Space-time admitting generalized conharmonic curvature tensor

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Abstract. The object of the present paper is to study space-time admitting generalized conharmonic curvature tensor. In this paper, we have studied the basic algebraic properties of generalized conharmonic curvature tensor. Next, it is proved that a $4-$dimensional relativistic generalized conharmonic flat space-time is an Einstein space-time and it is of constant curvature. Moreover, it is of $O-$type. It is also observed that in a $4-$dimensional relativistic perfect fluid generalized conharmonically flat space-time following Einstein's field equation in the absence of cosmological constant, energy momentum tensor is covariant constant. Finally, it is proved that a $4-$dimensional relativistic conservative generalized conharmonic space-time M with constant scalar function ψ is a GRW space-time.

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1 Introduction

The aim of the present work is to study certain investigations in general theory of relativity and cosmology by the coordinate free method of differential geometry. The basic difference between Riemannian and semi-Riemannian geometry are (i) the existence of null vector (i.e. $g(v,v) = 0$, for $v \neq 0$, where g is the metric tensor) in semi-Riemannian manifold but not in Riemannian manifold, (ii) the signature of metric tensor g in semi-Riemannian manifold is $(-,-,...-,+,+,...,+)$ but in a Riemannian manifold the signature of g is $(+,+,...,+)$. Lorentzian manifold is a spacial case of semi-Riemannian manifold. The signature of metric tensor g in Lorentzian manifold is $(-,+,+,...,+)$. A Lorentzian manifold consists of three types of vectors such as timelike (i.e. $g(v,v) < 0$), spacelike (i.e. $g(v,v) > 0$) and null vector (i.e. $g(v,v) = 0$, for $v \neq 0$). In general, a Lorentzian manifold (M,g) may not have a globally timelike vector field. If (M,g) admits a globally timelike vector field, it is called time orientable Lorentzian manifold, physically known as space-time. The foundation of general relativity is based on a $4-$dimensional space-time which is the

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stage of present modeling of the physical world a torsionless, time-oriented Lorentzian manifold (M, g).

An $n-$dimensional generalized Robertson-Walker (GRW) space-time with $n \geq 3$ is a Lorentzian manifold which is a warped product of an open interval I of $\mathbb{R}$ and an $(n-1)-$dimensional Riemannian manifold ([10], [11], [12]). These Lorentzian manifold broadly extends the classical Robertson-Walker (RW) space-time. RW space-time is regarded as cosmological model since it is spatially homogeneous and spatially isotropic whereas GRW space-time serve as inhomogeneous extension of RW space-times that admit an isotropic radiation [14].

In the general theory of relativity, the matter content of the space-time is described by the energy momentum tensor. The matter content is assumed to be a fluid having density and pressure and possessing dynamical and kinematical quantities like velocity, acceleration, vorticity, shear and expansion. In a perfect fluid space-time, the energy momentum tensor T of type $(0, 2)$ is of the form [13]

$$T(X, Y) = (\sigma + \rho)A(X)A(Y) + \rho g(X, Y), \tag{1.1}$$

where ρ is the isotropic pressure, σ is the energy density and A is a non-zero one-form such that $g(X, \mu) = A(X)$ for all X, μ, where μ is the velocity vector field such that $g(\mu, \mu) = -1$ and A is metrically equivalent to a unit space-like vector field. The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity. Perfect-fluid space-times in a language of differential geometry are called quasi-Einstein spaces [3]. If the isotropic pressure (ρ) vanishes in perfect fluid then it is said to be a dust fluid. In a dust fluid space-time, the energy momentum tensor T of type $(0, 2)$ is of the form [13]

$$T(X, Y) = \sigma A(X)A(Y). \tag{1.2}$$

The Einstein's field equation with cosmological constant is given by [13]

$$S(X, Y) - \frac{r}{2}g(X, Y) + \lambda g(X, Y) = kT(X, Y), \tag{1.3}$$

where S and r denotes the Ricci tensor and scalar curvature respectively, λ is the cosmological constant, $T(X, Y)$ is the energy momentum tensor and $k \neq 0$. Einstein's field equation without cosmological constant is given by [13]

$$S(X, Y) - \frac{r}{2}g(X, Y) = kT(X, Y). \tag{1.4}$$

The Einstein's field equations (1.3) and (1.4) imply that the energy-momentum tensor is conservative. This requirement is satisfied if the energy-momentum tensor is covariant constant [1]. Chaki and Ray [1] showed that a general relativistic space-time with covariant constant energy-momentum tensor is Ricci symmetric, i.e., $\nabla S = 0$.

2 Preliminaries

As a special subgroup of the conformal transformation group, in [8], Ishii introduced the notion of conharmonic transformation under which a harmonic function transforms into a harmonic function. The conharmonic curvature tensor $\mathcal{H}$ of type $(0, 4)$

on a Riemannian manifold (M^n, g) of dimension $n \geq 4$ is defined as follows:

$$\begin{aligned}\mathcal{H}(U,V,X,Y) = \mathcal{R}(U,V,X,Y) - \frac{1}{n-2}[&S(V,X)g(U,Y) - S(U,X)g(V,Y) \\ &+ S(U,Y)g(V,X) - S(V,Y)g(U,X)].\end{aligned} \tag{2.1}$$

In [15], Shaikh and Hui showed that the conharmonic curvature tensor satisfies the symmetric and skew-symmetric properties of the Riemannian curvature tensor as well as cyclic ones. Divergence of conharmonic curvature tensor is given by

$$\begin{aligned}(div\mathcal{H})(U,V)X =& \frac{n-3}{n-2}[(\nabla_U S)(V,X) - (\nabla_V S)(U,X)] \\ &- \frac{1}{2(n-2)}[g(V,X)dr(U) - g(U,X)dr(V)].\end{aligned} \tag{2.2}$$

Definition 2.1. In 2012, Mantica and Suh [9] introduced a new generalized $(0,2)$ symmetric tensor $\mathcal{Z}$ and studied various geometric properties of it on a Riemannian manifold. A new tensor $\mathcal{Z}$ is defined as:

$$\mathcal{Z}(X,Y) = S(X,Y) + \psi g(X,Y), \tag{2.3}$$

where ψ is an arbitrary scalar function named as generalized $\mathcal{Z}-$tensor.

Definition 2.2. A symmetric $(0,2)$ type tensor field E on a semi-Riemannian manifold (M^n, g) is said to be a Codazzi tensor if it satisfies the Codazzi equation

$$(\nabla_U E)(V,X) = (\nabla_V E)(U,X), \tag{2.4}$$

for arbitrary vector fields U, V and X. The geometrical and topological consequences of the existence of a non-trivial Codazzi tensor on a Riemannian manifold have been studied by Derdzinski and Shen [4].

Definition 2.3. A Riemannian manifold (M^n, g) is said to be of quasi-constant curvature [2] if the Riemannian curvature tensor $\mathcal{R}(U,V,X,Y)$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}\mathcal{R}(U,V,X,Y) =& p[g(V,X)g(U,Y) - g(U,X)g(V,Y)] \\ &+ q[g(U,Y)A(V)A(X) + g(V,X)A(U)A(Y) \\ &- g(U,X)A(V)A(Y) - g(V,Y)A(U)A(X)],\end{aligned} \tag{2.5}$$

where p, q are scalar functions and A is non-zero $1-$form. If $q = 0$, then the manifold reduces to manifold of constant curvature.

3 Generalized conharmonic curvature tensor

In view of equation (2.3), equation (2.1) takes the form

$$\begin{aligned}\mathcal{H}(U,V,X,Y) =& \mathcal{R}(U,V,X,Y) - \frac{1}{n-2}[\mathcal{Z}(V,X)g(U,Y) \\ &- \mathcal{Z}(U,X)g(V,Y) + \mathcal{Z}(U,Y)g(V,X) - \mathcal{Z}(V,Y)g(U,X)] \\ &+ \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)].\end{aligned} \tag{3.1}$$

Define

$$\mathcal{H}^*(U,V,X,Y) = \mathcal{R}(U,V,X,Y) - \frac{1}{n-2}[\mathcal{Z}(V,X)g(U,Y) - \mathcal{Z}(U,X)g(V,Y) + \mathcal{Z}(U,Y)g(V,X) - \mathcal{Z}(V,Y)g(U,X)]. \tag{3.2}$$

Thus from above equation, equation (3.1) reduces to

$$\mathcal{H}(U,V,X,Y) = \mathcal{H}^*(U,V,X,Y) + \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)],$$

which gives

$$\mathcal{H}^*(U,V,X,Y) = \mathcal{H}(U,V,X,Y) - \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)], \tag{3.3}$$

where $\mathcal{H}^*(U,V,X,Y)$ is called generalized conharmonic curvature tensor.

If $\psi = 0$, then from equation (3.3), we obtain

$$\mathcal{H}^*(U,V,X,Y) = \mathcal{H}(U,V,X,Y). \tag{3.4}$$

Thus we arrive at:

Note: A generalized conharmonic curvature tensor reduces to conharmonic curvature tensor provided that the scalar function ψ vanishes.

Theorem 3.1. *A generalized conharmonic curvature tensor on (M^n, g) is*
(1) *skew-symmetric in first two slots,*
(2) *skew-symmetric in last two slots,*
(3) *symmetric in pair of slots.*

Proof. Interchanging the places of U and V in equation (3.3), we obtain

$$\mathcal{H}^*(V,U,X,Y) = \mathcal{H}(V,U,X,Y) - \frac{2\psi}{(n-2)}[g(U,X)g(V,Y) - g(V,X)g(U,Y)]$$

i.e.

$$\mathcal{H}^*(V,U,X,Y) = -\mathcal{H}(U,V,X,Y) + \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)]$$

i.e.

$$\mathcal{H}^*(U,V,X,Y) = -\mathcal{H}^*(V,U,X,Y),$$

which gives

$$\mathcal{H}^*(U,V,X,Y) + \mathcal{H}^*(V,U,X,Y) = 0,$$

which shows that generalized conharmonic curvature tensor is skew-symmetric with respect to first two slots.

Now, interchanging the places of X and Y in equation (3.3), we obtain

$$\mathcal{H}^*(U,V,Y,X) = \mathcal{H}(U,V,Y,X) - \frac{2\psi}{(n-2)}[g(V,Y)g(U,X) - g(U,Y)g(V,X)]$$

i.e.

$$\mathcal{H}^*(U,V,Y,X) = -\mathcal{H}(U,V,X,Y) + \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)]$$

i.e.

$$\mathcal{H}^*(U,V,X,Y) = -\mathcal{H}^*(U,V,Y,X),$$

which gives

$$\mathcal{H}^*(U,V,X,Y) + \mathcal{H}^*(U,V,Y,X) = 0,$$

which shows that generalized conharmonic curvature tensor is skew-symmetric with respect to last two slots.

Again, interchanging pair of slots in equation (3.3), we obtain

$$\mathcal{H}^*(X,Y,U,V) = \mathcal{H}(X,Y,U,V) - \frac{2\psi}{(n-2)}[g(Y,U)g(X,V) - g(X,U)g(Y,V)]$$

i.e.

$$\mathcal{H}^*(X,Y,U,V) = \mathcal{H}(U,V,X,Y) - \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)]$$

i.e.

$$\mathcal{H}^*(U,V,X,Y) = \mathcal{H}^*(X,Y,U,V),$$

which gives

$$\mathcal{H}^*(U,V,X,Y) - \mathcal{H}^*(X,Y,U,V) = 0,$$

which shows that generalized conharmonic curvature tensor is symmetric on pair of slots. □

Theorem 3.2. *A generalized conharmonic curvature tensor on (M^n, g) satisfies Bianchi's first identity.*

Proof. Writing two more equations by the cyclic permutations of U, V and X from equation (3.3), we obtain

$$\mathcal{H}^*(V,X,U,Y) = \mathcal{H}(V,X,U,Y) - \frac{2\psi}{(n-2)}[g(X,U)g(V,Y) - g(V,U)g(X,Y)], \tag{3.5}$$

and

$$\mathcal{H}^*(X,U,V,Y) = \mathcal{H}(X,U,V,Y) - \frac{2\psi}{(n-2)}[g(U,V)g(X,Y) - g(X,V)g(U,Y)], \tag{3.6}$$

By using the fact, $\mathcal{H}(U,V,X,Y)+\mathcal{H}(V,X,U,Y)+\mathcal{H}(X,U,V,Y) = 0$ adding equations (3.3), (3.5) and (3.6), we obtain

$$\mathcal{H}^*(U,V,X,Y) + \mathcal{H}^*(V,X,U,Y) + \mathcal{H}^*(X,U,V,Y) = 0, \tag{3.7}$$

which shows that generalized conharmonic curvature tensor satisfied Bianchi's first identity. □

Theorem 3.3. *A generalized conharmonic curvature tensor on (M^n, g) satisfies Bianchi's second identity, if $\mathcal{Z}-$tensor is Codazzi tensor.*

Proof. Taking the covariant derivative of equation (3.2) along the vector field U, we obtain

$$\begin{aligned}(\nabla_U\mathcal{H}^*)(V,X,Y,W) &= (\nabla_U\mathcal{R})(V,X,Y,W)\\ &- \frac{1}{n-2}[g(V,W)(\nabla_U\mathcal{Z})(X,Y) - g(X,W)(\nabla_U\mathcal{Z})(V,Y)\\ &+ g(X,Y)(\nabla_U\mathcal{Z})(V,W) - g(V,Y)(\nabla_U\mathcal{Z})(X,W)].\end{aligned}\tag{3.8}$$

Writing two more equations by the cyclic permutations of U, V and X from equation (3.8), we obtain

$$\begin{aligned}(\nabla_V\mathcal{H}^*)(X,U,Y,W) &= (\nabla_V\mathcal{R})(X,U,Y,W)\\ &- \frac{1}{n-2}[g(X,W)(\nabla_V\mathcal{Z})(U,Y) - g(U,W)(\nabla_V\mathcal{Z})(X,Y)\\ &+ g(U,Y)(\nabla_V\mathcal{Z})(X,W) - g(X,Y)(\nabla_V\mathcal{Z})(U,W)]\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}(\nabla_X\mathcal{H}^*)(U,V,Y,W) &= (\nabla_X\mathcal{R})(U,V,Y,W)\\ &- \frac{1}{n-2}[g(U,W)(\nabla_X\mathcal{Z})(V,Y) - g(V,W)(\nabla_X\mathcal{Z})(U,Y)\\ &+ g(V,Y)(\nabla_X\mathcal{Z})(U,W) - g(U,Y)(\nabla_X\mathcal{Z})(V,W)].\end{aligned}\tag{3.10}$$

Adding equations (3.8), (3.9) and (3.10) with the fact that $(\nabla_U\mathcal{R})(V,X,Y,W)$ $+(\nabla_V\mathcal{R})(X,U,Y,W) + (\nabla_X\mathcal{R})(U,V,Y,W) = 0$, we get

$$\begin{aligned}(\nabla_U\mathcal{H}^*)(V,X,Y,W) &+ (\nabla_V\mathcal{H}^*)(X,U,Y,W) + (\nabla_X\mathcal{H}^*)(U,V,Y,W)\\ &= -\frac{1}{n-2}[g(V,W)\{(\nabla_U\mathcal{Z})(X,Y) - (\nabla_X\mathcal{Z})(U,Y)\}\\ &- g(X,W)\{(\nabla_U\mathcal{Z})(V,Y) - (\nabla_V\mathcal{Z})(U,Y)\}\\ &+ g(X,Y)\{(\nabla_U\mathcal{Z})(V,W) - (\nabla_V\mathcal{Z})(U,W)\}\\ &- g(V,Y)\{(\nabla_U\mathcal{Z})(X,W) - (\nabla_X\mathcal{Z})(U,W)\}\\ &- g(U,W)\{(\nabla_V\mathcal{Z})(X,Y) - (\nabla_X\mathcal{Z})(V,Y)\}\\ &+ g(U,Y)\{(\nabla_V\mathcal{Z})(X,W) - (\nabla_X\mathcal{Z})(V,W)\}].\end{aligned}\tag{3.11}$$

Assuming that $\mathcal{Z}-$tensor is codazzi tensor, then equation (3.11), reduces to

$$(\nabla_U\mathcal{H}^*)(V,X,Y,W) + (\nabla_V\mathcal{H}^*)(X,U,Y,W) + (\nabla_X\mathcal{H}^*)(U,V,Y,W) = 0.\tag{3.12}$$

which shows that generalized conharmonic curvature tensor satisfied Bianchi's second identity. □

4 Generalized conharmonic flat space-time

Theorem 4.1. *A 4−dimensional relativistic generalized conharmonic flat space-time is an Einstein space-time and it is of constant curvature.*

Proof. Let M be a 4-dimensional space-time of general relativity, then in view of equation (3.2), we have

$$\begin{aligned}\mathcal{H}^*(U,V,X,Y) = \mathcal{R}(U,V,X,Y) - \frac{1}{2}[\mathcal{Z}(V,X)g(U,Y) - \mathcal{Z}(U,X)g(V,Y) \\ + \mathcal{Z}(U,Y)g(V,X) - \mathcal{Z}(V,Y)g(U,X)].\end{aligned} \tag{4.1}$$

If $\mathcal{H}^*(U,V,X,Y) = 0$, then from equation (4.1), we have

$$\begin{aligned}\mathcal{R}(U,V,X,Y) = \frac{1}{2}[\mathcal{Z}(V,X)g(U,Y) - \mathcal{Z}(U,X)g(V,Y) \\ + \mathcal{Z}(U,Y)g(V,X) - \mathcal{Z}(V,Y)g(U,X)].\end{aligned} \tag{4.2}$$

Contracting V and Y in above equation by using equation (2.4), we obtain

$$S(U,X) = \frac{-(r+6\psi)}{4}g(U,X), \tag{4.3}$$

which on contraction gives $r = -3\psi$. Thus equation (4.3) reduces to

$$S(U,X) = \frac{r}{4}g(U,X). \tag{4.4}$$

This shows that, a generalized conharmonic flat space-time is an Einstein space-time. Again, using equation (4.4) in equation (2.3), we get

$$\mathcal{Z}(U,X) = \frac{-r}{12}g(U,X). \tag{4.5}$$

Now, using equation (4.5) in equation (4.2), we obtain

$$\mathcal{R}(U,V,X,Y) = \frac{-r}{24}[g(U,Y)g(V,X) - g(U,X)g(V,Y)], \tag{4.6}$$

which shows that M is of constant curvature. □

Corollary 4.2. *A 4−dimensional relativistic generalized conharmonic flat space-time is a conformally flat space-time.*

Proof. It is known that a Lorentzian manifold of constant curvature is a manifold of conformally flat. □

Theorem 4.3. *A 4−dimensional relativistic generalized conharmonic flat space-time is of O−type.*

Proof. In 1980, Kramer et al. [7] have been proved that a space-time is of $O-$type if the conformal curvature tensor vanishes on it. □

Theorem 4.4. *If a* $4-$*dimensional relativistic space-time having Einstein's field equation in the presence of cosmological constant is generalized conharmonically flat curvature tensor, then the space-time satisfies matter collineation along a vector field* ξ *if and only if* ξ *is a Killing vector field.*

Proof. Let $£_\xi$ be the Lie derivative operator along the vector field ξ generating the symmetry. The matter collineation defined by $(£_\xi T)(U,V) = 0$ represents the symmetry of energy momentum tensor T. In view of equation (4.4), equation (1.3) takes the form

$$(\frac{-r}{4} + \lambda)g(X,Y) = kT(X,Y). \tag{4.7}$$

If ξ be a Killing vector field on the space-time with generalized conharmonically flat curvature tensor, then

$$(£_\xi g)(X,Y) = 0. \tag{4.8}$$

Taking the Lie derivative of equation (4.7) along ξ, we obtain

$$(\frac{-r}{4} + \lambda)(£_\xi g)(X,Y) = k(£_\xi T)(X,Y). \tag{4.9}$$

In virtue of equation (4.8), equation (4.9) shows that $(£_\xi T)(X,Y) = 0$, which shows that the space-time admits matter collineation. Conversely, If $(£_\xi T)(X,Y) = 0$, it follows that from equation (4.9), that ξ is Killing vector field. □

Theorem 4.5. *In a* $4-$*dimensional relativistic space-time having Einstein's field equation in the presence of cosmological constant generalized conharmonically flat curvature tensor, a vector field* ξ *is conformal Killing vector field if and only if the energy-momentum tensor* T *has a symmetry inheritance property along* ξ.

Proof. Let us assume that the vector field ξ is a conformal Killing vector field, then we obtain

$$(£_\xi g)(X,Y) = 2\phi g(X,Y), \tag{4.10}$$

where ϕ is scalar, which in view of equation (4.9), gives

$$(\frac{-r}{4} + \lambda)2\phi g(X,Y) = k(£_\xi T)(X,Y). \tag{4.11}$$

Using equation (4.7) in equation (4.11), we obtain

$$(£_\xi T)(X,Y) = 2\phi T(X,Y). \tag{4.12}$$

From above equation, we see that the energy-momentum tensor has Lie inheritance property along ξ. Conversely, if equation (4.12) holds, then it follows that equation (4.10) holds, *i.e.* the vector field ξ is a conformal Killing vector field. □

Theorem 4.6. *In a* $4-$*dimensional relativistic perfect fluid generalized conharmonically flat space-time following Einstein's field equation in the absence of cosmological constant,* $\sigma + \rho = 0$ *and the isotropic pressure and energy density are constants. Moreover, energy momentum tensor is covariant constant.*

Proof. Using equations (1.1) and (4.4) in equation (1.4), we get

$$-(\frac{r}{4}+k\rho)g(X,Y)=k(\rho+\sigma)A(X)A(Y), \tag{4.13}$$

which on contraction gives

$$r=k(\sigma-3\rho). \tag{4.14}$$

Now, taking $X=Y=\mu$ in equation (4.13) and using $g(\mu,\mu)=-1$, we obtain

$$r=4k\sigma. \tag{4.15}$$

From equations (4.14) and (4.15), we have

$$\sigma+\rho=0. \tag{4.16}$$

which shows that the perfect fluid behaves as a cosmological constant. Thus, in view of equation (4.16), equation (1.1) reduces to

$$T(X,Y)=\rho g(X,Y). \tag{4.17}$$

For a generalized conharmonically flat space-time, the scalar curvature is constant. Thus σ is constant. Consequently, ρ is constant. Therefore, on taking covariant derivative of equation (4.17), we obtain

$$(\nabla_U T)(X,Y)=0, \tag{4.18}$$

which shows that the energy momentum tensor is covariant constant. □

Theorem 4.7. *In a 4−dimensional relativistic perfect fluid space-time with flat generalized conharmonic curvature tensor having Einstein's field equation in the absence of cosmological constant,* $div\mu=0$, $\nabla_\mu\mu=0$ *and* $\sigma+\rho=0$.

Proof. Since the energy momentum tensor is divergence free in a space-time, therefore from equation (4.17), ρ is constant and consequently from equation (4.16), σ is also constant. This implies

$$\mu\sigma=-(\sigma+\rho)div\mu \tag{4.19}$$

and

$$(\sigma+\rho)\nabla_\mu\mu=-grad\rho-(\mu\rho)\mu. \tag{4.20}$$

Equations (4.19) and (4.20) gives $div\mu=0$ (expansion scalar vanishes) and $\nabla_\mu\mu=0$ (acceleration vector vanishes) as both pressure ρ and energy density σ are constants. □

Theorem 4.8. *A 4−dimensional relativistic generalized conharmonically flat space-time having Einstein's field equation in the absence of cosmological constant for a purely electromagnetic distribution is an Euclidean space.*

Proof. Taking the frame-field after contraction over X and Y of equation (1.4),we obtain

$$r = -kt, \tag{4.21}$$

where t is $tr(T)$. Therefore, equation (1.4) can be written as

$$S(X,Y) = k[T(X,Y) - \frac{t}{2}g(X,Y)]. \tag{4.22}$$

Einstein's field equation in the absence of cosmological constant for a purely electromagnetic distribution takes the form [13]

$$S(X,Y) = kT(X,Y). \tag{4.23}$$

From equations (4.22) and (4.23), we obtained $t = 0$. Thus from equation (4.21), we get $r = 0$. Therefore, from equation (4.6), we obtain $R(U,V,X,Y) = 0$, which shows that the space is flat. □

5 Conservative generalized conharmonic space-time

Definition 5.1. A 4−dimensional relativistic space-time is said to be conservative generalized conharmonic space-time if $(div\mathcal{H}^*)(U,V)X = 0$, where "div" denotes the divergence.

Theorem 5.1. *A 4−dimensional relativistic conservative generalized conharmonic space-time M with constant scalar function ψ is conservative conharmonic curvature tensor, provided that the Ricci tensor is codazzi tensor.*

Proof. From equation (3.3), generalized conharmonic curvature tensor is given by

$$\mathcal{H}^*(U,V)X = \mathcal{H}(U,V)X - \frac{2\psi}{(n-2)}[g(V,X)U - g(U,X)Y], \tag{5.1}$$

The divergence of $\mathcal{H}^*(U,V)X$ is defined as

$$(div\mathcal{H}^*)(U,V)X = g((\nabla_{e_i}\mathcal{H}^*)(U,V)X, e_i)$$

i.e.

$$(div\mathcal{H}^*)(U,V)X = g((\nabla_{e_i}\mathcal{H})(U,V)X, e_i) - \frac{1}{n-2}[g((\nabla_{e_i}\psi)\{g(V,X)U - g(U,X)V\}, e_i)],$$

which gives

$$(div\mathcal{H}^*)(U,V)X = (div\mathcal{H})(U,V)X - \frac{1}{(n-2)}[(U\psi)g(V,X) - (V\psi)g(U,X)]. \tag{5.2}$$

If scalar function ψ is constant then from equation (5.2), we obtain

$$(div\mathcal{H}^*)(U,V)X = (div\mathcal{H})(U,V)X. \tag{5.3}$$

From equations (2.2) and (5.3), we obtain

$$(5.4)\qquad \begin{aligned}(div\mathcal{H}^*)(U,V)X =&\frac{n-3}{n-2}[(\nabla_U S)(V,X) - (\nabla_V S)(U,X)]\\ &- \frac{1}{2(n-2)}[g(V,X)dr(U) - g(U,X)dr(V)].\end{aligned}$$

If $(div\mathcal{H}^*)(U,V)X = 0$ then from equation (5.4), we obtain

$$(5.5)\quad (\nabla_U S)(V,X) - (\nabla_V S)(U,X) - \frac{1}{2(n-3)}[g(V,X)dr(U) - g(U,X)dr(V)] = 0,$$

which gives

$$(div\mathcal{H})(U,V)X = 0.$$

From equation (5.5) we conclude that the Ricci tensor is Codazzi tensor. □

Theorem 5.2. *A* $4-$*dimensional relativistic conservative generalized conharmonic space-time* M *with constant scalar function* ψ *is a Yang Pure space.*

Proof. Guifoyle and Nolan [6], gave "Yang Pure Space", a 4-dimensional Lorentzian manifold (M^n, g) whose metric tensor solves Yang's equation

$$(\nabla_U S)(V,X) - (\nabla_V S)(U,X) = 0.$$

□

Theorem 5.3. *A* $4-$*dimensional relativistic conservative generalized conharmonic space-time* M *with constant scalar function* ψ *is a RW space-time.*

Proof. Since we known that [6], a $4-$dimensional relativistic perfect fluid space- time with $\sigma + \rho \neq 0$ is a Yang Pure space-time if and only if space-time is RW space-time. □

Theorem 5.4. *A* $4-$*dimensional relativistic space-time* M *satisfying* $(divC)(U,V)X = 0$ *if and only if* $(div\mathcal{H}^*)(U,V)X = 0$ *with constant scalar function* ψ*.*

Proof. It is known that divergence of conformal (Weyl) curvature tensor can be written as

$$(5.6)\qquad \begin{aligned}(divC)(U,V)X =& \frac{n-3}{n-2}[(\nabla_U S)(V,X) - (\nabla_V S)(U,X)\\ &- \frac{1}{2(n-1)}\{g(V,X)dr(U) - g(U,X)dr(V)\}].\end{aligned}$$

In virtue of equations (5.3) and (5.6), we observe that $(divC)(U,V)X = 0$ if and only if $(div\mathcal{H}^*)(U,V)X = 0$ with constant scalar function ψ. □

Theorem 5.5. *A* $4-$*dimensional relativistic conservative generalized conharmonic space-time* M *with constant scalar function* ψ *is a GRW space-time.*

Proof. Since a perfect fluid space- time of dimension n $(n \geq 3)$ satisfies conservative conformal curvature tensor and the velocity vector field irrotational (see [10], [11] and [12]), then the space-time is a GRW space-time with $A(C(U,V)X) = 0$. □

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