Study of Kähler structure on 4-dimensional space-time

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Abstract. In this paper, the Einstein equation with cosmological constant has been studied in a perfect fluid space-time space under the consideration of Kähler structure and it is proved that the space is an Einstein space. In such spaces the perfect fluid reduced to dust and the space reduced to the vacuum. A relation between associated vector fields has been obtained. Finally, it is proved that the scalar curvature vanishes in a weakly Ricci symmetric Kähler space-time.

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1 Introduction

The study of space-time is associated with 4-dimensional pseudo-Riemannian manifolds equipped with Lorentzian metric g having signature (-,+,+,+). In last few decades, the space-time is extended by U. C. De and G. C. Ghosh [6], A. A. Shaikh, D. W. Yoon and S. K. Hui [14], A. De, C. Özgür and U. C. De [5], U. C. De and S. Mallick [7, 10], U. C. De and L. Velimirovic [8] and by the several authors.

In 2004 U. C. De and G. C. Ghosh [6] studied the weakly Ricci symmetric space-time and obtained a relation between cosmological constant and scalar curvature. In the same paper, they [6] have also proved that weakly Ricci-symmetric space-time can not admit heat flux and the integral curves of the velocity vector are geodesics.

An n-dimensional (pseudo) Riemannian manifold is said to be weakly symmetric [15] if the curvature tensor R of type (0,4) of the manifold satisfies

$$
\begin{aligned}
(\nabla_X R)(Y,Z,U,V) = & A(X)R(Y,Z,U,V) \\
& + B(Y)R(X,Z,U,V) + C(Z)R(Y,X,U,V) \\
& + D(U)R(Y,Z,X,V) + E(V)R(Y,Z,U,X),
\end{aligned} \tag{1.1}
$$

and if the Ricci tensor S of the manifold satisfies

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + C(Z)S(Y,X), \tag{1.2}$$

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then manifold is called weakly Ricci symmetric manifold, where A, B, C, D, E are simultaneously non-vanishing 1-forms and X, Y, Z, U, V are vector fields.

In 1995, Prvanovic [13] proved that if the manifold is weakly symmetric manifold satisfying (1.1) then $B = C = D = E$. In this paper we have taken $B = C = D = E = \omega$ (say), therefore the equations (1.1) and (1.2) can be written as

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) =& A(X)R(Y,Z,U,V)\\ &+ \omega(Y)R(X,Z,U,V) + \omega(Z)R(Y,X,U,V)\\ &+ \omega(U)R(Y,Z,X,V) + \omega(V)R(Y,Z,U,X),\end{aligned} \tag{1.3}$$

and

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + \omega(Y)S(X,Z) + \omega(Z)S(Y,X), \tag{1.4}$$

where, A and ω are arbitrary 1-forms with respect to associated vector fields α and ρ defined by $g(X,\alpha) = A(X)$ and $g(X,\rho) = \omega(X)$.

Recently, A. De, C. Özgür and U. C. De [5] (2012) extended the theory of space-time by studying conformally flat almost pseudo Ricci-symmetric manifolds. In 2012 and 2014, Mallick and De [7, 10] gave examples of weakly symmetric and conformally flat weakly Ricci symmetric space-time and proved that a conformally flat weakly Ricci symmetric space-time with non-zero scalar curvature is the Robertson-Walker space-time and the vorticity and the shear vanish. Also, they have shown that a weakly symmetric perfect fluid space-time with cyclic parallel Ricci tensor can not admit heat flux. In 2014, De and Velimirovic [8] explained the space-time with semi-symmetric energy-momentum tensor and proved that a space-time with semi-symmetric energy momentum tensor is Ricci semi-symmetric. Chaturvedi and Pandey [1, 2] also studied weakly symmetric and weakly Ricci symmetric manifolds in 2015. In [1], they have proved that in a weakly symmetric kähler manifold equipped with special type of semi-symmetric metric connection scalar curvature vanishes. Further, it is proved that if the manifold is conformally flat Riemannian manifold equipped with special type of semi-symmetric metric connection then manifold is of quasi-constant curvature. In [2], they have constructed some examples on weakly symmetric and weakly Ricci symmetric manifolds.

In 2000, L. Tamassy, U. C. De and T. Q. Binh [16] studied weakly symmetric and weakly Ricci symmetric Kähler manifolds and proved that if the scalar curvature is non-zero constant then sum of associated vector fields is zero. Also, the Einstein field's equation on Kähler manifolds has been discussed by I. Hinterleitner and V. Kiosak [9] (2010). They [9] have classified the Kähler manifold on the basis of energy momentum tensor. In last two decades, weakly symmetric and weakly Ricci-symmetric manifolds are studied by several authors with different point of aspect [3, 4, 12]. Some of them also considered these spaces with Kähler manifolds [16] but not in space-time. Therefore, we are interested in Kähler space-time. In this paper, we have considered 4-dimensional Kähler manifold with Lorentzian metric and called Kähler space-time.

A 4-dimensional space-time is said to be Kähler space-time if the following conditions hold:

$$F^2(X) = -X, \quad g(\overline{X},\overline{Y}) = g(X,Y), \quad (\nabla_X F)(Y) = 0, \tag{1.5}$$

where, F is a tensor field of type (1,1) such that $F(X) = \overline{X}$, g is a pseudo-Riemannian metric and ∇ is a Levi-Civita connection.

We know that in a Kähler manifold the Ricci tensor S satisfies

$$S(\overline{X}, \overline{Y}) = S(X, Y). \tag{1.6}$$

2 Perfect fluid Kähler space-time

We know that the Einstein equation with cosmological constant for perfect fluid space-time is given by

$$S(X, Y) - \frac{r}{2}g(X, Y) + \lambda g(X, Y) = k[(\sigma + p)\omega(X)\omega(Y) + pg(X, Y)], \tag{2.1}$$

where k is gravitational constant, σ is energy density, p is isotropic pressure of the fluid, λ is cosmological constant, r is scalar curvature of the manifold and ω is 1-form defined by $\omega(X) = g(X, \rho)$ for time-like vector field ρ. The time-like vector field ρ is called velocity of the fluid and satisfies $g(\rho, \rho) = -1$.
Now replacing X and Y by $\overline{X}$ and $\overline{Y}$ respectively in (2.1) and using (1.5) and (1.6), we get

$$S(X, Y) - \frac{r}{2}g(X, Y) + \lambda g(X, Y) = k[(\sigma + p)\omega(\overline{X})\omega(\overline{Y}) + pg(X, Y)]. \tag{2.2}$$

Subtracting (2.1) from (2.2), we have

$$k(\sigma + p)[\omega(\overline{X})\omega(\overline{Y}) - \omega(X)\omega(Y)] = 0. \tag{2.3}$$

Putting $Y = \rho$ in (2.3), we obtained

$$k(\sigma + p)\omega(X) = 0, \tag{2.4}$$

since $k \neq 0$ and $\omega(X) \neq 0$, we have

$$\sigma + p = 0, \tag{2.5}$$

Now using (2.5), equation (2.1) gives

$$S(X, Y) = (\frac{r}{2} - \lambda + k.p)g(X, Y). \tag{2.6}$$

Contracting X and Y in (2.6), we get

$$\lambda - k.p = \frac{r}{4}. \tag{2.7}$$

Equations (2.6) and (2.7) together yields

$$S(X, Y) = \frac{r}{4}g(X, Y). \tag{2.8}$$

Hence, we can state the following theorem:

Theorem 2.1. *Let M be a perfect fluid Kähler space-time satisfying Einstein equation with cosmological term then space-time is an Einstein space.*

Since an Einstein manifold is the manifold of constant scalar curvature r, therefore, equation (2.7) implies the pressure p is constant and hence from (2.5) we have the energy density σ is constant.

It is well known [11] that the Energy equation for perfect fluid is given by

$$\rho.\sigma = -(\sigma + p)div\rho. \tag{2.9}$$

Using (2.5) in (2.9), we get

$$\rho.\sigma = 0. \tag{2.10}$$

Above equation implies $\sigma = 0$ as $\rho \neq 0$. Because if $\rho = 0$ then we have contradiction that $g(\rho, \rho) = -1$. But then the equation (2.5) gives $p = 0$ and hence the energy momentum tensor $T(X, Y) = (\sigma + p)\omega(X)\omega(Y) + pg(X, Y)$ vanishes. Since, we know that if energy density σ vanishes then content matter of the fluid is not pure, if pressure p vanishes then the fluid becomes dust and if energy momentum tensor vanishes then space-time is vacuum.
Thus from above discussion, we conclude the following:

Theorem 2.2. *Let M be a perfect fluid Kähler space-time satisfying Einstein equation with cosmological constant then*
(i) content matter of the fluid is not pure.
(ii) perfect fluid is dust.
(iii) space-time is vacuum.

3 Weakly symmetric perfect fluid Kähler space-time

Let M is a weakly symmetric Kähler manifold then we have

$$R(\overline{Y}, \overline{Z}, U, V) = R(Y, Z, U, V). \tag{3.1}$$

Taking covariant derivative of (3.1), we get easily

$$(\nabla_X R)(\overline{Y}, \overline{Z}, U, V) = (\nabla_X R)(Y, Z, U, V). \tag{3.2}$$

Using (1.3) in (3.2), we have

$$\begin{aligned} \omega(Y)R(X, Z, U, V) + \omega(Z)R(Y, X, U, V) = \omega(\overline{Y})R(X, \overline{Z}, U, V) \\ + \omega(\overline{Z})R(\overline{Y}, X, U, V). \end{aligned} \tag{3.3}$$

Putting $Z = U = e_i, 1 \leq i \leq 4$ in (3.3) and taking summation over i, we obtained

$$\omega(Y)S(X, V) - R(Y, X, V, \rho) = \omega(\overline{Y})S(X, \overline{V}) + R(\overline{Y}, X, V, \overline{\rho}). \tag{3.4}$$

By using (2.8), equation (3.4) implies

$$\frac{r}{4}\omega(Y)g(X, V) - R(Y, X, V, \rho) = \frac{r}{4}\omega(\overline{Y})g(X, \overline{V}) + R(\overline{Y}, X, V, \overline{\rho}). \tag{3.5}$$

Putting $X = V = e_i, 1 \leq i \leq 4$ in (3.5) and taking summation over i, we get

$$S(Y, \rho) = \frac{r}{2}\omega(Y). \tag{3.6}$$

or

$$S(Y, \rho) = \frac{r}{2}g(Y, \rho). \tag{3.7}$$

Replacing ρ by $\overline{\rho}$ in (3.7), we can write

$$S(Y, \overline{\rho}) = \frac{r}{2}g(Y, \overline{\rho}). \tag{3.8}$$

Hence from (3.7) and (3.8), we can state:

Theorem 3.1. *Let M be a weakly symmetric perfect fluid Kähler space-time satisfying Einstein equation with cosmological constant then ρ and $\overline{\rho}$ are eigen vectors of the Ricci tensor S with respect to eigen value $\frac{r}{2}$.*

Now putting $X = V = e_i, 1 \leq i \leq 4$ in (1.3) and summing over i, we have

$$\begin{aligned}(divR)(Y, Z)U =& R(Y, Z, U, \alpha) + \omega(Y)S(Z, U) \\ &+ \omega(Z)S(Y, U) + R(Y, Z, U, \rho).\end{aligned} \tag{3.9}$$

From (2.8), it can be easily obtained

$$(\nabla_Y S)(Z, U) = \frac{r}{4}(\nabla_Y g)(Z, U) = 0. \tag{3.10}$$

Using (3.10) in Bianchi second identity, we can write

$$(divR)(Y, Z)U = (\nabla_Y S)(Z, U) - (\nabla_Z S)(Y, U) = 0. \tag{3.11}$$

By using (3.11), the equation (3.9) gives

$$R(Y, Z, U, \alpha) + \omega(Y)S(Z, U) - \omega(Z)S(Y, U) + R(Y, Z, U, \rho) = 0. \tag{3.12}$$

Putting $Z = U = e_i, 1 \leq i \leq 4$ in (3.12) and taking summation over i, we have

$$S(Y, \alpha) + r\omega(Y) = 0. \tag{3.13}$$

Using (2.8) in (3.13), we can write

$$\frac{r}{4}g(Y, \alpha) + rg(Y, \rho) = 0. \tag{3.14}$$

Replacing Y by ρ in (3.14), we get

$$r[g(\alpha, \rho) - 4] = 0, \tag{3.15}$$

which implies either $r = 0$ or $g(\alpha, \rho) = 4$.
Hence, we have

Theorem 3.2. *Let M be a weakly symmetric perfect fluid Kähler space-time satisfying Einstein equation with cosmological constant then for non-vanishing scalar curvature tensor the associated vector fields α and ρ are related by $g(\alpha, \rho) = 4$.*

4 Weakly Ricci symmetric perfect fluid Kähler space-time

From equation (2.8) it is clear that if the manifold is perfect fluid Kähler space-time then $(\nabla_X S)(Y,Z) = 0$, therefore from (1.4), we have

$$A(X)S(Y,Z) + \omega(Y)S(Z,X) + \omega(Z)S(X,Y) = 0, \tag{4.1}$$

for weakly Ricci symmetric perfect fluid Kähler space-time. Using (2.8) in (4.1), we can write

$$\frac{r}{4}[A(X)g(Y,Z) + \omega(Y)g(Z,X) + \omega(Z)g(X,Y)] = 0. \tag{4.2}$$

equation (4.2) implies either scalar curvature $r = 0$ or

$$A(X)g(Y,Z) + \omega(Y)g(Z,X) + \omega(Z)g(X,Y) = 0. \tag{4.3}$$

Now, if (4.3) holds then by replacing Y and Z by $\overline{Y}$ and $\overline{Z}$ in (4.3) and using (1.5), we get

$$A(X)g(Y,Z) + \omega(\overline{Y})g(\overline{Z},X) + \omega(\overline{Z})g(X,\overline{Y}) = 0. \tag{4.4}$$

Subtracting (4.3) from (4.4), we have

$$\omega(\overline{Y})g(X,\overline{Z}) - \omega(Y)g(X,Z) + \omega(\overline{Z})g(X,\overline{Y}) - \omega(Z)g(X,Y) = 0. \tag{4.5}$$

Putting $X = Z = e_i, 1 \leq i \leq 4$ in (4.5) and taking summation over i, we get

$$\omega(Y) = 0, \tag{4.6}$$

which is not possible because $g(\rho,\rho) = -1$, therefore $r = 0$.
Hence, we can state that

Theorem 4.1. *Let M be a weakly Ricci symmetric perfect fluid Kähler space-time satisfying Einstein equation with cosmological constant then its scalar curvature tensor vanishes.*

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