Abstract. This work describes the Minkowski sum and difference of sets and some of their important geometric properties. As a basic result, necessary and sufficient condition have created for the existence of the Minkowski difference of the squares given on the plane \( \mathbb{R}^2 \). Also, the calculation formula and the exact method of finding the Minkowski difference of the squares given by the vectors corresponding to the side on the plane \( \mathbb{R}^2 \) are introduced.

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1 Introduction

The Minkowski difference and the Minkowski sum are more complex and unique than other operations on sets, and these operations depend on the nature of the elements that make up the sets. The first information about these concepts can be found in the works of the famous German mathematician Hermann Minkowski[5]. The computation of Minkowski sum and Minkowski difference is crucial for many applications, such as robot motion planning, morphological image analysis, computer-aided design and manufacturing, etc. Minkowski sum and geometric difference were also used in differential games to obtain sufficient conditions to complete the game [1],[4],[7].

Finding the Minkowski difference of convex polygons is much more complicated than finding the Minkowski sum. Z.R.Gabidullina, D.Velichova, L.Montejano dealt with this problem. However, so far there are no necessary and sufficient conditions for the existence or non-existence of the Minkowski difference of arbitrary convex polygons. The following is a summary of the problem and the results obtained for the Minkowski difference of squares on the plane.

2 Definition and properties of the Minkowski operations

Definition 2.1. The Minkowski sum of the two sets \( S_1 \) and \( S_2 \) given in the \( n \)-dimensional \( \mathbb{R}^n \) Euclidean space is said to be the set \( S \subset \mathbb{R}^n \) satisfying the following

\[
S = \{ x + y : x \in S_1, y \in S_2 \}
\]
equation:

\[ S = S_1 + S_2 = \{ z \in \mathbb{R}^n | z = x + y, x \in S_1, y \in S_2 \}. \]

In particular, if each of sets \( S_1 \) and \( S_2 \) consists of a single element, then this operation corresponds to the operation of the usual addition of vectors.

**Definition 2.2.** The Minkowski sum of any vector \( x \in \mathbb{R}^n \) and nonempty set \( S \subset \mathbb{R}^n \) in the \( n \)-dimensional \( \mathbb{R}^n \) Euclidean space is defined to be the set

\[ x + S = \{ x + z : z \in S \}. \]

From Definition 2.2, we can see that the set \( x + S \) is formed by moving the set \( S \) in parallel along the vector \( x \). The following operation, called the Minkowski difference or geometric difference of sets, is determined by the Minkowski sum.

**Definition 2.3.** The Minkowski difference of the two sets \( S_1 \) and \( S_2 \) given in the \( n \)-dimensional \( \mathbb{R}^n \) Euclidean space is said to be the set \( D \subset \mathbb{R}^n \) satisfying the following equation:

\[ D = S_1 - S_2 = \{ d \in \mathbb{R}^n | d + S_2 \subset S_1 \}. \]

As you can see from the definition of these operations, they are different and more complex than the other operations on the sets which we know. To do these operations, the elements of the two sets must be of the same nature. For example, if set \( S_1 \) is a set of polynomials whose level does not exceed \( n - 1 \), and \( S_2 \) is a set of square matrices whose number of rows and columns is \( n - 1 \), then the Minkowski difference and sum operations described above cannot performed on these sets. Therefore, if set \( S_1 \) belongs to a vector space, then set \( S_2 \) must also belong to that vector space.

Nevertheless, these operations have the following properties associated with the union and intersection operations, which are well known to us on sets.

1) \( S_1 - (S_2 + S_3) = S_1 - S_2 - S_3; \)
2) \( S_1 - S_2 + S_2 \subset S_1; \)
3) \( S_1 \subset S_1 + S_2 - S_2; \)
4) \( S_1 \subset S_2 - (S_2 \cdot S_1); \)
5) If \( S_1 \subset S_2, P_1 \subset P_2 \), then \( S_1 + P_1 \subset S_2 + P_2; \)
6) If \( S_1 \subset S_2, P_1 \supset P_2 \), then \( S_1 - P_1 \subset S_2 - P_2; \)
7) \( (S_1 \cdot S_3) \cap (S_2 \cdot S_3) = (S_1 \cap S_2) \cdot S_3; \)
8) \( (z + S_1) \cup (z + S_2) = z + (S_1 \cup S_2); \)
9) \( (S_1 \cdot S_3) \cap (S_1 \cdot S_3) = S_1 \cdot (S_2 \cap S_3); \)
10) \( (S_1 \cap S_2) + (S_1 \cap S_3) \subset (S_1 + S_4) \cap (S_2 + S_4). \)

Here \( S_1, S_2, S_3, S_4, P_1, P_2 \) are sets taken from the space \( \mathbb{R}^n \) and \( z \) is the point (vector) taken from the space \( \mathbb{R}^n \). Proofs of these properties are described in detail in the work [3].
3 Theorem on the Minkowski difference of squares

If sets $S_1$ and $S_2$ are convex sets on a plane $\mathbb{R}^2$, from definition 3 above, set $D$ determines how much it is possible to move set $S_2$ without going beyond set $S_1$. [2] presents the conditions for the existence of the Minkowski difference of segments and circles in a straight line and a plane, and the rules of calculation. For Minkowski difference of intervals $X = (a, b)$ and $Y = (a_1, b_1)$ on the straight line $\mathbb{R}$ following relation is true [2, lemma 2]:

$$X \ast Y = \begin{cases} [a - a_1, b - b_1] & \text{if } a - a_1 < b - b_1, \\ \{a - a_1\} & \text{if } a - a_1 = b - b_1, \\ \emptyset & \text{if } a - a_1 > b - b_1. \end{cases}$$

So, for the Minkowski difference of the $X = (a, b)$ and $Y = (a_1, b_1)$ intervals to exist, the length of the $X$ interval must not be less than the length of the $Y$ interval.

[2] shows a way to calculate the Minkowski difference of any circle in a plane. According to it, to subtract circle $C_2$ from circle $C_1$, we move circle $C_2$ by touching its boundary without leaving circle $C_1$. Then the circle (set) bounded by the line drawn by the center of circle $C_2$ is equal to the difference we are looking for. In work [6] considered the problem of calculating the Minkowski sum of polygons using vectors parallel to its sides and equal in length to its sides. The directions of these vectors are chosen such that, the vectors along this direction rotate clockwise around the given polygon. For example, the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5$ are placed on the side of the polygon $P$ as shown in Fig. 1. We also use vectors corresponding to its sides to find the Minkowski difference of squares, that is, we consider the problem of finding the difference of these squares when given by the coordinates of these vectors.

Any square on a plane $\mathbb{R}^2$ can be defined by a vector corresponding to one side of it. In this case, the vectors corresponding to the remaining sides are found by rotating the given vector to the angles $-90^\circ, 180^\circ, 90^\circ$. Let $S_1$ and $S_2$ be the squares in the plane $\mathbb{R}^2$. According to the rule of rotation, the vectors $\vec{a}_2(a^2,-a^1), \vec{a}_3(-a^1,-a^2), \vec{a}_4(-a^2,a^1)$ corresponding to the remaining sides of the square $S_1$, determined by the vector $\vec{a}_1(a^1,a^2)$. Similarly, the vectors $\vec{b}_2(b^2,-b^1), \vec{b}_3(-b^1,-b^2), \vec{b}_4(-b^2,b^1)$ corresponding to the other sides of the square $S_2$, determined by the vector $\vec{b}_1(b^1,b^2)$. Then, the vectors corresponding to the diagonals of the square are in the form $\vec{d}_1(b^1 + b^2, -b^1 + b^2)$ and $\vec{d}_2(-b^1 + b^2, -b^1 - b^2)$. 

![Fig. 1.](image)
Theorem 3.1. It is necessary and sufficient that the length of the orthogonal projection of the vectors \( \mathbf{d}_1 \) and \( \mathbf{d}_2 \) on the vector \( \mathbf{a}_1 \) is not greater than the length of the vector \( \mathbf{a}_1 \) so that the difference \( S_1 \triangle S_2 \) is not empty.

Proof. Suppose that the ends of squares \( S_1 \) and \( S_2 \) are \( A_1B_1C_1D_1 \) and \( A_2B_2C_2D_2 \), respectively. There can be two cases when finding the Minkowski difference of two squares. In the first case, the vectors \( \mathbf{a}_1 \) and \( \mathbf{b}_1 \) are parallel to each other, then the lengths of the orthogonal projections of the vectors \( \mathbf{d}_1 \) and \( \mathbf{d}_2 \) on the vector \( \mathbf{a}_1 \) are equal to \( |\mathbf{b}_1| \), that is \( \text{proj}_{\mathbf{a}_1} \mathbf{d}_1 = \text{proj}_{\mathbf{a}_1} \mathbf{d}_2 = |\mathbf{b}_1| \) (Fig. 2). According to the condition of the theorem \( |\mathbf{a}_1| \geq |\mathbf{b}_1| \). It follows that square \( S_2 \) can be placed inside square \( S_1 \). Therefore, \( S_1 \triangle S_2 \neq \emptyset \). In the second case, the vectors \( \mathbf{a}_1 \) and \( \mathbf{b}_1 \) are not parallel to each other (Fig. 3). Then, the lengths of the orthogonal projections of the vectors \( \mathbf{d}_1(b_1 + b_2, -b_1 + b_2) \) and \( \mathbf{d}_2(-b_1 + b_2, -b_1 - b_2) \) on the vector \( \mathbf{a}_1(a_1, a_2) \) can be calculated as follows:

\[
\begin{align*}
|\text{proj}_{\mathbf{a}_1} \mathbf{d}_1| &= \frac{|(\mathbf{a}_1, \mathbf{d}_1)|}{|\mathbf{a}_1|} = \frac{|a_1(b_1 + b_2) + a_2(-b_1 + b_2)|}{\sqrt{(a_1)^2 + (a_2)^2}}; \\
|\text{proj}_{\mathbf{a}_1} \mathbf{d}_2| &= \frac{|(\mathbf{a}_1, \mathbf{d}_2)|}{|\mathbf{a}_1|} = \frac{|a_1(-b_1 + b_2) + a_2(-b_1 - b_2)|}{\sqrt{(a_1)^2 + (a_2)^2}}.
\end{align*}
\]

These positive numbers are the distances calculated in the direction of the vector \( \mathbf{a}_1 \) between the outermost points of the square \( S_2 \). Then we can draw a square \( S'_2 \), the sides of which are parallel to the sides of the square \( S_1 \) and the ends of the square \( S_2 \) lie on these sides. It is known from the construction of the square that \( S_2 \) contains the square, and the length of the side is equal to whichever of the projections \( |\text{proj}_{\mathbf{a}_1} \mathbf{d}_1|, |\text{proj}_{\mathbf{a}_1} \mathbf{d}_2| \) is longer. Thus

\[
S_2 \subset S'_2
\]
According to the condition of the theorem, projections \( \text{proj}_{\vec{a}_1} \vec{d}_1 \), \( \text{proj}_{\vec{a}_1} \vec{d}_2 \) are not longer than \( |\vec{a}_1| \), similar that not longer than \( |\vec{a}_2| \) too (because \( |\vec{a}_1| = |\vec{a}_2| \)). It means that, square \( S'_2 \) can be placed inside square \( S_1 \). Therefore \( S_1 \ast S'_2 \neq \emptyset \). Since (3.3), it follows \( S_1 \ast S_2 \neq \emptyset \). The theorem has been proved.

During the proof of the above theorem, the formula for finding the Minkowski difference of squares on the plane \( \mathbb{R}^2 \) determined by the vectors \( \vec{a}_1(a^1, a^2) \) and \( \vec{b}_1(b^1, b^2) \) has been formed. Accordingly, this difference consists of either a single point or a square with a vector \( \vec{c}_1 \parallel \vec{a}_1 \) corresponding to its side. The length of the vector \( \vec{c}_1 \) is

\[
(3.4) \quad |\vec{a}_1| - \max_{n \in \vec{d}_1, \vec{d}_2} (|\text{proj}_{\vec{a}_1} n|).
\]

Here \( \vec{d}_1(b^1 + b^2, -b^1 + b^2), \vec{d}_1(b^1 + b^2, -b^1 + b^2) \).

4 Conclusion

The results lead to scientific research on the exact calculation of the geometric difference of arbitrary convex polygons in the plane. The existence of the Minkowski difference in cubes in three-dimensional space and its calculation can be thought of in the same way as above. But the number and direction of the vectors corresponding to the sides of the cube increase, and therefore the problem becomes more complicated. Therefore, it is more efficient to work with planes instead of vectors in three-dimensional space.
References


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