Eta star-Ricci solitons on Sasakian manifolds

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Abstract. The paper deals with eta star Ricci solitons on Sasakian manifolds. In this paper, we study and investigate the geometric properties of eta star Ricci solitons on such manifolds and also provide an example of such solitons on five dimensional Sasakian manifold.

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1 Introduction

In 1982, Hamilton [14] introduced the notion of the Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}. \tag{1.1}$$

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) [6]. A Ricci soliton is a triple (g, V, λ_*) with g a Riemannian metric, V a vector field (called the potential vector field), and λ_* a real scalar such that

$$\pounds_V g + 2S + 2\lambda_* g = 0, \tag{1.2}$$

where S is a Ricci tensor of M and $\pounds_V$ denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ_* is negative, zero and positive, respectively [15]. A Ricci soliton with V zero is reduced to Einstein equation. Metrics satisfying (1.2) is interesting and useful in physics and is often referred as quasi-Einstein [7, 8]. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$, projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contributions in this direction is due to Friedmann [11], who discusses some aspects of it. Ricci solitons were introduced in Riemannian geometry [14], as the

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self-similar solutions of the Ricci flow which play an important role in understanding its singularities.

We recommend the reference [10] for more details about the Ricci flow and Ricci soliton. In this connection, we mention that within the framework of contact geometry Ricci solitons were first considered by Sharma in [22].

As a generalization of Ricci soliton, the notion of η_*-Ricci soliton introduced by Cho and Kimura [9], which was treated by Calin and Crasmareanu on Hopf hypersurfaces in complex space forms [6]. An η_*-Ricci soliton is a tuple (g, V, λ_*, μ_*), where V is vector field and λ_*, μ_* are constants, g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda_* g + 2\mu_* \eta_* \otimes \eta_* = 0, \tag{1.3}$$

where S is the Ricci tensor associated to g.

In particular, if $\mu_* = 0$, then the notion of η_*-Ricci solitons (g, V, λ_*, μ_*) reduces to the notion of Ricci solitons (g, V, λ_*). If $\mu_* \neq 0$, then the η_*-Ricci solitons are called proper η_*-Ricci solitons. An η_*-Ricci solitons in quasi-Sasakian 3-manifolds studied by [16].

The notion of $*$-Ricci solitons on almost Hermitian manifolds was introduced by Tachibana [20], in 1959. Hamada [13] studied $*$-Ricci flat real hypersurfaces in non flat complex space forms. The $*$-Ricci tensor in contact metric manifold is given by [15]

$$S^*(X,Y) = \frac{1}{2}(trace\{\varphi_* o R(X, \varphi_* Y)\}),$$

for all vector fields X, Y on M and φ_* is a $(1,1)$-tensor field. Further, if $S^*(X,Y) = \lambda_* g(X,Y)$ for all vector fields $X, Y \perp \xi_*$ and λ_* is constant then we say that M is $*$-Einstein.

Recently, Kaimakamis-Panagiotidou [18] introduced the notion of $*$-Ricci soliton where they essentially modified the definition of Ricci soliton by replacing the Ricci tensor S in (1.1) with the $*$-Ricci tensor S^* .

Definition 1.1. A Riemannian metric g on M is called a $*$-Ricci soliton if there exists a constant λ_* and a vector field V such that

$$\frac{1}{2}(\pounds_V g)(X,Y) + S^*(X,Y) + \lambda_* g(X,Y) = 0, \tag{1.4}$$

for all vector fields X, Y on M.

If λ_* in (1.2) is a smooth function, then we say that it is an almost $*$-Ricci soliton. Moreover, if the vector field V is a gradient of a smooth function f, then we say that it is almost $*$-Ricci soliton. Note that a $*$-Ricci soliton is trivial if the vector field V is Killing, and in this case the manifold becomes $*$-Einstein. Equation (1.2) has been studied by Kaimakamis-Panagiotidou (see [18]) within the framework of real hypersurfaces of a complex space form (e.g. see [13], [17]). So far we know that the notion of $*$-Ricci tensor appears on complex and contact manifolds only. However, some classifications of complex and contact manifolds are available in the literature in terms of the $*$-Ricci tensor. Making use of the formulas of Sasakian manifolds one can easily deduce an expression of the $*$-Ricci tensor. Due to the presence of some extra terms in the expression of $*$-Ricci tensor the defining condition of the $*$-Ricci soliton is different from Ricci soliton.

In 2018, Ghosh and Patra [12] first undertook the study of $*$-Ricci solitons on almost contact metric manifolds. In their paper, the authors proved that if the metric of Sasakian manifold is a $*$-Ricci soliton, then it is either positive Sasakian, or null-Sasakian. Furthermore, they also proved that if a complete Sasakian metric is a gradient almost $*$-Ricci soliton, then it is positive-Sasakian and isometric to a unit sphere S^{2n+1}. Here, we also mention the works of Prakasha and Veeresha [21] within the frame-work of paracontact geometry. Majhi et al. [19] studied $*$-Ricci solitons and $*$-gradient Ricci solitons on three dimensional Sasakian manifolds. Ricci solitons have been studied in many contexts by several authors. Inspired by above-mentioned works, here, we consider an eta star-Ricci soliton in the framework of Sasakian manifold.

Therefore, It is an interesting and natural to see the condition in case of an eta star-Ricci soliton. From equations (1.3) and (1.4) we are introducing the notion of an eta star-Ricci solitons by the following definition:

Definition 1.2. A Riemannian metric g on M is called an eta star-Ricci soliton if there exists constants λ_*, μ_* and a vector field V such that

$$\frac{1}{2}(\pounds_V g)(X,Y) + S^*(X,Y) + \lambda_* g(X,Y) + \mu_* \eta_*(X)\eta_*(Y) = 0, \tag{1.5}$$

for all vector fields X, Y on M.

In particular $\mu_* = 0$ then the data (g, ξ_*, λ_*) is a $*$-Ricci soliton.

The present paper is organized as follows:

In section 2, we reminisce some fundamental formulas and properties of Sasakian manifolds. In section 3, we study a Sasakian manifolds whose metric is an eta star-Ricci soliton, then it is either positive Sasakian (in case the soliton vector field is Killing), or null-Sasakian (in case the soliton vector field leaves φ_* invariant).

2 Preliminaries

In this section, we collect some basic facts about contact metric manifolds. We refer to [2] for a more detailed treatment. An odd-dimensional differentiable manifold M is called a contact manifold if there exists a globally defined 1-form η_* such that $(d\eta_*)^n \wedge \eta_* \neq 0$. On a contact manifold there exists a unique global vector field ξ_* satisfy following conditions:

$$d\eta_*(\xi_*, X) = 0, \eta_*(\xi_*) = 1 \tag{2.1}$$

for any vector field X tangent to M.

Moreover, it is well-known that there exist a $(1,1)$-tensor field φ_*, a Riemannian metric g which satisfy following conditions:

$$\varphi_*^2 = -I + \eta_* \otimes \xi_*, \tag{2.2}$$

$$g(\varphi_* X, \varphi_* Y) = g(X,Y) - \eta_*(X)\eta_*(Y), \tag{2.3}$$

(2.4) $$d\eta_*(X,Y) = g(X,\varphi_* Y)$$

(2.5) $$g(X,\xi_*) = \eta_*(X),$$

for all X,Y tangent to M. As a consequence of the above relations, we have

(2.6) $$\varphi_*\xi_* = 0, \eta_* \circ \varphi_* = 0,$$

The structure $(\varphi_*,\xi_*,\eta_*,g)$ is called a contact metric structure and the metric g is called an associated metric. A Riemannian manifold M together with the structures $(\varphi_*,\xi_*,\eta_*,g)$ is said to be a contact metric manifold and we denote it by $M(\varphi_*,\xi_*,\eta_*,g)$. On a contact metric manifold (e.g., [2], p.84)

(2.7) $$\nabla^*_X\xi_* = -\varphi_* X - h\varphi_* X,$$

for any vector field X on M and ∇^* denotes the operator of covariant differentiation of g. If the vector field ξ_* is Killing (equivalently, $h = 0$ ([2], p.87)) with respect to g, then the contact metric structure on M is said to be K-contact. An almost contact metric structure on M is said to be normal if the almost complex structure J on $M \times R$ defined by (e.g., see Blair [2] p.80):

$$J(X, fd/dtX) = (\varphi_* X - f\xi_*, \eta_*(X)d/dt),$$

for any vector field X on M; where f is a real valued function on $M\times R$, is integrable. A normal contact metric manifold is said to be Sasakian. Also, a contact metric structure on M is said to be Sasakian if the metric cone $C(M)(dr^2 + r^2 g, d(r^2\eta_*))$ is Kaehler (e.g., [3]). A Sasakian manifold is K-contact but the converse is true only in dimension 3 (e.g., [2], p.87). On Sasakian manifold the equation (2.2) give

(2.8) $$\nabla^*_X\xi_* = -\varphi_* X,$$

for all vector fields X on M.

Also, Sasakian manifolds satisfy :

(2.9) $$(\nabla^*_X\eta_*)Y = g(\varphi_* X, Y),$$

(2.10) $$(\nabla^*_X\varphi_*)Y = g(X,Y)\xi_* - \eta_*(Y)X,$$

where ∇^* denotes the operator of covariant differentiation with respect to the metric g on M.

Moreover, on a Sasakian manifold the following relations are hold ;

(2.11) $$R(\xi_*, X)Y = [g(X,Y)\xi_* - \eta_*(Y)X],$$

(2.12) $$R(X,Y)\xi_* = [\eta_*(Y)X - \eta_*(X)Y],$$

(2.13) $$R(\xi_*, X)\xi_* = [\eta_*(X)\xi_* + X],$$

$$S(X, \xi_*) = 2n\eta_*(X), \tag{2.14}$$

$$Q\xi_* = 2n\xi_*, \tag{2.15}$$

$$S(\xi_*, \xi_*) = 2n, \tag{2.16}$$

where α is some constant, R is the Riemannian curvature, S is the Ricci tensor and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$, for all $X, Y \in M$.

A contact metric manifold M is said to be η_*-Einstein, if there exists two smooth functions α, β such that the Ricci tensor can be expressed as

$$S(X,Y) = \alpha g(X,Y) + \beta\eta_*(X)\eta_*(Y),$$

for all vector fields X, Y on M. If M is a K-contact manifold of dimension > 3, then the functions α and β become constant (Okumura [20]). It is well-known that a D-homothetic deformation (see Tanno [23]):

$$\overline{\eta_*} = a\eta_*, \overline{\xi_*} = \frac{1}{a}\xi_*, \overline{\varphi_*} = \varphi_*, \overline{g} = g + a(a-1)\eta_* \otimes \eta_*,$$

for a positive real constant a, transforms a contact metric structure into a new contact metric structure, and preserves many basic properties like being K-contact (in particular, Sasakian). Using the formulas for D-homothetic deformation [23], one can easily verify that, a K-contact η_*-Einstein manifold transforms to a K-contact η_*-Einstein manifold such that $\overline{\alpha} = \frac{\alpha+2-2a}{a}$ a and $\beta = 2n - \overline{\alpha}$. In particular, D-homothetic deformation on a unit sphere S^{2n+1} (which is Einstein and K-contact) gives rise to a family of η_*-Einstein K-contact structures for each positive real number a. It is interesting to note that the particular value: $\alpha = -2$ remains fixed under a D-homothetic deformation, and as $\alpha + \beta = 2n$, we have $\beta = 2n + 2$ which is also fixed. So, we adopt the following definition.

Definition 2.1. A K-contact η_*-Einstein manifold with $\alpha = -2$ is said to be D-homothetically fixed.

Let J denote the restriction of φ_* on D. Then J defines an almost complex structure on D. For a Sasakian manifold $(D, J, d\eta_*)$ defines a Kaehler metric on D, with the transverse Kaehler metric g^T related to the Sasakian metric g as $g = g^T + \eta_* \otimes \eta_*$. By a direct computation it is easy to derive that the transverse Ricci tensor Ric^T of g^T and the Ricci tensor S of g are connected by

$$S^T(X,Y) = S(X,Y) + 2g(X,Y),$$

for arbitrary vector fields X, Y in D. The Ricci form ρ and transverse Ricci form ρT are defined by

$$\rho(X,Y) = S(X, \varphi_*Y), \quad \rho^T(X,Y) = S^T(X, \varphi_*Y),$$

for $X, Y \in D$. The basic first Chern class $2\pi c_1^B$ of D is represented by ρT . The condition $c_1^B = 0$ provides an example of the so-called transverse Calabi- Yau structure.

An η_*-Einstein Sasakian manifold with $\alpha = -2$ and $\beta = 2n+2$ is known as null-Sasakian, which is characterized by $c_1^B = 0$. A simple example of a null-Sasakian manifold is a Sasakian space-form R^{2n+1} with constant φ_*- sectional curvature -3, identifiable with a $(2n+1)$-dimensional Heisenberg group. Further, an η_*-Einstein Sasakian manifold with $\alpha > -2$ is known as positive-Sasakian, which is characterized by $c_1^B > 0$. In this case, the transverse geometry is Fano and the η_*-Einstein condition implies that the transverse geometry is Käehler-Einstein with positive scalar curvature. For details we refer to [4].

Definition 2.2. A vector field V on a contact metric manifold M is said to be an infinitesimal contact transformation if

$$\pounds_V \eta_* = f\eta_*, \tag{2.17}$$

for some smooth function f on M. V is said to be an infinitesimal automorphism of the contact metric structure if it leaves all the structure tensor $\eta_*, \xi_*, g, \varphi_*$ invariant ([23]).

3 Eta star Ricci soliton on Sasakian manifold

First, we need the following lemmas to obtain the further results.

Lemma 3.1. *An odd dimensional Sasakian manifold satisfies*

$$(\nabla^*_X Q)\xi_* = Q\varphi_* X - 2n\varphi_* X. \tag{3.1}$$

Proof. Note that (2.5) implies $Q\xi_* = 2n\xi_*$. Covariant differentiating this and recalling (2.3) provides (3.1). □

Lemma 3.2. *Let $M(\varphi_*, \xi_*, \eta_*, g)$ be a Sasakian manifold. Then*

$$\nabla^*_{\xi_*} Q = 0. \tag{3.2}$$

Lemma 3.3. *The star-Ricci tensor on a Sasakian manifold $M(\varphi_*, \xi_*, \eta_*, g)$ is given by*

$$S^*(X,Y) = S(X,Y) - (2n-1)g(X,Y) - \eta_*(X)\eta_*(Y), \tag{3.3}$$

for all vector fields X, Y on M.

Theorem 3.4. *Let $M(\varphi_*, \xi_*, \eta_*, g)$ be a Sasakian manifold. If g is an eta star-Ricci soliton, then M is an η_*-Einstein and the Ricci tensor can be expressed as*

$$S(X,Y) = (2n + \frac{\mu_*}{2} - \frac{\lambda_*}{2} - 1)g(X,Y) + (\frac{\lambda_*}{2} - \frac{\mu_*}{2} + 1)\eta_*(X)\eta_*(Y), \tag{3.4}$$

for all vector fields X, Y on M.

Proof. Feeding the expression of $*$-Ricci tensor as given by (3.3) into the eta star-Ricci soliton equation (1.3), it follows that

$$\begin{aligned}(\pounds_V g)(X,Y) \quad = \quad & -2S(X,Y) \\ & +2(2n-1-\lambda_*)g(X,Y) - 2(\mu_*-1)\eta_*(X)\eta_*(Y).\end{aligned} \tag{3.5}$$

Taking covariant derivative of (3.5) along an arbitrary vector field Z, we get

$$(\nabla^*_Z \pounds_V g)(X,Y) = -2(\nabla^*_Z S)(X,Y) + 2(\mu_*-1)[g(X,\varphi_* Z)\eta_*(Y) + g(Y,\varphi_* Z)\eta_*(X)]. \tag{3.6}$$

From Yano [24], we know the following well known commutation formula:

$$\begin{aligned} & (\pounds_V \nabla^*_X g - \nabla^*_X \pounds_V g - \nabla^*_{[V,X]} g)(Y,Z) \\ = & -g((\pounds_V \nabla^*)(X,Y),Z) - g((\pounds_V \nabla^*)(X,Z),Y), \end{aligned}$$

for all $X,Y,Z \in M$. Since $\nabla^* g = 0$, the above equation gives

$$(\nabla^*_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla^*)(X,Y),Z) + g((\pounds_V \nabla^*)(X,Z),Y), \tag{3.7}$$

for all $X,Y,Z \in M$. As $\pounds_V \nabla^*$ is a symmetric, it follows from (3.7) that

$$g((\pounds_V \nabla^*)(X,Y),Z) = \frac{1}{2}(\nabla^*_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla^*_Y \pounds_V g)(Z,X) - \frac{1}{2}(\nabla^*_Z \pounds_V g)(X,Y). \tag{3.8}$$

Using (3.6) in (3.8), we have

$$g((\pounds_V \nabla^*)(X,Y),Z) = (\nabla^*_Z S)(X,Y) - (\nabla^*_X S)(Y,Z) - (\nabla^*_Y S)(Z,X) + 2(\mu_*-1)[g(Z,\varphi_* X)\eta_*(Y) + g(Z,\varphi_* Y)\eta_*(X)]. \tag{3.9}$$

Plugging $Y = \xi_*$ in the above equation and using (3.1) and (3.2), we have

$$(\pounds_V \nabla^*)(X,\xi_*) = -2Q\varphi_* X + 2(2n+\mu_*-1)\varphi_* X. \tag{3.10}$$

Covariant differentiating the above equation along Y and using (2.3), we obtain

$$(\nabla^*_Y \pounds_V \nabla^*)(X,\xi_*) = (\pounds_V \nabla^*)(X,\varphi_* Y) - 2(\nabla^*_Y Q)\varphi_* X + 2\eta_*(X)QY - 2(2n+\mu_*-1)\eta_*(X)Y + 2(\mu_*-1)g(X,Y)\xi_*. \tag{3.11}$$

Feeding the above obtained expression into the following well known formula (see Yano [24])

$$(\pounds_V R)(X,Y)Z = (\nabla^*_X \pounds_V \nabla^*)(Y,Z) - (\nabla^*_Y \pounds_V \nabla^*)(X,Z), \tag{3.12}$$

one can derive

$$\begin{aligned} (\pounds_V R)(X,Y)\xi_* = & (\pounds_V \nabla^*)(Y,\varphi_* X) - (\pounds_V \nabla^*)(X,\varphi_* Y) \\ & + 2(\nabla^*_Y Q)\varphi_* X - 2(\nabla^*_X Q)\varphi_* Y \\ & + 2\eta_*(Y)QX - 2\eta_*(X)QY \\ & + 2(2n+\mu_*-1)\{\eta_*(X)Y - \eta_*(Y)X\}. \end{aligned} \tag{3.13}$$

Substituting ξ_* for Y in (3.13) and then using $(2.4), (2.7), (3.4), (3.12)$ and $\varphi_*\xi_* = 0$, we deduce

$$(\pounds_V R)(X,\xi_*)\xi_* = 4\{QX + (\mu_*-1)\eta_*(X)\xi_* - (2n+\mu_*-1)X\}. \tag{3.14}$$

Setting $Y=\xi_*$ in (3.8) and using (2.4) it follows that

$$(\pounds_V g)(X,\xi_*) = -2(\lambda_* + \mu_*)\eta_*(X).$$

Lie-differentiating the second term of (2.1) along V and by virtue of the last equation, we find

$$(\pounds_V \eta_*)(X) - g(\pounds_V \xi_*, X) + 2(\lambda_* + \mu_*)\eta_*(X) = 0. \tag{3.15}$$

From (3.15), we also obtain $(\pounds_V \xi_*) = (\lambda_* + \mu_*)$ and $(\pounds_V \eta_*)(\xi_*) = -(\lambda_* + \mu_*)$, where we have used the Lie-derivative of $g(\xi_*, \xi_*) = 1$ along V.

Next, Lie-derivating the equation $R(X,\xi_*)\xi_* = X - \eta_*(X)\xi_*$ (follows from (2.6)) along V and taking into account (2.6), (3.14) and $(\pounds_V \xi_*) = (\lambda_* + \mu_*)$ one can derive

$$\begin{aligned}(\pounds_V \eta_*)(X)\xi_* &= g(\pounds_V \xi_*, X)\xi_* - QX + (4n + 2\mu_* - 2\lambda_* - 4)X \\ &\quad -4(\mu_* - 1)\eta_*(X)\xi_*.\end{aligned} \tag{3.16}$$

for all vector fields X on M. Making use of (3.15), We obtain

$$S(X,Y) = (2n + \frac{\mu_*}{2} - \frac{\lambda_*}{2} - 1)g(X,Y) + (\frac{\lambda_*}{2} - \frac{\mu_*}{2} + 1)\eta_*(X)\eta_*(Y).$$

□

Theorem 3.5. *Let $M(\varphi_*, \xi_*, \eta_*, g)$ be a Sasakian manifold. If g is an eta star-Ricci soliton, then either M is positive Sasakian, or null- Sasakian. In the first case, the soliton vector field is Killing and in the second case, the soliton vector field leaves φ_* invariant.*

Proof. Using (3.7) in the soliton equation (3.8) takes the form

$$(\pounds_V g)(X,Y) = -(\lambda_* + \mu_*)[g(X,Y) + \eta_*(X)\eta_*(Y)]. \tag{3.17}$$

for all vector fields X, Y on M. Covariant differentiation of (3.7) along an arbitrary vector field Z on M and then using (2.3) we can find

$$(\nabla^*_Z S)(X,Y) = (\frac{\lambda_*}{2} - \frac{\mu_*}{2} + 1)[g(Z,\varphi_* X)\eta_*(Y) + g(Z,\varphi_* Y)\eta_*(X)]. \tag{3.18}$$

In view of this, equation (3.11) transforms into

$$g((\pounds_V \nabla^*)(X,Y), Z) = (\lambda_* + \mu_*)[\eta_*(Y)\varphi_* X + \eta_*(X)\varphi_* Y]. \tag{3.19}$$

Taking covariant derivative of (3.17) along an arbitrary vector field Z on M and making use of (2.3) implies

$$\begin{aligned}(\nabla^*_Z \pounds_V \nabla^*)(X,Y) &= (\lambda_* + \mu_*)[g(Z,\varphi_* X)\varphi_* Y + g(Z,\varphi_* Y)\varphi_* X \\ &\quad +\eta_*(Y)(\nabla^*_Z \varphi_*)X + \eta_*(X)(\nabla^*_Z \varphi_*)Y].\end{aligned} \tag{3.20}$$

Making use of the last equation in (3.13) and using (2.3), (2.5) we deduce

$$\begin{aligned}(\pounds_V R)(Z,X)Y &= (\lambda_*+\mu_*)[2g(Z,\varphi_* X)\varphi_* Y + g(Z,\varphi_* Y)\varphi_* X \\ &\quad -g(X,\varphi_* Y)\varphi_* Z + 2\eta_*(Z)\eta_*(Y)X \\ &\quad -2\eta_*(X)\eta_*(Y)Z + \eta_*(X)g(Z,Y)\xi_* \\ &\quad -\eta_*(Z)g(X,Y)\xi_*]. \end{aligned} \tag{3.21}$$

Contracting the foregoing equation over Z, we have

$$(\pounds_V S)(X,Y) = 2(\lambda_*+\mu_*)\{g(X,Y)-(2n+1)\eta_*(X)\eta_*(Y)\}. \tag{3.22}$$

Next, Lie-differentiation of (3.7) along the vector field V and using (3.16) yields

$$\begin{aligned}(\pounds_V S)(X,Y) &= (2n+\frac{\mu_*}{2}-\frac{\lambda_*}{2}-1)(\pounds_V g)(X,Y) \\ &\quad +(\frac{\lambda_*}{2}-\frac{\mu_*}{2}+1)[(\pounds_V\eta_*)(X)\eta_*(Y)+\eta_*(X)(\pounds_V\eta_*)(Y)]. \end{aligned} \tag{3.23}$$

Comparing (3.18) with (3.19), and using (3.16) provides

$$\begin{aligned} & 2(\lambda_*+\mu_*)\{g(X,Y)-(2n+1)\eta_*(X)\eta_*(Y)\} \\ =\; & -(\lambda_*+\mu_*)(2n+\frac{\mu_*}{2}-\frac{\lambda_*}{2}-1)\{g(X,Y)+\eta_*(X)\eta_*(Y)\} \\ & +(\frac{\lambda_*}{2}-\frac{\mu_*}{2}+1)[(\pounds_V\eta_*)(X)\eta_*(Y)+\eta_*(X)(\pounds_V\eta_*)(Y)]. \end{aligned} \tag{3.24}$$

At this point, we replace X by $\varphi_*^2 X$ and Y by $\varphi_*^2 Y$ in (3.20) to deduce

$$(\lambda_*+\mu_*)(2n+1+\frac{\mu_*}{2}-\frac{\lambda_*}{2})d\eta_*(X,Y)=0.$$

Since $d\eta_*$ is non-vanishing everywhere on M, we can conclude $(\lambda_*+\mu_*)(2n+1+\frac{\mu_*}{2}-\frac{\lambda_*}{2})=0$. Thus, we have either $\lambda_*+\mu_*=0$ or $2n+1+\frac{\mu_*}{2}-\frac{\lambda_*}{2}=0$.

Case I: In this case, we see that V is Killing (follows from (3.16)) and M is η_*-Einstein for $\lambda_*\neq -1$ (follows from (3.7)), i.e., the Ricci tensor S is of the form

$$S(X,Y)=(2n-1-\lambda_*)g(X,Y)+(\lambda_*+1)\eta_*(X)\eta_*(Y),$$

for all vector fields X,Y on M.

Case II: Using $\lambda_*-\mu_*=2(2n+1)$

In (42), setting $Y=\xi_*$ and then replacing $X=\varphi_* X$ it follows that

$$(\lambda_*-\mu_*+2)(\pounds_V\eta_*)(\varphi_* X)=0,$$

for all vector field X on M. If $\lambda_*-\mu_*=-2$, then as $\lambda_*-\mu_*=2(2n+1)$, we arrive at a contradiction.

Thus, we have $(\pounds_V\eta_*)(\varphi_* X)=0$ for all vector field X on M. Further, using $\lambda_*-\mu_*=2(2n+1)$ in (3.7) it follows that

$$S(X,Y)=-2g(X,Y)+2(n+1)\eta_*(X)\eta_*(Y), \tag{3.25}$$

for all vector fields X, Y on M. This shows that M is null-Sasakian (i.e., D-homothetically fixed transverse Calabi-Yau).

Next, replacing X by $\varphi_* X$ in $(\pounds_V \eta_*)(\varphi_* X) = 0$ gives

$$(\pounds_V \eta_*)X = \eta_*(X)(\pounds_V \eta_*)\xi_* = -(\lambda_* + \mu_*)\eta_*(X).$$

Taking exterior differentiation d on above equation

$$(\pounds_V d\eta_*)(X, Y) = -(\lambda_* + \mu_*)g(X, \varphi_* Y). \tag{3.26}$$

As d commutes with $\pounds_V$. Taking the Lie-derivative of $d\eta_*(X, Y) = g(X, \varphi_* Y)$ along the soliton vector field V provides

$$(\pounds_V d\eta_*)(X, Y) = (\pounds_V g)(X, \varphi_* Y) + g(X, (\pounds_V \varphi_*)Y). \tag{3.27}$$

Using (3.18), we obtain

$$(\pounds_V g)(X, \varphi_* Y) = -(\lambda_* + \mu_*)g(X, \varphi_* Y). \tag{3.28}$$

Using above with (3.22) in (3.23), we find

$$(\pounds_V \varphi_*) = 0. \tag{3.29}$$

Therefore, soliton vector field V leaves φ_* invariant. □

4 Example

We consider 5-dimensional manifold M, where $M = \{(x_1, x_2, y_1, y_2, z) \in R^5\}$, where (x_1, x_2, y_1, y_2, z) are standard coordinates in R^5. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ be linearly independent frame fields on M given by

$$\sigma_1 = z\left(y_1 \frac{\partial}{\partial z} - \frac{\partial}{\partial x_1}\right), \sigma_2 = z\frac{\partial}{\partial y_1},$$

$$\sigma_3 = z\left(y_2 \frac{\partial}{\partial z} - \frac{\partial}{\partial x_2}\right), \sigma_4 = z\frac{\partial}{\partial y_2}, \sigma_5 = -z\frac{\partial}{\partial z}.$$

$$[\sigma_1, \sigma_2] = 2\sigma_5, [\sigma_1, \sigma_3] = 0, [\sigma_1, \sigma_4] = 0, [\sigma_1, \sigma_5] = 0,$$

$$[\sigma_2, \sigma_3] = 0, [\sigma_3, \sigma_4] = 2\sigma_5, [\sigma_2, \sigma_4] = 0, [\sigma_2, \sigma_5] = 0,$$

$$[\sigma_3, \sigma_5] = 0, [\sigma_4, \sigma_5] = 0.$$

Let g be a Riemannian metric defined by

$$\begin{aligned} g(\sigma_i, \sigma_j) &= 1 \text{ if } i - j \\ &= 0 \text{ if } i \neq j,\ i, j = 1, 2, 3, 4, 5 \end{aligned}$$

Let η_* be the 1-form defined by $\eta_*(X) = g(X, \sigma_5)\ \forall\ X \in \chi(M)$, where $\chi(M)$ be the set of all $C^\infty-$ vector fields defined on M. Let φ_* be $(1, 1)$ tensor field defined by

$$\varphi_* \sigma_1 = \sigma_2, \varphi_* \sigma_2 = -\sigma_1, \varphi_* \sigma_3 = \sigma_4, \varphi_* \sigma_4 = -\sigma_3, \varphi_* \sigma_5 = 0.$$

Let ∇^* be a Levi-Civita connection with respect to the Riemannian metric g, then the Koszul formula

$$\begin{aligned} 2g(\nabla^*_X Y, Z) &= X\{g(Y,Z)\} + Y\{g(Z,X)\} - Z\{g(X,Y)\} \\ &\quad -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]). \end{aligned}$$

We have

$$\begin{aligned} \nabla^*_{\sigma_1}\sigma_1 &= 0, \nabla^*_{\sigma_1}\sigma_2 = \sigma_5, \nabla^*_{\sigma_1}\sigma_3 = 0, \nabla^*_{\sigma_1}\sigma_4 = 0, \nabla^*_{\sigma_1}\sigma_5 = -\sigma_2, \\ \nabla^*_{\sigma_2}\sigma_1 &= -\sigma_5, \nabla^*_{\sigma_2}\sigma_2 = 0, \nabla^*_{\sigma_2}\sigma_3 = 0, \nabla^*_{\sigma_2}\sigma_4 = 0, \nabla^*_{\sigma_2}\sigma_5 = \sigma_1, \\ \nabla^*_{\sigma_3}\sigma_1 &= 0, \nabla^*_{\sigma_3}\sigma_2 = 0, \nabla^*_{\sigma_3}\sigma_3 = 0, \nabla^*_{\sigma_3}\sigma_4 = \sigma_5, \nabla^*_{\sigma_3}\sigma_5 = -\sigma_4, \\ \nabla^*_{\sigma_4}\sigma_1 &= 0, \nabla^*_{\sigma_4}\sigma_2 = 0, \nabla^*_{\sigma_4}\sigma_3 = -\sigma_5, \nabla^*_{\sigma_4}\sigma_4 = 0, \nabla^*_{\sigma_4}\sigma_5 = \sigma_3, \\ \nabla^*_{\sigma_5}\sigma_1 &= -\sigma_2, \nabla^*_{\sigma_5}\sigma_2 = \sigma_1, \nabla^*_{\sigma_5}\sigma_3 = -\sigma_4, \nabla^*_{\sigma_5}\sigma_4 = \sigma_3, \nabla^*_{\sigma_5}\sigma_5 = 0. \end{aligned}$$

It can be easily obtain that for $\sigma_5 = \xi_*, (\varphi_*, \xi_*, \eta_*, g)$ is a Sasakian structure.
Also, the Riemannian curvature tensor R is given by,

$$R(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z.$$

Now, we can easily obtain the components of the curvature tensors R as follows:

$$\begin{gathered} R(\sigma_1,\sigma_2)\sigma_1 = 3\sigma_2,\ R(\sigma_1,\sigma_3)\sigma_1 = 0,\ R(\sigma_2,\sigma_4)\sigma_1 = \sigma_3,\ R(\sigma_2,\sigma_5)\sigma_1 = 0, \\ R(\sigma_4,\sigma_5)\sigma_1 = 0,\ R(\sigma_1,\sigma_2)\sigma_2 = 3\sigma_1,\ R(\sigma_1,\sigma_4)\sigma_2 = -\sigma_3,\ R(\sigma_2,\sigma_3)\sigma_2 = 0, \\ R(\sigma_4,\sigma_5)\sigma_2 = 0,\ R(\sigma_1,\sigma_3)\sigma_3 = 0,\ R(\sigma_2,\sigma_3)\sigma_1 = -\sigma_4,\ R(\sigma_3,\sigma_4)\sigma_3 = 3\sigma_4, \\ R(\sigma_4,\sigma_5)\sigma_4 = -\sigma_5,\ R(\sigma_1,\sigma_2)\sigma_5 = 0,\ R(\sigma_1,\sigma_4)\sigma_5 = 0,\ R(\sigma_2,\sigma_4)\sigma_5 = 0, \\ R(\sigma_4,\sigma_5)\sigma_5 = \sigma_4,\ R(\sigma_1,\sigma_4)\sigma_5 = 0,\ R(\sigma_3,\sigma_5)\sigma_5 = \sigma_3,\ R(\sigma_2,\sigma_5)\sigma_5 = \sigma_2, \end{gathered}$$

Then, the Ricci tensor S is given by,

$$S(X,Y) = \sum_{i=1}^{n} (gR(\sigma_i, X)Y, \sigma_i), \tag{4.1}$$

Using the values of R, we obtain

$$\begin{aligned} r = \sum_{i=1}^{5} S(\sigma_i,\sigma_i) &= S(\sigma_1,\sigma_1) + S(\sigma_2,\sigma_2) + S(\sigma_3,\sigma_3) + S(\sigma_4,\sigma_4) + S(\sigma_5,\sigma_5), \\ &= -2 - 2 + 2 + 2 + 4 = 4. \end{aligned}$$

Which satisfies equation (1.5) and also verified Theorem 1. So, g defines an eta star Ricci soliton on a 5 dimensional Sasakian manifold and also it is η_*-Einstein manifold.

References

[1] A. M. Blaga, *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat., 30(2016), no. 2, 489-496.

[2] D.E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhauser, Boston, 2010.

[3] C.P. Boyer and K. Galicki, *Einstein manifolds and contact geometry*, Proc. Amer. Math. Soc., 129, (2001), 2419-2430.

[4] C.P. Boyer, K. Galicki and P. Matzeu, *On Einstein Sasakian geometry*, Commun. Math. Phys., 262, (2006), 177-208.

[5] C. Cálin and M. Crasmareanu, *η-Ricci solitons on Hopf hypersurfaces in complex space forms*, Rev. Roumaine Math. Pures Appl., 57, (2012), no. 1, 55-63.

[6] C. Cálin and M. Crasmareanu, *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds*, Bull. Malaysian Math. Sci. Soc., 33, (2010), no. 3, 361-368.

[7] T. Chave and G. Valent, *Quasi-Einstein metrics and their renormalizability properties*, Helv. Phys. Acta. 69, (1996), 344-347.

[8] T. Chave and G. Valent, *On a class of compact and non-compact quasi-Einstein metrics and their renormalizability properties*, Nuclear Phys. B., 478, (1996), 758-778.

[9] J.T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J., 61, (2009), no.2, 205-212.

[10] B. Chow and D. Knopf, *The Ricci flow: an introduction, Mathematical Surveys and Monographs*, 110, American Mathematical Society, 2004.

[11] D. Friedan, *Non linear models in $2+\epsilon$ dimensions*, Ann. Phys., 163, (1985), 318-419.

[12] A. Ghosh and D.S. Patra, *Ricci Soliton within the frame-work of Sasakian and (k,μ)-contact manifold*. Int. J. Geom. Methods Mod. Phys., 15 (7), (2018).

[13] T. Hamada, "*Real hypersurfaces of complex space forms in terms of Ricci $*$ tensor*," Tokyo J. Math., 25, (2002), 473-483 .

[14] R.S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff . Geom., 17, (1982), no. 2, 255-306.

[15] R.S. Hamilton, *The Ricci flow on surfaces, Mathematics and general relativity*, (Contemp. Math. Santa Cruz, CA, 1986), American Math. Soc., (1988), 237-263.

[16] A. Haseeb, S. Pandey and R. Prasad, *Some results on η-Ricci solitons in quasi-Sasakian 3-manifolds*, Commun. Korean Math. Soc., 36, (2021), No.2, 377-387.

[17] T. Ivey and P.J. Ryan, *The CH_n $*$-Ricci tensor for hypersurfaces in CP_n and CH_n*, Tokyo J. Math., 34, (2011), 445-47.

[18] G. Kaimakamis, and K. Panagiotidou, *$*$-Ricci solitons of real hypersurface in non-flat complex space forms*. J. Geom. Phy. 76, (2014), 408-413 .

[19] P. Majhi, U.C. De, and Y.J. Shu, "*$*-$Ricci soliton Sasakian 3 manifolds* ," Publ. Math. Debrecen, 93, (2018).

[20] M. Okumura, *Some remarks on space with a certain contact structure*, Tohoku Math. J., 14,(1962), 135-145.

[21] D.G. Prakasha and P. Veeresha, *Para-Sasakian manifolds and $*$-Ricci solitons*. arXiv:1801.01727v1.

[22] R. Sharma, *Certain results on K -contact and (k,μ)-contact manifolds*. J. Geom., 89, (2008) 138-147 .

[23] S. Tanno, *Some transformations on manifolds with almost contact and contact metric structures*, II, Tohoku Math. J.(2), 15 (1963), 322-331.

[24] K. Yano, *Integral formulas in Riemannian geometry*. New York, Marcel Dekker, 1970.

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