been considered in the last section.

2 Preliminaries

Let $\bar{M}$ be an n-dimensional (n is odd) smooth differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a (1,1)-tensor field, ξ is a vector field, η is a one form and g is a compatible Riemannian metric on $\bar{M}$. For such manifolds, we know [1]

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

$$\eta(X) = g(X, \xi), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$\phi\xi = 0, \quad \eta o \phi = 0, \quad g(X, \phi Y) = -g(\phi X, Y), \tag{2.4}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\Gamma(T\bar{M})$ denotes the Lie algebra of all vector fields on $\bar{M}$. A connected manifold $\bar{M}$ endowed with almost contact metric structure (ϕ, ξ, η, g) is called a trans-Sasakian manifold [14] if $(\bar{M} \times R, J, G)$ belongs to the class W_4 [10], where J is an almost complex structure on $\bar{M} \times R$ which is defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

for any vector field X on $\bar{M}$ and the smooth function f on $\bar{M} \times R$, and G is the usual product metric on $\bar{M} \times R$. According to [2], an almost contact metric manifold is a trans-Sasakian manifold if and only if

$$(\bar{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.5}$$

for smooth functions α, β on $\bar{M}$, where $\bar{\nabla}$ denote the covariant derivative with respect to g. Generally, $\bar{M}$, is said to be a trans-Sasakian manifold of type (α, β). From the equation (2.5), it follows that

$$\bar{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(x)\xi), \tag{2.6}$$

$$(\bar{\nabla}_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.7}$$

Let M be the sub-manifold of an n-dimensional almost contact metric manifold $\bar{M}$. Let ∇ and $\bar{\nabla}$ are the Levi-Civita connections of M and $\bar{M}$, respectively. Then for any vector fields $X, Y \in \Gamma(TM)$, the second fundamental form σ is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y). \tag{2.8}$$

A submanifold of a trans-Sasakian manifold is called totally geodesic if

$$\sigma(X, Y) = 0, \qquad \text{for} \qquad X, Y \in \Gamma(TM).$$

Furthermore, for any section N of normal bundle $T^{\perp}M$, we have

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp} N. \tag{2.9}$$

Where $\nabla^{\perp}$ denotes the normal bundle connection of M. The second fundamental form σ and shape operator A_N are related by

$$g(A_N X, Y) = g(\sigma(X, Y), N). \tag{2.10}$$

For details see [4].

On a Riemannian manifold M, for a $(0, k)$-type tensor field $T (k \geq 1)$ and a $(0, 2)$-type tensor field E, we denote by $Q(E, T)$ a $(0, k+2)$-type tensor field ([19]) defined as follows

$$\begin{aligned} Q(E,T)(X_1, X_2, ..., X_k; X, Y) = & - T((X \wedge_E Y)X_1, X_2, ..., X_n) \\ & - T(X_1, (X \wedge_E Y)X_2, ..., X_k) - ... \\ & - T(X_1, ..., (X \wedge_E Y)X_k), \end{aligned} \tag{2.11}$$

where $(X \wedge_E Y)Z = E(Y, Z)X - E(X, Z)Y$.

From Gauss and Codazzi equations for submanifolds, we get [3]

$$\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(\sigma(X, W), \sigma(Y, Z)) + g(\sigma(X, Z), \sigma(Y, W)). \tag{2.12}$$

$$(\bar{R}(X, Y)Z)^{\perp} = (\bar{\nabla}_X \sigma)(Y, Z) - (\bar{\nabla}_Y \sigma)(X, Z). \tag{2.13}$$

Here $\bar{R}$ is the curvature tensor of the ambient manifold. $(\bar{R}(X, Y)Z)^{\perp}$ is the normal component of $\bar{R}$.

From [6], we get for three-dimensional trans-Sasakian manifolds

$$\bar{R}(X, Y)\xi = (4(\alpha^2 - \beta^2) - \frac{r}{2})(\eta(Y)X - \eta(X)Y). \tag{2.14}$$

3 Invariant submanifolds of trans-Sasakian manifolds

In this section we shall study a three dimensional submanifold M of a trans-Sasakian manifold $\bar{M}$ such that the characteristic vector field ξ is tangential to M. Generally, a submanifold M is said to be invariant submanifold of $\bar{M}$ if $\phi(TM) \subset TM$. On an invariant submanifold M of $\bar{M}$, it follows that $\xi \in \Gamma(TM)$. We see that

$$\begin{aligned}(\bar{\nabla}_X\phi)Y &= \bar{\nabla}_X\phi Y - \phi(\nabla_X Y)\\ &= \nabla_X\phi Y - \sigma(X,\phi Y) - \phi(\bar{\nabla}_X Y - \sigma(X,Y))\\ &= (\nabla_X\phi)Y - \sigma(X,\phi Y) + \phi(\sigma X, Y).\end{aligned}$$

From (2.4) and the above equation we get by considering the submanifold as invariant and comparing tangential components

$$(\nabla_X\phi)Y = \alpha(g(X,Y)\xi + \eta(Y)X) - \beta(g(\phi X,Y)\xi - \eta(Y)\phi X). \tag{3.1}$$

Thus we have the following lemma

Lemma 3.1. *An invariant submanifold of a trans-Sasakian manifold is tarns-Sasakian.*

This Lemma is also proved in [5].

From the equation (2.6) we directly can establish the following lemma.

Lemma 3.2.[16] *Let M be the invariant submanifold of a trans-Sasakian manifold $\bar{M}$. Then the following equations hold:*

$$\nabla_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \qquad \sigma(X,\xi) = 0,$$

$$\sigma(X,\phi Y) = \sigma(\phi X, Y) = \phi\sigma(X,Y), \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$.

Lemma 3.3.[7] *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$ of dimension greater than three, then we have*

$$R(X,\xi)\xi = (\alpha^2 - \beta^2 - \xi\beta)(X - \eta(X)\xi) + 2\alpha\beta\phi X + (\xi\alpha)\phi X, \tag{3.3}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \beta^2)\eta(X) - (n-2)X\beta - \eta(X)\xi\beta - (\phi X)\alpha, \tag{3.4}$$

$$S(\xi,\xi) = (n-1)(\alpha^2 - \beta^2 - \xi(\beta)), \tag{3.5}$$

where any $X \in \Gamma(TM)$. Here R and S are respectively the Riemann curvature and Ricci curvature of the submanifold.

The Projective curvature tensor P of type (1,3) on a Riemannian manifold (M, g) of dimension n is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{3.6}$$

for any $X, Y, Z \in \Gamma(TM)$.

Theorem 3.1. *An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if $Q(g, P.\sigma) = 0$, provided that $(2\alpha\beta + \xi\alpha) \neq 0$.*

Proof. Assume $Q(g,P.\sigma)=0$, then

$$Q(g,P(X,Y).\sigma)(W,K;U,V)=0,$$

for the vector fields $X,Y,W,K,U,V\in\Gamma(TM)$. Using (2.11) we have

$$\begin{aligned} 0 = &- g(V,W)(P(X,Y).\sigma)(U,K)+g(U,W)(P(X,Y).\sigma)(V,K)\\ &- g(V,K)(P(X,Y).\sigma)(W,U)+g(U,K)(P(X,Y).\sigma)(W,V)\end{aligned}$$

$$\begin{aligned} = &\ -g(V,W)[P^{\perp}(X,Y)\sigma(U,K)-\sigma(P(X,Y)U,K)-\sigma(P(X,Y)K,U)]\\ + &\ g(U,W)[P^{\perp}(X,Y)(\sigma V,K)-\sigma(P(X,Y)V,K)-\sigma(P(X,Y)K,V)]\\ - &\ g(V,K)[P^{\perp}(X,Y)(\sigma W,U)-\sigma(P(X,Y)W,U)-\sigma(P(X,Y)U,W)]\\ + &\ g(U,K)[P^{\perp}(X,Y)(\sigma W,V)-\sigma(P(X,Y)W,V)-\sigma(P(X,Y)V,W)].\end{aligned}$$

Using Lemma 3.2 and putting $Y=K=U=W=\xi$ in the above equation we have

$$\sigma(P(X,\xi)\xi,V)=0. \tag{3.7}$$

By the Lemma 3.4 and the equation (3.12) we have

$$\sigma(P(X,\xi)\xi,V)=(2\alpha\beta+\xi\alpha)\sigma(V,\phi X). \tag{3.8}$$

By the equations (3.13), (3.14) and the assumed condition $(2\alpha\beta+\xi\alpha)\neq 0$, we have

$$\sigma(V,\phi X)=0.$$

Hence by Lemma 3.2,

$$\sigma(V,X)=0$$

for any $X,Y\in\Gamma(TM)$. Thus the submanifold is totally geodesic. Converse part is trivially true. This completes the proof. □

Remark 3.1. *The above theorem is also true for invariant submanifolds of Sasakian and Kenmotsu manifolds.*

Theorem 3.2. *An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if* $Q(S,P.\sigma)=0$*, provided that* $(n-1)(\alpha^2-\beta^2+\xi(\beta))(2\alpha\beta+\xi\alpha)\neq 0$.

Proof. Assume $Q(S,P.\sigma)=0$, then

$$Q(g,P(X,Y).\sigma)(W,K;U,V)=0$$

for the vector fields $X,Y,W,K,U,V\in\Gamma(TM)$. Using (2.11) we have

$$\begin{aligned} 0 = &- S(V,W)(P(X,Y).\sigma)(U,K)+S(U,W)(P(X,Y).\sigma)(V,K)\\ &- S(V,K)(P(X,Y).\sigma)(W,U)+S(U,K)(P(X,Y).\sigma)(W,V)\end{aligned}$$

$$\begin{aligned} = &\ -S(V,W)[P^{\perp}(X,Y)\sigma(U,K)-\sigma(P(X,Y)U,K)-\sigma(P(X,Y)K,U)]\\ + &\ S(U,W)[P^{\perp}(X,Y)(\sigma V,K)-\sigma(P(X,Y)V,K)-\sigma(P(X,Y)K,V)]\\ - &\ S(V,K)[P^{\perp}(X,Y)(\sigma W,U)-\sigma(P(X,Y)W,U)-\sigma(P(X,Y)U,W)]\\ + &\ S(U,K)[P^{\perp}(X,Y)(\sigma W,V)-\sigma(P(X,Y)W,V)-\sigma(P(X,Y)V,W)].\end{aligned}$$

Using Lemma 3.2 and putting $Y = K = U = W = \xi$ in the above equation we have

$$S(\xi,\xi)\sigma(P(X,\xi)\xi,V) = 0. \tag{3.9}$$

By the Lemma 3.4 and the equation (3.12) we have

$$S(\xi,\xi)\sigma(P(X,\xi)\xi,V) = (n-1)(\alpha^2-\beta^2+\xi(\beta))(2\alpha\beta+\xi\alpha)\sigma(V,\phi X). \tag{3.10}$$

By the equations (3.15), (3.16) and the given condition $(n-1)(\alpha^2-\beta^2+\xi(\beta))(2\alpha\beta+\xi\alpha) \neq 0$ we have

$$\sigma(V,\phi X) = 0.$$

Hence by the Lemma 3.2,

$$\sigma(V,X) = 0$$

for any $X, Y \in \Gamma(TM)$. Thus the submanifold is totally geodesic. Converse part is trivially true. This completes the proof. □

Remark 3.2. *The above theorem is also true for invariant submanifolds of Sasakian and Kenmotsu manifolds.*

Example 3.1. Let us consider the five dimensional differentiable manifold [11] $M = \{(x_1,x_2,x_3,x_4,t) \in \mathbb{R}^5 : t \neq 0\}$, where (x_1,x_2,x_3,x_4,t) are the standard coordinates of $\mathbb{R}^5$. We choose the vector fields

$$e_1 = e^{-t}\frac{\partial}{\partial x_1}, \quad e_2 = e^{-t}\frac{\partial}{\partial x_2}, \quad e_3 = e^{-t}\frac{\partial}{\partial t}, \quad e_4 = e^{-t}\frac{\partial}{\partial x_4}, \quad e_5 = e^{-t}\frac{\partial}{\partial t},$$

which are linearly independent at each point of $\bar{M}$. We define g by

$$g = e^{2t}K,$$

where K is the Euclidean metric on $\mathbb{R}$. Hence $\{e_1,e_2,e_3,e_4,e_5\}$ is orthonormal basis of $\bar{M}$ i.e.,

$$\begin{aligned} g(e_i,e_j) &= 1 \quad \text{if} \quad i=j, \\ &= 0 \quad \text{if} \quad i\neq j, \quad \text{where } 1\leq i,j\leq 5. \end{aligned}$$

We consider an 1-form η defined by

$$\eta(X) = g(X,e_5), \quad X \in T\bar{M}.$$

i.e., we choose $e_5 = \xi$. We define the (1.1) tensor field ϕ by

$$\phi\Big(\sum_{i=1}^{2}\big(x_i\frac{\partial}{\partial x_i} + x_{i+2}\frac{\partial}{\partial x_{i+2}}\big) + t\frac{\partial}{\partial t}\Big) = \sum_{i=1}^{2}\big(x_i\frac{\partial}{\partial x_{i+2}}\big) - x_{i+2}\frac{\partial}{\partial x_i}\big).$$

Thus we have

$$\phi(e_1) = e_3, \quad \phi(e_2) = e_4, \quad \phi(e_3) = -e_1, \quad \phi(e_4) = -e_2, \quad \phi(e_5) = 0.$$

The linear property of g and ϕ shows that

$$\eta(e_5) = 1, \qquad \phi^2(X) = -X + \eta(X)e_5,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on $\bar{M}(\phi, \xi, \eta, g)$ defines an almost contact manifold with $e_5 = \xi$. Moreover, let $\bar{\nabla}$ is the Levi-Civita connection with respect to metric g. Then we have

$$[e_i, e_5] = e^{-t}e_i \quad i = 1, 2, 3, 4, 5, \quad [e_i, e_j] = 0, \quad 1 \leq i, j \leq 4.$$

By Koszul formula, we obtain the following

$$\bar{\nabla}_{e_1}e_1 = -e^{-t}e_5, \quad \bar{\nabla}_{e_2}e_2 = -e^{-t}e_5, \quad \bar{\nabla}_{e_3}e_3 = -e^{-t}e_5, \quad \bar{\nabla}_{e_4}e_4 = -e^{-t}e_5,$$
$$\bar{\nabla}_{e_5}e_5 = 0, \quad \bar{\nabla}_{e_5}e_i = 0, \quad \bar{\nabla}_{e_i}e_5 = e^{-t}e_i, \quad \text{for } 1 \leq i \leq 4,$$
$$\bar{\nabla}_{e_i}e_j = 0, \quad \text{otherwise.}$$

Thus we see that $\bar{M}$ is a trans-Sasakian manifold of type $(0, e^{-t})$.

Let M be a sub set of $\bar{M}$ and consider the isometric immersion $f : M \to \bar{M}$ defined by

$$f(x_1, x_3, t) = (x_1, 0, x_3, 0, t).$$

It is easy to prove that $M = \{(x_1, x_3, t) \in \mathbb{R}^3 : t \neq 0\}$, where (x_1, x_3, t) are the standard co-ordinate of $\mathbb{R}^3$. We choose the vector fields

$$e_1 = e^{-t}\frac{\partial}{\partial x_1}, \qquad e_3 = e^{-t}\frac{\partial}{\partial x_3}, \qquad e_5 = e^{-t}\frac{\partial}{\partial t},$$

which are linearly independent at each point of M. We define g_1 by

$$g_1 = e^{2t}K_1,$$

where K_1 is the Euclidean metric on $\mathbb{R}$. Hence $\{e_1, e_3, e_5\}$ are orthonormal basis of $\bar{M}$ i.e., $g(e_i, e_j) = 1$ if $i = j$ and 0 otherwise. Here $i = 1, 3, 5$.

We define 1-form η_1 and (1,1) tensor ϕ_1 respectively by $\eta_1 = g_1(X, e_5)$,

$$\phi_1(x_1\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_3} + t\frac{\partial}{\partial t}) = (x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1}).$$

Thus we have

$$\phi_1(e_1) = e_3, \qquad \phi_1(e_3) = -e_1, \qquad \phi_1(e_5) = 0.$$

The linear property of g_1 and ϕ_1 shows that

$$\eta_1(e_5) = 1, \qquad \phi_1^2(X) = -X + \eta_1(X)e_5,$$

$$g_1(\phi_1 X, \phi_1 Y) = g_1(X, Y) - \eta_1(X)\eta_1(Y)$$

for any vector fields X, Y on $M(\phi_1, \xi, \eta_1, g_1)$. It is seen that M is an invariant submanifold of $\bar{M}$ with $e_5 = \xi$. Moreover, let ∇ be the Levi-Civita connection with respect to the metric g_1. Then we have

$$[e_i, e_5] = e^{-t} e_i, \quad [e_i, e_j] = 0, \quad i, j = 1, 3, 5.$$

By using Kouszul formula, we obtain

$$\nabla_{e_1} e_1 = -e^{-t} e_5, \quad \nabla_{e_3} e_3 = -e^{-t} e_5, \quad \nabla_{e_5} e_5 = 0 \quad , \nabla_{e_1} e_5 = e^{-t} e_1,$$
$$\nabla_{e_3} e_5 = e^{-t} e_3, \quad \nabla_{e_5} e_1 = 0, \quad \nabla_{e_5} e_3 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_3} e_1 = 0.$$

Using the above results we see that $\sigma(X, Y) = 0$. So the submanifold is totally geodesic.

4 Totally umbilical submanifolds of a trans-Sasakian manifold

Let M be an n-dimensional totally umbilical submanifold of a trans -Sasakian manifold $\bar{M}$. Here, we take $n \geq 5$. The second fundamental form σ of M is given by $\sigma(X, Y) = g(X, Y)H$ where $X, Y \in \Gamma(TM)$ and H is mean curvature vector [8].

If we set $\mu = ||H||^2$, then for the umbilical submanifold M with mean curvature parallel in the normal bundle, we have $X.\mu = 0$ for any $X \in \Gamma(TM)$, that is, μ is constant.

If $\mu \neq 0$, define a unit vector $e \in \nu$ in the normal bundle, by setting $H = \sqrt{\mu} e$. The normal bundle can be split into the direct sum $\mu = \{e\} \oplus \{e\}^{\perp}$, where $\{e\}^{\perp}$ is the orthogonal compliment of the line sub-bundle e spanned by e. For each $X \in \Gamma(TM)$. Set

$$\phi X = \psi(X) - A(X)e + P(X), \qquad \phi e = t + F, \tag{4.1}$$

where $\psi(x)$ is the tangential components of ϕX, while $A(X)$ and $P(X)$ are the $\{e\}$ and $\{e\}^{\perp}$ components, respectively. t and F are the $\{e\}$ and $\{e\}^{\perp}$ components of ϕe, respectively, in view of the skew-symmetry of ϕ.

Lemma 4.1. *Let M be a totally umbilical submanifold of a trans-Sasakian manifold $\bar{M}$ with curvature vector parallel to the normal bundle. If $\mu \neq 0$, then for any $X \in \Gamma(TM)$ following hold:*
(i) $\bar{\nabla}_X e = -\sqrt{\mu} X$,
(ii) $\nabla_X t = -\sqrt{\mu}\psi(X) - \alpha g(e, \xi)X - \beta g(e, \xi)\psi(X)$,
(iii)$\nabla^{\perp}_X F = -\sqrt{\mu} P(X) - \beta A(X)\xi$.

Proof. Taking inner product with respect to Y, in both sides of the equation (2.8), we obtain

$$\bar{\nabla}_X N = -g(H, N)X + \nabla^{\perp}_X N.$$

Putting $N = e$ in the above equation, we obtain

$$\bar{\nabla}_X e = -\sqrt{\mu}X.$$

Thus (i) is proved. Next we compute $(\bar{\nabla}_X \phi)e$, for $X \in \Gamma(TM)$. Using the equations (2.5) and (4.1) we obtain

$$\begin{aligned} \nabla_X t + \nabla^{\perp}_X F \quad &+ \quad \sqrt{\mu}(\psi(X) - A(X)e + P(X)) + \sigma(X, \phi(e)) \\ &= \quad \alpha(g(X,e)\xi - \eta(e)X) + \beta(g(\phi X, e)\xi - \eta(e)\phi(X)). \end{aligned}$$

Next comparing the tangential part we have

$$\nabla_X t = -\sqrt{\mu}\phi(X) - \alpha g(e,\xi)X - \beta g(e,\xi)\psi(X).$$

Thus (ii) is proved. Now comparing $\{e\}^{\perp}$ component and using the result $A(X) = g(X,t)$ we obtain

$$\nabla^{\perp}_X F = -\sqrt{\mu}P(X) - \beta A(X)\xi.$$

Thus (iii) is proved. □

Lemma 4.2. *Let M be a totally umbilical sub-manifold of a tans-Sasakian manifold $\bar{M}$ with mean curvature vector parallel in the normal bundle. If $\mu \neq 0$, and $\xi \perp e$, then, setting $\xi = \xi_1 + \xi_2$, where ξ_1 is the tangential component and ξ_2 is the $\{e\}^{\perp}$-component of ξ, we have*

(i) $\nabla_X \xi_1 = -\alpha\psi(X) + \beta(X - \eta(X)\xi_1)$,

(ii)$(\nabla_X \psi)Y = (\alpha - \frac{\mu}{\alpha})(g(X,Y)\xi_1 - \eta(Y)X) + \beta(g(\phi X, Y)\xi_1 - \eta(Y)\psi(X))$.

Proof. Putting $\xi = \xi_1 + \xi_2$ in the equation (2.6) and (4.1) we have

$$\nabla_X \xi_1 + \nabla_X \xi_2 + \sigma(X,\xi) = -\alpha(\psi X - A(X)e + P(X)) + \beta(X - g(X,\xi)\xi).$$

Comparing tangential part we have (i), and comparing e component, we have $\sigma(X,\xi) = \alpha A(X)e$ i.e.,

$$\sqrt{\mu}\eta(X) = \alpha A(X), \qquad \sqrt{\mu}\xi_1 = \alpha t. \tag{4.2}$$

Now using the equations (2.5) and (4.1) we have

$$\begin{aligned} &\nabla_X(\psi Y) - \nabla_X(AY)e - A(Y)(\nabla_X e) - \psi(\nabla_X Y) + A(\nabla_X Y)e - \\ &\quad P(\nabla_X Y) = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X). \end{aligned}$$

Using the Lemma 4.1, we obtain from the above equation

$$\begin{aligned} &(\nabla_X \psi)Y + (\nabla_X P)Y + \sqrt{\mu}g(X,Y)(\frac{\sqrt{\mu}}{\alpha}\xi_1 + F) + \frac{\sqrt{\mu}\beta}{\alpha}g(X,Y)e - \\ &\quad \frac{\mu}{\alpha}g(X,Y)e = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X). \end{aligned}$$

Comparing the tangential part we obtain (ii). □

It is known that a trans-Sasakian manifold of dimension greater or equal to five is either α-Sasakian or β-Kenmotsu[12]. So let us first study umbilical submanifolds of α-Sasakian manifolds, then umbilical submanifolds of β-Kenmotsu manifold of dimension greater or equal to five. We shall now deduce the following:

Theorem 4.1. *Let M be a totally umbilical submanifold of an α-Sasakian manifold of dimension greater or equal to five with mean curvature vector parallel in the normal bundle. Then one of the following hold* :
(i) *M is totally geodesic*
(ii) *M is isometric to a sphere*
(iii) *M is homothetic to a Sasakian manifold.*

Proof. Since H is parallel in the normal bundle, μ is a constant. If $\mu = 0$, then $H = 0$, and consequently $\sigma(X, Y) = 0$, $X, Y \in \Gamma(TM)$. Thus the submanifold M is totally geodesic, which proves the first part of the theorem.

Next we assume that $\mu \neq 0$. Define a smooth function $f : M \to R$ by $f = g(e, \xi), X \in \Gamma(TM)$. Then Lemma 4.1, and equations (2.6), (2.8), (2.9), imply that

$$\begin{aligned} Xf &= g(\nabla_X \xi, e) + g(\xi, \nabla_X e) \\ &= \alpha g(X, t) - \sqrt{\mu} g(\xi, X). \end{aligned}$$

So, by using the equations (2.6) and the Lemma 4.1, we have,

$$XYf - (\nabla_X Y)f = -\alpha^2 f g(X, Y).$$

Then

$$g(\nabla_X \mathrm{grad} f, Y) = -\alpha^2 f g(X, Y). \tag{4.3}$$

Taking trace of this equation we have

$$\Delta f = -\alpha^2 n f. \tag{4.4}$$

If f is non-constant function, according to [15], the equation (4.4) is the differential equation whose existence ensures necessary and sufficient condition for M to be isometric to a sphere of radius $\frac{1}{\alpha}$.

If f is a constant, then equation (4.4) gives $-n\alpha^2 f = 0$, α is non-zero and consequently $f = 0$, that is $\xi \perp e$.

Now define a smooth function $G : M \to R$ by

$$G = \frac{1}{2} tr.\psi^2. \tag{4.5}$$

Note that (4.1) gives $g(\psi Y, X) = -g(\psi Y, X), \quad X, Y \in \Gamma(TM)$.

Let ω be a 1-form defined by $\omega = dG$. For each $p \in M$ we can choose a local orthonormal frame $\{e_1, ..., e_n\}$ of M such that $\nabla e_i(p) = 0$. Thus, for any $Z \in \Gamma(TM)$, we have

$$\omega(Z) = ZG = \sum_{i=1}^{n} g((\nabla_Z \psi)(e_i), \psi(e_i)). \tag{4.6}$$

Using the Lemma 4.2, we obtain

$$\omega(Z) = 2(\alpha - \frac{\mu}{\alpha})g(\psi Z, \xi_1). \tag{4.7}$$

The first covariant derivative of (4.7) is

$$(\nabla\omega)(Y,Z) = 2(\alpha - \frac{\mu}{\alpha})(-\alpha g(\psi Y, \psi Z)) + 2(\alpha - \frac{\mu}{\alpha})^2(g(\xi_1,\xi_1)g(Y,Z) - g(Y,\xi)g(Z,\xi_1)).$$

And consequently using the equation (4.7) and the above equation we have

$$(\nabla^2\omega)(X,Y,Z) + k^2(2g(Y,Z)\omega(X) + g(X,Y)\omega(Z) + g(X,Z)\omega(Y)) = 0. \tag{4.8}$$

where $k^2 = (\alpha - \frac{\mu}{\alpha})$. According to Tanno [18], existence of the differential equation (4.8) in which, G being non-constant, is the necessary and sufficient condition for M to be isometric to a sphere. This again leads to case (ii). Suppose G is constant function. Then equation (4.7) gives $\psi(\xi_1) = 0$. Define a smooth function $G_1 : M \to R$ by

$$G_1 = g(\xi_1, \xi_1).$$

Then using the Lemma 4.2, we get $X\alpha = 0$, $X \in \Gamma(TM)$. In others words ξ_1 has constant length. Taking the covariant derivative in (i) of Lemma 4.2 and using (ii), we get

$$\nabla_X\nabla_Y\xi_1 - \nabla_{\nabla_X Y}\xi_1 = k^2(g(X,Y)\xi_1 - g(Y,\xi_1)X). \tag{4.9}$$

where $k^2 = (\alpha - \frac{\mu}{\alpha})$. Furthermore, from (i) of the Lemma 4.2, it follows that ξ_1 is a Killing vector field. Since $k \neq 0$ as $(\alpha - \frac{\mu}{\alpha}) \neq 0$ and ξ_1 is a Killing vector field of constant length, which satisfies (4.9), a result of Okumura [13] states that, if $\xi_1 \neq 0$, then M is homothetic to a Sasakian manifold. which is (iii). Thus to complete the proof we have only to show that $\xi_1 = 0$ cannot happen.

We note that if $\xi_1 = 0$ then $\xi \in \{e\}^\perp$ as $\xi \perp e$. Lemma 4.2 gives $\psi(X) = 0$, i.e. ϕX is normal to M for all $X \in \Gamma(TM)$. Again, equation (4.2) gives $t = 0$, i.e. $\phi e = F \in \{e\}^\perp$, and $g(\phi X, \phi e) = g(X, e) - \eta(X)\eta(e) = 0$, $X \in \Gamma(TM)$, $g(\phi e, \xi) = 0$. Thus the dim of $\nu \geq$ dim$\{M\}$ +dim$\{\xi\}$ + dim$\{e\}$ + dim$\{\phi e\}$ - 1, which is impossible as dim$\{\bar{M}\}$=$2n+1$. This completes the proof. □

Theorem 4.2. *Let M be a totally umbilical submanifold of a β-Kenmotsu manifold of dimension greater or equal to five with mean curvature vector parallel in the normal bundle. Then one of the following hold:*
(i) M is totally geodesic
(ii) M is isometric to a sphere
(iii) M is homothetic to a Sasakian manifold.

Proof. Since H is parallel in the normal bundle, μ is a constant. If $\mu = 0$, then $H = 0$, and consequently $\sigma(X,Y) = 0$, $X, Y \in \Gamma(TM)$. Thus the submanifold M is totally geodesic, which proves the first part of the theorem.

Next we assume that $\mu \neq 0$. Define a smooth function $f : M \to R$ by $f = g(e, \xi), X \in \Gamma(TM)$. Then Lemma 4.1, and equations (2.6), (2.8), (2.9), imply that

$$\begin{aligned} Xf &= g(\nabla_X \xi, e) + g(\xi, \nabla_X e) \\ &= \beta g(Y, e) - g(\xi, Y)(\beta f + \sqrt{\mu}). \end{aligned}$$

So, by using the equation (2.6) and the Lemma 4.1, we have,

$$XYf - (\nabla_X Y)f = -2\sqrt{\mu}\beta g(X, Y) - \beta^2 g(X, Y)f + 2\beta\eta(X)\eta(Y)(\beta f + \sqrt{\mu}).$$

Then

$$g(\nabla_X \mathrm{grad} f, Y) = -2\sqrt{\mu}\beta g(X, Y) - \beta^2 g(X, Y)f + 2\beta\eta(X)\eta(Y)(\beta f + \sqrt{\mu}). \tag{4.10}$$

Taking trace of this equation we have

$$\Delta f = 2\sqrt{\mu}\beta(n + 1) - \beta^2(n + 2)f. \tag{4.11}$$

Replace f_1 by f, where f_1 is defined by

$$f_1 = f + \frac{2\sqrt{\mu}\beta(n + 1)}{\beta^2(n + 2)}. \tag{4.12}$$

If β is constant then $\Delta f = \Delta f_1$, and the Equation (4.11) gives

$$\Delta f_1 = -\beta^2(n + 2)f_1. \tag{4.13}$$

If f_1 is non constant function, then the equation (4.13) is the differential equation in [15], which is necessary and sufficient condition for M to be isometric to a sphere of radius $\frac{1}{\beta^2(n+2)}$.

Now if f_1 is a non-constant function then by similar calculation done to prove the previous theorem we can prove this theorem. □

5 Totally umbilical submanifolds of three-dimensional trans-Sasakian manifolds

In the previous section, we have studied submanifolds of trans-sasakian manifolds of dimension greater or equal to five. In this section, we shall study totally umbilical submanifolds of a three-dimensional trans-Sasakian maifold. Here we prove the following:

Theorem 5.1. *A totally umbilical submanifold of a three-dimensional trans-Sasakian manifold is totally geodesic.*

Proof. From the equation (2.4), we get

$$\bar{R}(X, Y)Z = (4(\alpha^2 - \beta^2) - \frac{r}{2})(\eta(Y)X - \eta(X)Y).$$

Since the right hand side of the above equation is a vector field in the tangent bundle of the submanifold, we get

$$(\bar{R}(X,Y)Z)^{\perp} = 0. \tag{5.1}$$

Now if the submanifold is totally umbilical then

$$\sigma(X,Y) = g(X,Y)H.$$

Here H is mean curvature vector. Hence

$$(\nabla_W \sigma)(X,Y) = g(X,Y)\nabla^{\perp}_W H. \tag{5.2}$$

Hence from (2.13)

$$(\bar{R}(X,Y)Z)^{\perp} = g(Y,Z)\nabla^{\perp}_X H - g(X,Z)\nabla^{\perp}_Y H. \tag{5.3}$$

In view of (5.1) and (5.3)

$$g(Y,Z)\nabla^{\perp}_X H = g(X,Z)\nabla^{\perp}_Y H.$$

Putting $Z = \xi$ and replacing X by ϕX, we get

$$\nabla^{\perp}_{\phi X} H. \tag{5.4}$$

So form (5.2)

$$(\nabla_{\phi W}\sigma)(X,Y) = 0.$$

Hence

$$\nabla^{\perp}_{\phi W}\sigma(X,Y) - \sigma(\nabla_{\phi W}X,Y) - \sigma(X,\nabla_{\phi W}Y) = 0.$$

Putting $Y = \xi$ and using the Lemma 3.2 we have

$$\alpha\sigma(X,W) + \beta\sigma(X,\phi W) = 0. \tag{5.5}$$

Replacing W by ϕW we get

$$-\beta\sigma(X,W) + \alpha\sigma(X,\phi W) = 0. \tag{5.6}$$

From (5.5) and (5.6) we get

$$\sigma(X,W) = 0. \tag{5.7}$$

This proves the theorem. □

Remark 5.1. *The submanifolds of a three-dimensional trans-Sasakian manifold are either of dimension one or two. One dimensional submanifolds are trivial. Two dimensional submanifolds are hypersurfaces. So, we can state the following* :

Corollary 5.1. *A totally umbilical hypersurface of a three-dimensional trans-Sasakian manifold is totally geodesic.*

References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds,* Birkhauser, 2005.

[2] D. E. Blair and J. A. Oubina, *Conformal and related changes of matric on the product of two almost contact metric manifolds,* Publ. Math. **34** (1990), 199-207.

[3] B. Y. Chen, *Totally umbilical submanifolds*, Soochow J. Math., 5 (1979), 9-37.

[4] B. Y. Chen, *Geometry of Submanifolds*, Dover Publications, 2019.

[5] D. Chinea, and P. S. Prestelo, *Invariant submanifolds of a trans-Sasakian manifold,* Publ. Math. Debrecen, **38** (1991), 103-109.

[6] U. C. De and A. Sarkar, *On three dimensional trans-Sasakian manifolds,* Extracta Mathematicae, **23** (2008), 265-277.

[7] A. De, *Totally geodesic submanifolds of a trans-Sasakian manifold,* Proc. Est. Acad. Sci., **62** (2013), 249-257.

[8] S. Deshmukh and M. A. Al-Gwiz, *Totally umbilical submanifolds of Sasakian manifolds,* New Zeland J. Math., **22** (1993), 43-47.

[9] F. Erkekogulu, E. Garcia-Rio, D. Kupeli and B. Unal, *Characterizing specific Riemannian manifolds by differential equation*, Acta Applicandae Mathematicae, **76** (2003), 195-219.

[10] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants,* Ann. Mat. Pura Appl., **123** (1980), 35-58.

[11] C. Hu, Y. Wang, *A note on invariant Submanifolds of trans-Sasakian manifolds*, Int. Elect. J. of Geom., **9** (2016) 27-35.

[12] J. C. Marrero and D. Chinea, *On trans-Sasakian manifolds*, In Proceedings of the XIVth Spanish-Portuguese Conference on Mathematics, Vol. I-III (Spanish) (Puerto de la Cruz, 1989), Univ. La Laguna, 1990, 655-659.

[13] M. Okumura, *Totally umbilical submanifolds of a Kaeher manifold,* J. Math. Soc. Japan, **19** (1964), 371-327.

[14] J. A. Oubina, *New classes of almost contact metric structures,* Publ. Math. Debrecen, **32** (1985), 187-193.

[15] M. Obata, *Riemannian manifolds admitting a solution of a certain system of differential equations,* Proc. U.S.-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965, 101-114.

[16] A. Sarkar, M. Sen, *On invariant submanifolds of LP-Sasakian manifolds,* **27** (2012), 35-58.

[17] Y. Tashiro, *Complex Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc., **117** (1965), 251-275.

[18] S. Tanno, *Some differential equations on Riemannian manifolds*, J. Math. Soc., Japan, **30** (1978), 509-531.

[19] L. Verstraelen, *Comments on pseudosymmetry in the sense of R. Deszcz, Geometry and Topology of Submanifolds,* **6** (1994), 199-209.

Authors' address:

Avijit Sarkar and Nirmal Biswas
Department of Mathematics, University of Kalyani
Kalyani 741235, West Bengal, India.
E-mail:avjaj@yahoo.co.in , nirmalbiswas.maths@gmail.com