# On quasi bi-slant $\xi^\perp-$Riemannian submersions

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**Abstract.** In this paper, we introduce quasi bi-slant $\xi^\perp-$Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. We study the geometry of leaves of distributions that are involved in the definition of the submersion. We also obtain the necessary and sufficient condition for quasi bi-slant $\xi^\perp-$Riemannian submersions to be totally geodesic and provide an example for this setting.

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**Key words**: Riemannian submersions; Semi-invariant submersions; Quasi bi-slant $\xi^\perp-$Riemannian submersions.

## 1 Introduction

A Riemannian submersion $F : (M^m, g_M) \longrightarrow (N^n, g_N)$ is a differentiable map between Riemannian manifolds $(M^m, g_M)$ and $(N^n, g_N)$, where $\dim M = m, \dim N = n$ and $m > n$, satisfying the following axioms:

(a) $F$ has the maximal rank.

(b) The derivative map $F_*$ preserves the lengths of horizontal vectors.

The theory of Riemannian submersion has acquired prominence due to its applications in the theory of robotics, physics and mechanics. N. Bedrossian and M. Spong [5] showed the existence of a class of robotic chains having Riemannian curvature that is locally vanishing (once potential energy and friction phenomena are ignored). C. Altafini [3] commenced use of the notion of Riemannian submersions for the modeling and control of redundant robotic chain ( redundant means that robotic chain has more than six degrees of freedom). He also showed that Riemannian submersion gives a close relationship between inverse kinematic in robotics and the pull back vectors, called the horizontal lift. On the other hand, Bourguignon and Lawson [8] used Riemannian submersions as a special tool in obtaining the general solution of recent model which can be expressed in harmonic maps satisfying Einstein equation. However, a very general class of solution is given by Riemannian submersion from the extra dimensional space onto the space in which the scalar fields take values (see [10] for details). Riemannian submersion theory has also applications in the Yang-Mills theory [7], Supergravity and superstring theories [14], [17]. These broad applications of this topic make it an interesting field of research for geometers.

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It is known that the study of Riemannian submersion between Riemannian manifolds was initiated by O′Neill [20] and Gray [11]. Watson [28] studied almost Hermitian submersions and he obtained that the base manifold and each fiber has the same kind of structure as the total space in most cases. The notion of almost Hermitian submersion has been extended to different kinds of sub-classes of almost contact manifolds by D. Chinea [9]. Several good results related to Riemannian submersions can be found in [10] and [25]. The semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds were introduced by B. Sahin [22]. These were the generalizations of holomorphic submersions and anti-invariant submersions. Further, Sahin [23] introduced the notion of slant submersions from almost Hermitian manifolds onto arbitrary Riemannian manifolds. Different kinds of Riemannian submersions between Riemannian manifolds have been studied by several authors ([12],[13],[19],[21],[23],[24],[26]) etc. Park and Prasad [18] defined and studied semi-slant submersions from an almost Hermitian manifold onto a Riemannian manifold. Tastan, Sahin and Yanan [27] introduced hemi-slant Riemannian submersions from Hermitian manifolds onto Riemannian manifolds and also gave a decomposition theorem for such submersions.

On the other hand, Lee [15] study anti invariant $\xi^{\perp}-$Riemannian submersions and obtain a few interesting results on the geometry of these submersions. As a generalization of anti invariant $\xi^{\perp}-$ Riemannian submersions, Akyol, Sari and Aksoy [1] introduce semi-invariant $\xi^{\perp}-$Riemannian submersions and semi-slant $\xi^{\perp}-$Riemannian submersions [2] and they investigate the geometry of the total space and the base space for the existence of such submersions. The main scope of this paper is to introduce quasi bi-slant $\xi^{\perp}-$Riemannian submersions from Sasakian manifolds onto Riemannian manifolds.

The present paper is organized as follows : In section 2, we mention basic definitions and some properties of Riemannian submersions, the second fundamental form and Sasakian manifolds. In section 3, we define quasi-bi slant $\xi^{\perp}-$Riemannian submersions and obtained some basic properties of these submersions. The necessary and sufficient conditions for integrability of distributions and totally geodesicness for a quasi-bi slant $\xi^{\perp}-$Riemannian submersions are also obtained in this section. In the last section, we provide worthy example of such submersions.

## 2 Preliminaries

An odd-dimensional smooth manifold $M_1$ is said to have an almost contact structure $(\phi, \xi, \eta)$ if there exist on $M_1$, a tensor field $\phi$ of type $(1, 1)$, a structural vector field $\xi$ and $1-$form $\eta$ such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \ \eta \circ \phi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

where $I$ denote the identity tensor. The manifold $M_1$ with an almost contact structure is called an almost contact manifold.

If there exists a Riemannian metric $g_1$ on an almost contact manifold $M_1$ satisfying the following conditions;

$$\begin{aligned} g_1(\phi Z_1, \phi Z_2) &= g_1(Z_1, Z_2) - \eta(Z_1)\eta(Z_2), \\ g_1(Z_1, \phi Z_2) &= -g_1(\phi Z_1, Z_2), \end{aligned} \tag{2.3}$$

$$g_1(Z_1, \xi) = \eta(Z_1), \tag{2.4}$$

where $Z_1, Z_2$ are the vector fields on $M_1$, then structure $(\phi, \xi, \eta, g_1)$ is called almost contact metric structure and the manifold $M_1$ is called an almost contact metric manifold. An almost contact manifold $M_1$ with almost contact metric structure $(\phi, \xi, \eta, g_1)$ is denoted by $(M_1, \phi, \xi, \eta, g_1)$. Further, an almost contact structure $(\phi, \xi, \eta)$ is said to be normal if $N + d\eta \otimes \xi = 0$, where $N$ is the Nijenhuis tensor of $\phi$. The fundamental $2-$form $\Phi$ is defined by $\Phi(Z_1, Z_2) = g_1(Z_1, \phi Z_2)$.

An almost contact metric manifold $(M_1, \phi, \xi, \eta, g_1)$ is said to be Sasakian manifold ([29]) if it satisfies the following condition;

$$(\nabla_{Z_1}\phi)Z_2 = g_1(Z_1, Z_2)\xi - \eta(Z_2)Z_1, \tag{2.5}$$

where $\nabla$ denotes the Riemannian connection of metric $g_1$ on $M_1$.

For a Sasakian manifold $M_1$, we have

$$\nabla_{Z_1}\xi = -\phi Z_1, \tag{2.6}$$

for any vector fields $Z_1$ and $Z_2$ on $M_1$.

**Example 2.1.** [6] Let $R^{2k+1}$ with Cartesian coordinates $(x_i, y_i, z)$ $(i = 1, 2.....k)$ and its usual contact form

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{k} y_i dx_i).$$

The characteristic vector field $\xi$ is given by $2\frac{\partial}{\partial z}$ and its Riemannian metric $g_{R^{2k+1}}$ and tensor field $\phi$ are given by

$$g_{R^{2k+1}} = (\eta \otimes \eta) + \frac{1}{4}\sum_{i=1}^{k}((dx_i)^2 + (dy_i)^2), \;\; \phi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_i & 0 \end{bmatrix}, \quad i, j = 1, ..., k.$$

This gives a contact metric structure on $R^{2k+1}$. The vector fields $E_i = 2\frac{\partial}{\partial y_i}$, $E_{k+i} = 2(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z})$ and $\xi = 2\frac{\partial}{\partial z}$ form a $\phi-$basis for the contact metric structure. On the other hand, it can be shown that $(R^{2k+1}, \phi, \xi, \eta, g_{R^{2k+1}})$ is a Sasakian manifold.

**Definition 2.2.** [25] Let $F : (M_1, g_1, J) \to (M_2, g_2)$ be a Riemannian submersion from an almost Hermitian manifold onto a Riemannian manifold. Then we say that $F$ is an invariant Riemannian submersion if the vertical distribution is invariant with respect to the complex structure $J$, i.e.,

$$J(\ker F_*) = \ker F_*.$$

**Definition 2.3.** [25] Let $F : M_1 \to M_2$ be a Riemannian submersion from an almost Hermitian manifold $(M_1, g_1, J)$ onto a Riemannian manifold $(M_2, g_2)$. Then we say that $F$ is a semi-invariant Riemannian submersion if there is a distribution $\mathfrak{D}_1 \subseteq \ker F_*$ such that

$$\ker F_* = \mathfrak{D}_1 \oplus \mathfrak{D}_2, J(\mathfrak{D}_1) = \mathfrak{D}_1,$$

where $D_1 \oplus D_2$ is an orthogonal decomposition of $\ker F_*$.

Let $\mu$ denotes the complementary orthogonal subbundle to $J(\ker F_*)$ in $(\ker F_*)^\perp$. Then, we have

$$(\ker F_*)^\perp = \omega(\mathfrak{D}_2) \oplus \mu.$$

Obviously $\mu$ is an invariant subbundle of $(\ker F_*)^\perp$ with respect to the complex structure $J$.

**Definition 2.4.** [23] Let $F : (M_1, g_1, J) \to (M_2, g_2)$ be a Riemannian submersion from an almost Hermitian manifold onto a Riemannian manifold. If for any non-zero vector field $Z_1 \in (\ker F_*)_p$, $p \in M_1$, the angle $\theta(Z_1)$ between $JZ_1$ and the space $(\ker F_*)_p$ is constant, i.e., it is independent of the choice of the point $p \in M_1$ and the tangent vector $Z_1$ in $\ker F_*$, then we say that $F$ is a slant submersion. In this case, the angle $\theta$ is called the slant angle of the submersion. If the slant angle $0 < \theta < \frac{\pi}{2}$, then the submersion is called a proper slant submersion.

**Definition 2.5.** [18] Let $(M_1, g_1, J)$ be an almost Hermitian manifold and $(M_2, g_2)$ be a Riemannian manifold. A Riemannian submersion $F : (M_1, g_1, J) \to (M_2, g_2)$ is called a semi-slant Riemannian submersion if there are two distributions $D_1, D_2 \subset \ker F_*$ such that

$$\ker F_* = D_1 \oplus D_2, \phi(D_1) = D_1,$$

and the angle $\theta = \theta(Z_1)$ between $JZ_1$ and the space $(D_2)_p$ is constant for non-zero vector fields $Z_1 \in (D_2)_p$ and $p \in M_1$, where $D_1 \oplus D_2$ is an orthogonal decomposition of $\ker F_*$. The angle $\theta$ is called semi$-$slant angle of the submersion.

**Definition 2.6.** [27] Let $(M_1, g_1, J)$ be an almost Hermitian manifold and $(M_2, g_2)$ be a Riemannian manifold. A Riemannian submersion $F : (M_1, g_1, J) \to (M_2, g_2)$ is called a hemi-slant submersion if the vertical distribution $\ker F_*$ of $F$ admits two orthogonal complementary distributions $D^\theta$ and $D^\perp$ such that $D^\theta$ is slant with angle $\theta$ and $D^\perp$ is anti-invariant, i.e, we have

$$\ker F_* = D^\theta \oplus D^\perp.$$

In this case, the angle $\theta$ is called the hemi-slant angle of the submersion.

**Definition 2.7.** [16] Let $(M_1, g_1, J)$ be an almost Hermitian manifold and $(M_2, g_2)$ a Riemannian manifold. Suppose that a Riemannian submersion $F : (M_1, g_1, J) \to (M_2, g_2)$ such that its vertical distribution $\ker F_*$ admits three orthogonal distributions $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ which are invariant, slant and anti-invariant respectively, i.e.,

$$\ker F_* = \mathcal{D} \oplus \mathcal{D}_1 \oplus \mathcal{D}_2, J(\mathcal{D}) = \mathcal{D},$$

with $J\mathcal{D} = \mathcal{D}$, the angle $\theta$ between $J\mathcal{D}_1$ and $\mathcal{D}_1$ being constant and $J\mathcal{D}_2 \subseteq (\ker F_*)^\perp$. Then we say that $F$ is called Quasi-hemi-slant Riemannian submersion and the angle $\theta$ is called the quasi-hemi-slant angle of the submersion.

**Definition 2.8.** [19] Let $(M_1, g_1, J)$ be an almost Hermitian manifold and $(M_2, g_2)$ a Riemannian manifold. A Riemannian submersion $F : (M_1, g_1, J) \to (M_2, g_2)$ is called a bi-slant submersion if

$(a)$ for any non-zero vector field $Z_1 \in (D_1)_q$ and $q \in M_1$, the angle $\theta_1$ between $JZ_1$ and the space $(D_1)_q$ are constant,
$(b)$ for any non-zero vector field $Z_2 \in (D_2)_p$ and $p \in M_1$, the angle $\theta_2$ between $JZ_2$ and the space $(D_2)_p$ are constant, $F^*([Z_1, Z_2]^{\mathcal{H}})$,
$(c)$ $JD_1 \perp D_2$ and $JD_2 \perp D_1$,
such that $\ker F_* = D_1 \oplus D_2$, $F_*$ is called proper if its bi-slant angles satisfy $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

**Definition 2.9.** [19] A Riemannian submersion $F$ from an almost Hermitian manifold $(M_1, g_1, J)$ onto a Riemannian manifold $(M_2, g_2)$ is called a quasi bi-slant submersion if there exist three mutually orthogonal distributions $D, D_1$ and $D_2$ and such that

$(i)$ $\ker F_* = D \oplus D_1 \oplus D_2$,
$(ii)$ $J(D) = D$ i.e., $D$ is invariant,
$(iii)$ $J(D_1) \perp D_2$ and $J(D_2) \perp D_1$,
$(iv)$ for any non-zero vector field $X_1 \in (D_1)_p$, $p \in M_1$, the angle $\theta_1$ between $JX_1$ and $(D_1)_p$ is constant and independent of the choice of point $p$ and $X_1$ in $(D_1)_p$,
$(v)$ for any non-zero vector field $Z_1 \in (D_2)_q$, $q \in M_1$, the angle $\theta_2$ between $JZ_1$ and $(D_2)_q$ is constant and independent of the choice of point $q$ and $Z_1$ in $(D_2)_q$.
These angles $\theta_1$ and $\theta_2$ are called slant angles of the submersion.

**Definition 2.10.** [15] Let $F : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a Riemannian submersion from an almost contact metric manifold onto a Riemannian manifold. Suppose that there exists a Riemannian submersion $F$ such that $\xi$ is normal to $\ker F_*$ and $\ker F_*$ is anti-invariant with respect to $\phi$ i.e., $\phi(\ker F_*) \subset (\ker F_*)^{\perp}$. Then we say that $F$ is an anti-invariant $\xi^{\perp}-$Riemannian submersion.

A Riemannian submersion $F : (M_1, g_1) \to (M_2, g_2)$ characterized by two fundamental O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ on $M_1$ is defined by the following formulae

$$\mathcal{A}_{E_1} E_2 = \mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}E_2, \tag{2.7}$$

$$\mathcal{T}_{E_1} E_2 = \mathcal{H}\nabla_{\mathcal{V}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{V}E_1}\mathcal{H}E_2, \tag{2.8}$$

for any vector fields $E_1, E_2$ on $M_1$, where $\nabla$ is the Levi-Civita connection of $g_1$. It is easy to see that $\mathcal{T}_{E_1}$ and $\mathcal{A}_{E_1}$ are skew-symmetric operators on the tangent bundle of $M_1$ reversing the vertical and the horizontal distributions.

From equations (2.7) and (2.8), we have

$$\nabla_{X_1} X_2 = \mathcal{T}_{X_1} X_2 + \mathcal{V}\nabla_{X_1} X_2, \tag{2.9}$$

$$\nabla_{X_1} Z_1 = \mathcal{T}_{X_1} Z_1 + \mathcal{H}\nabla_{X_1} Z_1, \tag{2.10}$$

$$\nabla_{Z_1} X_1 = \mathcal{A}_{Z_1} X_1 + \mathcal{V}\nabla_{Z_1} X_1, \tag{2.11}$$

$$\nabla_{Z_1} Z_2 = \mathcal{H}\nabla_{Z_1} Z_2 + \mathcal{A}_{Z_1} Z_2, \tag{2.12}$$

for all $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^\perp$, where $\mathcal{H}\nabla_{X_1} Z_1 = \mathcal{A}_{Z_1} X_1$, if $Z_1$ is basic. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution.

It is seen that for $q \in M_1$, $X_1 \in \mathcal{V}_q$ and $Z_1 \in \mathcal{H}_q$ the linear operators $\mathcal{A}_{Z_1}$, $\mathcal{T}_{X_1}$ : $T_q M_1 \to T_q M_1$ are skew-symmetric, that is

$$g_1(\mathcal{A}_{Z_1} E_1, E_2) = -g_1(E_1, \mathcal{A}_{Z_1} E_2) \text{ and } g_1(\mathcal{T}_{X_1} E_1, E_2) = -g_1(E_1, \mathcal{T}_{X_1} E_2), \tag{2.13}$$

for each $E_1, E_2 \in T_q M_1$. Since $\mathcal{T}_{\mathcal{V}}$ is skew-symmetric, we observe that $F$ has totally geodesic fibers if and only if $\mathcal{T} \equiv 0$.

We recall that a differentiable map $F$ between two Riemannian manifolds is totally geodesic [4] if $(\nabla F_*)(X_1, X_2) = 0$, for all $X_1, X_2 \in \Gamma(TM_1)$.

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Let $F : (M_1, g_1) \to (M_2, g_2)$ is a smooth map between Riemannian manifolds. Then the differential map $F_*$ of $F$ can be observed a section of the bundle $Hom(TM_1, F^{-1}TM_2) \to M_1$, where $F^{-1}TM_2$ is the bundle which has fibers $(F^{-1}TM_2)_x = T_{F(x)} M_2$ has a connection $\nabla$ induced from the Riemannian connection and $\nabla^F$ the pullback connection. Then the second fundamental form of $F$ is given by

$$(\nabla F_*)(X_1, X_2) = \nabla^F_{X_1} F_*(X_2) - F_*(\nabla^{M_1}_{X_1} X_2), \tag{2.14}$$

for vector field $X_1, X_2 \in \Gamma(TM_1)$. We know that the second fundamental form is symmetric.

**Lemma 2.1.** *Let $(M, g_M)$ and $(N, g_N)$ are Riemannian manifolds. If $f : M \to N$ be a Riemannian submersion, then for any horizontal vector fields $U, V$ and vertical vector fields $X, Y$, we have*

(*i*) $(\nabla f_*)(U, V) = 0$,
(*ii*) $(\nabla f_*)(X, Y) = -f_*(\mathcal{T}_X Y) = -f_*(\nabla_X Y)$,
(*iii*) $(\nabla f_*)(U, X) = -f_*(\nabla_U X) = -f_*(\mathcal{A}_U X)$.

# 3 Quasi bi-slant $\xi^\perp-$Riemannian submersions

Now, we introduce the notion of a quasi bi-slant submersion from almost contact metric manifolds onto Riemannian manifolds.

**Definition 3.1.** Let $(M_1, \phi, \xi, \eta, g_1)$ be an almost contact metric manifold and $(M_2, g_2)$ is a Riemannian manifold. A Riemannian submersion $F : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ is called a quasi bi-slant $\xi^\perp-$Riemannian submersion if there exist three mutually orthogonal distributions $D, D_1$ and $D_2$ such that

(*i*) $\ker F_* = D \oplus D_1 \oplus D_2$,
(*ii*) $\phi(D) = D$ i.e., $D$ is invariant,
(*iii*) $\phi(D_1) \perp D_2$ and $\phi(D_2) \perp D_1$,

(*iv*) for any non-zero vector field $X_1 \in (D_1)_p$, $p \in M_1$, the angle $\theta_1$ between $\phi X_1$ and $(D_1)_p$ is constant and independent of the choice of point $p$ and $X_1$ in $(D_1)_p$,
(*v*) for any non-zero vector field $Z_1 \in (D_2)_q$, $q \in M_1$, the angle $\theta_2$ between $\phi Z_1$ and $(D_2)_q$ is constant and independent of the choice of point $q$ and $Z_1$ in $(D_2)_q$.
These angles $\theta_1$ and $\theta_2$ are called slant angles of the submersion.

Let $F$ be quasi bi-slant $\xi^{\perp}-$Riemannian submersion from an almost contact metric manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then we have

$$TM_1 = \ker F_* \oplus (\ker F_*)^{\perp}. \tag{3.1}$$

Now, for any vector field $X_1 \in \Gamma(\ker F_*)$, we put

$$X_1 = PX_1 + QX_1 + RX_1, \tag{3.2}$$

where $P, Q$ and $R$ are projection morphisms of $\ker F_*$ onto $D, D_1$ and $D_2$, respectively.

For all $Z_1 \in \Gamma(\ker F_*)$, we get

$$\phi Z_1 = \psi Z_1 + \omega Z_1, \tag{3.3}$$

where $\psi Z_1 \in \Gamma(\ker F_*)$ and $\omega Z_1 \in \Gamma(\omega D_1 \oplus \omega D_2)$.

The horizontal distribution $(\ker F_*)^{\perp}$ is decomposed as

$$(\ker F_*)^{\perp} = \omega D_1 \oplus \omega D_2 \oplus \mu. \tag{3.4}$$

Here $\mu$ is an invariant distribution of $\omega D_1 \oplus \omega D_2$ in $\ker F_*$ and contains $\xi$.

From equations (3.2) and (3.3), we have

$$\begin{aligned} \phi X_1 &= \phi(PX_1) + \phi(QX_1) + \phi(RX_1), \\ &= \psi(PX_1) + \omega(PX_1) + \psi(QX_1) + \omega(QX_1) + \psi(RX_1) + \omega(RX_1). \end{aligned}$$

Since $\phi D = D$, we get $\omega PX_1 = 0$.

Hence above equation reduces to

$$\phi X_1 = \psi(PX_1) + \psi QX_1 + \omega QX_1 + \psi RX_1 + \omega RX_1. \tag{3.5}$$

Thus we have the following decomposition

$$\phi(\ker F_*) = D \oplus (\psi D_1 \oplus \psi D_2) \oplus (\omega D_1 \oplus \omega D_2), \tag{3.6}$$

where $\oplus$ denotes orthogonal direct sum.

Further, let $X_1 \in \Gamma(D_1)$ and $X_2 \in \Gamma(D_2)$. Then, $g_1(X_1, X_2) = 0$.

From definition 3.1(*iii*), we have

$$g_1(\phi X_1, X_2) = -g_1(X_1, \phi X_2) = 0.$$

Now, consider

$$g_1(\psi X_1, X_2) = g_1(\phi X_1 - \omega X_1, X_2) = g_1(\phi X_1, X_2) = 0.$$

Similarly, we get $g_1(X_1, \psi X_2) = 0$.

Let $V_1 \in \Gamma(D)$ and $V_2 \in \Gamma(D_1)$. Then we have

$$g_1(\psi V_1, V_2) = g_1(\phi V_1 - \omega V_1, V_2) = g_1(\phi V_1, V_2) = -g_1(V_1, \phi V_2) = 0,$$

as $D$ is invariant, i.e., $\phi V_1 \in \Gamma(D)$.

Similarly, for $Z_1 \in \Gamma(D)$ and $Z_2 \in \Gamma(D_2)$, we obtain

$$g_1(\psi Z_2, Z_1) = 0.$$

From above equations, we have

$$g_1(\psi Y_1, \psi Y_2) = 0, g_1(\omega Y_1, \omega Y_2) = 0,$$

for all $Y_1 \in \Gamma(D_1)$ and $Y_2 \in \Gamma(D_2)$.

So, we can write $\psi D_1 \cap \psi D_2 = \{0\}$, $\omega D_1 \cap \omega D_2 = \{0\}$.

If $\theta_2 = \frac{\pi}{2}$, then $\psi R = 0$ and $D_2$ is anti-invariant, i.e., $\phi(D_2) \subseteq (\ker F_*)^\perp$. In this case we denote $D_2$ by $D^\perp$.

We also have

$$\phi(\ker F_*) = D \oplus \psi D_1 \oplus \omega D_1 \oplus \phi D^\perp. \tag{3.7}$$

Since $\omega D_1 \subseteq (\ker F_*)^\perp$, $\omega D_2 \subseteq (\ker F_*)^\perp$. So we can write

$$(\ker F_*)^\perp = \omega D_1 \oplus \omega D_2 \oplus \mathcal{V},$$

where $\mathcal{V}$ is orthogonal complement of $(\omega D_1 \oplus \omega D_2)$ in $(\ker F_*)^\perp$.

For any non-zero vector field $X_1 \in \Gamma(\ker_* F)^\perp$, we have

$$\phi X_1 = BX_1 + CX_1, \tag{3.8}$$

where $BX_1 \in \Gamma(\ker F)$ and $CX_1 \in \Gamma(\mathcal{V})$.

**Lemma 3.1.** *Let $F$ be a quasi bi-slant $\xi^\perp-$Riemannian submersion from an almost contact metric manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then we have*

$$\psi^2 Z_1 + B\omega Z_1 = -Z_1, \omega\psi Z_1 + C\omega Z_1 = 0,$$
$$\omega B Z_2 + C^2 Z_2 = -Z_2 + \eta(Z_2)\xi, \psi B Z_2 + BCZ_2 = 0,$$

for all $Z_1 \in \Gamma(\ker F_*)$ and $Z_2 \in \Gamma(\ker F_*)^\perp$.

*Proof.* The proof follows from a straightforward calculation using equations (2.2), (2.4), (3.3) and (3.8). □

The proof of the following result is the same as theorem [25] therefore, we omit its proof.

**Lemma 3.2.** *Let $F$ be a quasi bi-slant $\xi^\perp-$Riemannian submersion from an almost contact metric manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then we have*

*(i)* $\psi^2 X_1 = -(\cos^2\theta_i)X_1$
*(ii)* $g_1(\psi X_1, \psi X_2) = \cos^2\theta_i g_1(X_1, X_2)$,
*(iii)* $g_1(\omega X_1, \omega X_2) = \sin^2\theta_i g_1(X_1, X_2)$,
for all $X_1, X_2 \in \Gamma(D_i)$, where $i = 1, 2$.

**Lemma 3.3.** *Let $F$ be a quasi bi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then we have*

$$\mathcal{V}\nabla_{X_1}\psi X_2 + \mathcal{T}_{X_1}\omega X_2 = B\mathcal{T}_{X_1}X_2 + \psi\mathcal{V}\nabla_{X_1}X_2, \tag{3.9}$$

$$\mathcal{T}_{X_1}\psi X_2 + \mathcal{H}\nabla_{X_1}\omega X_2 - g_1(X_1, X_2)\xi = C\mathcal{T}_{X_1}X_2 + \omega\mathcal{V}\nabla_{X_1}X_2, \tag{3.10}$$

$$\mathcal{V}\nabla_{X_1}BZ_1 + \mathcal{T}_{X_1}CZ_1 + \eta(Z_1)X_1 = \psi\mathcal{T}_{X_1}Z_1 + B\mathcal{H}\nabla_{X_1}Z_1, \tag{3.11}$$

$$\mathcal{T}_{X_1}BZ_1 + \mathcal{H}\nabla_{X_1}CZ_1 = \omega\mathcal{T}_{X_1}Z_1 + C\mathcal{H}\nabla_{X_1}Z_1, \tag{3.12}$$

$$\mathcal{V}\nabla_{Z_1}\psi X_1 + \mathcal{A}_{Z_1}\omega X_1 = B\mathcal{A}_{Z_1}X_1 + \psi\mathcal{V}\nabla_{Z_1}X_1, \tag{3.13}$$

$$\mathcal{A}_{Z_1}\psi X_1 + \mathcal{H}\nabla_{Z_1}\omega X_1 = \omega\mathcal{V}\nabla_{Z_1}X_1 + C\mathcal{A}_{Z_1}X_1, \tag{3.14}$$

$$\mathcal{V}\nabla_{Z_1}BZ_2 + \mathcal{A}_{Z_1}CZ_2 = \psi\mathcal{A}_{Z_1}Z_2 + B\mathcal{H}\nabla_{Z_1}Z_2, \tag{3.15}$$

$$\mathcal{A}_{Z_1}BZ_2 + \mathcal{H}\nabla_{Z_1}CZ_2 - g_1(Z_1, Z_2)\xi + \eta(Z_2)Z_1 = \omega\mathcal{A}_{Z_1}Z_2 + C\mathcal{H}\nabla_{Z_1}Z_2, \tag{3.16}$$

for all $Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}$ and $X_1, X_2 \in \Gamma(\ker F_*)$.

*Proof.* Using equations $(2.4), (2.5), (2.9), (2.10), (2.11), (2.12), (3.3)$ and $(3.8)$, we get the above lemma completely . □

Now, we define

$$(\nabla_{X_1}\psi)X_2 = \mathcal{V}\nabla_{X_1}\psi X_2 - \psi\mathcal{V}\nabla_{X_1}X_2, \tag{3.17}$$

$$(\nabla_{X_1}\omega)X_2 = \mathcal{H}\nabla_{X_1}\omega X_2 - \omega\mathcal{V}\nabla_{X_1}X_2, \tag{3.18}$$

$$(\nabla_{Z_1}C)Z_2 = \mathcal{H}\nabla_{Z_1}CZ_2 - C\mathcal{H}\nabla_{Z_1}Z_2, \tag{3.19}$$

$$(\nabla_{Z_1}B)Z_2 = \mathcal{V}\nabla_{Z_1}BZ_2 - B\mathcal{H}\nabla_{Z_1}Z_2, \tag{3.20}$$

for any $X_1, X_1 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}$.

**Lemma 3.4.** *Let $F$ be a quasi bi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then we have*

$$(\nabla_{X_1}\psi)X_2 = B\mathcal{T}_{X_1}X_2 - \mathcal{T}_{X_1}\omega X_2,$$

$$(\nabla_{X_1}\omega)X_2 = C\mathcal{T}_{X_1}X_2 - \mathcal{T}_{X_1}\psi X_2 + g_1(X_1, X_2)\xi,$$

$$(\nabla_{Z_1}C)Z_2 = \omega\mathcal{A}_{Z_1}Z_2 - \mathcal{A}_{Z_1}BZ_2 + g_1(Z_1, Z_2)\xi + \eta(Z_2)Z_1,$$

$$(\nabla_{Z_1}B)Z_2 = \psi\mathcal{A}_{Z_1}Z_2 - \mathcal{A}_{Z_1}CZ_2,$$

*for any vectors $X_1, X_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}$.*

*Proof.* The proof of the above lemma is straightforward, so we omit its proof. □

If the tensors $\phi$ and $\omega$ are parallel with respect to the linear connection $\nabla$ on $M_1$ respectively, then

$$B\mathcal{T}_{X_1}X_2 = \mathcal{T}_{X_1}\omega X_2,$$

and

$$C\mathcal{T}_{X_1}X_2 = \mathcal{T}_{X_1}\psi X_2 - g_1(X_1, X_2)\xi,$$

for any $X_1, X_2 \in \Gamma(TM_1)$.

**Theorem 3.5.** *Let $F$ be a proper quasi bi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then the invariant distribution $D$ is integrable if and only if*

$$\begin{aligned}(3.21)\qquad & g_1(\mathcal{T}_{X_2}\phi X_1 - \mathcal{T}_{X_1}\phi X_2, \omega QZ_1 + \omega RZ_1) \\ = \; & g_1(\mathcal{V}\nabla_{X_1}\phi X_2 - \mathcal{V}\nabla_{X_2}\phi X_1, \psi QZ_1 + \psi RZ_1),\end{aligned}$$

*for all $X_1, X_2 \in \Gamma(D)$ and $Z_1 \in \Gamma(D_1 \oplus D_2)$.*

*Proof.* For all $X_1, X_2 \in \Gamma(D)$, $Z_1 \in \Gamma(D_1 \oplus D_2)$, and $Z_2 \in (\ker F_*)^{\perp}$, since $[X_1, X_2] \in (\ker F_*)$, we have $g_1([X_1, X_2], Z_2) = 0$. Thus $D$ is integrable $\Leftrightarrow g_1([X_1, X_2], Z_1) = 0$. Now, using equations (2.1), (2.3), (2.4), (2.5), (2.9), (3.2) and (3.3), we obtain

$$\begin{aligned} & g_1([X_1, X_2], Z_1) \\ = \; & g_1(\phi\nabla_{X_1}X_2, \phi Z_1) - g_1(\phi\nabla_{X_2}X_1, \phi Z_1), \\ = \; & g_1(\nabla_{X_1}\phi X_2, \phi Z_1) - g_1(\nabla_{X_2}\phi X_1, \phi Z_1), \\ = \; & g_1(\mathcal{T}_{X_1}\phi X_2 - \mathcal{T}_{X_2}\phi X_1, \omega QZ_1 + \omega RZ_1) \\ & + g_1(\mathcal{V}\nabla_{X_1}\phi X_2 - \mathcal{V}\nabla_{X_2}\phi X_1, \psi QZ_1 + \psi RZ_1),\end{aligned}$$

which completes the assertion. □

**Theorem 3.6.** *Let $F$ be a proper quasi bi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then the distribution $D_1$ is integrable if and only if*

$$\begin{aligned}(3.22)\qquad & g_1(\mathcal{T}_{Z_1}\omega\psi Z_2 - \mathcal{T}_{Z_2}\omega\psi Z_1, V_1) \\ = \; & g_1(\mathcal{T}_{Z_1}\omega Z_2 - \mathcal{T}_{Z_2}\omega Z_1, \phi PV_1 + \psi RV_1) + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2 - \mathcal{H}\nabla_{Z_2}\omega Z_1, \omega RV_1),\end{aligned}$$

for all $Z_1, Z_2 \in \Gamma(D_1)$ and $V_1 \in \Gamma(D \oplus D_2)$.

*Proof.* For all $Z_1, Z_2 \in \Gamma(D_1)$, $V_1 \in \Gamma(D \oplus D_2)$ and $V_2 \in (\ker F_*)^{\perp}$, since $[X_1, X_2] \in (\ker F_*)$, we have $g_1([X_1, X_2], V_2) = 0$. Thus $D_1$ is integrable $\Leftrightarrow g_1([X_1, X_2], V_1) = 0$. Using equations (2.3), (2.4), (2.5), (2.6), (2.9), (2.10), (3.2), (3.3) and lemma 3.2, we derive

$$\begin{aligned}
& g_1([Z_1, Z_2], V_1) \\
= \;& g_1(\nabla_{Z_1}\phi Z_2, \phi V_1) - g_1(\nabla_{Z_2}\phi Z_1, \phi V_1), \\
= \;& g_1(\nabla_{Z_1}\psi Z_2, \phi V_1) + g_1(\nabla_{Z_1}\omega Z_2, \phi V_1) - g_1(\nabla_{Z_2}\psi Z_1, \phi V_1) - g_1(\nabla_{Z_2}\omega Z_1, \phi V_1), \\
= \;& \cos^2\theta_1 g_1(\nabla_{Z_1} Z_2, V_1) - \cos^2\theta_1 g_1(\nabla_{Z_2} Z_1, V_1) - g_1(\mathcal{T}_{Z_1}\omega\psi Z_2 - \mathcal{T}_{Z_2}\omega\psi Z_1, V_1) \\
& + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2 + \mathcal{T}_{Z_1}\omega Z_2, \phi P V_1 + \psi R V_1 + \omega R V_1) \\
& - g_1(\mathcal{H}\nabla_{Z_2}\omega Z_1 + \mathcal{T}_{Z_2}\omega Z_1, \phi P V_1 + \psi R V_1 + \omega R V_1).
\end{aligned}$$

Thus we have

$$\begin{aligned}
& \sin^2\theta_1 g_1([Z_1, Z_2], V_1) \\
= \;& g_1(\mathcal{T}_{Z_1}\omega Z_2 - \mathcal{T}_{Z_2}\omega Z_1, \phi P V_1 + \psi R V_1) + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2 - \mathcal{H}\nabla_{Z_2}\omega Z_1, \omega R V_1) \\
& - g_1(\mathcal{T}_{Z_1}\omega\psi Z_2 - \mathcal{T}_{Z_2}\omega\psi Z_1, V_1),
\end{aligned}$$

which proves the assertion. □

The proof of the following theorem is similar to the one given in theorem 3.6.

**Theorem 3.7.** *Let $F$ be a proper quasi bi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then the distribution $D_2$ is integrable if and only if*

$$\begin{aligned}
& g_1(\mathcal{T}_{Z_1}\omega\psi Z_2 - \mathcal{T}_{Z_2}\omega\psi Z_1, W_1) \\
= \;& g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2 - \mathcal{H}\nabla_{Z_2}\omega Z_1, \omega Q W_1) \\
& + g_1(\mathcal{T}_{Z_1}\omega Z_2 - \mathcal{T}_{Z_2}\omega Z_1, \phi P W_1 + \psi Q W_1),
\end{aligned}$$

*for all $Z_1, Z_2 \in \Gamma(D_2)$ and $W_1 \in \Gamma(D \oplus D_1)$.*

**Theorem 3.8.** *Let $F$ be a proper quasi bi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then the distribution $(\ker F_*)^{\perp}$ is integrable if and only if*

$$\begin{aligned}
& g_1(\mathcal{V}\nabla_{X_1} B X_2 - \mathcal{V}\nabla_{X_2} B X_1, \phi Z_1) \\
= \;& -g_2(F_*(C X_2), (\nabla F_*)(X_1, \phi Z_1)) + g_2(F_*(C X_1), (\nabla F_*)(X_2, \phi Z_1)),
\end{aligned}$$

$$\begin{aligned}
& g_1(\mathcal{A}_{X_1} B X_2 - \mathcal{A}_{X_2} B X_1, \omega Q Z_2) \\
= \;& g_2((\nabla F_*)(X_1, C X_2), F_*(\omega Q Z_2)) - g_2((\nabla F_*)(X_2, C X_1), F_*(\omega Q Z_2)) \\
& - \eta(X_2) g_1(X_1, \omega Q Z_2) + \eta(X_1) g_1(X_2, \omega Q Z_2),
\end{aligned}$$

$$\begin{aligned}
& g_1(\mathcal{A}_{X_1} B X_2 - \mathcal{A}_{X_2} B X_1, \omega Q Z_3) \\
= \;& g_2((\nabla F_*)(X_1, C X_2), F_*(\omega Q Z_3)) - g_2((\nabla F_*)(X_2, C X_1), F_*(\omega Q Z_3)) \\
& - \eta(X_2) g_1(X_1, \omega Q Z_3) + \eta(X_1) g_1(X_2, \omega Q Z_3),
\end{aligned}$$

*for all $X_1, X_2 \in \Gamma(\ker F_*)^{\perp}$, $Z_1 \in \Gamma(D)$, $Z_2 \in \Gamma(D_1)$ and $Z_3 \in \Gamma(D_2)$.*

*Proof.* For $X_1, X_2 \in \Gamma(\ker F_*)^\perp$, $Z_1 \in \Gamma(D)$, $Z_2 \in \Gamma(D_1)$ and $Z_3 \in \Gamma(D_2)$ and using equations (2.3), (2.4), (2.5), (2.6), (2.11), (2.12), (3.2), (3.3), (3.8) and lemma 3.2, we have

$$\begin{aligned} & g_1([X_1, X_2], Z_1) \\ = \; & g_1(\nabla_{X_1}\phi X_2, \phi Z_1) - g_1(\nabla_{X_2}\phi X_1, \phi Z_1) \\ & +\eta(X_2)g_1(X_1, \phi Z_1) - \eta(X_1)g_1(X_2, \phi Z_1), \\ = \; & g_1(\mathcal{V}\nabla_{X_1}BX_2 - \mathcal{V}\nabla_{X_2}BX_1, \phi Z_1) \\ & -g_1(CX_2, \nabla_{X_1}\phi Z_1) + g_1(CX_1, \nabla_{X_2}\phi Z_1). \end{aligned}$$

Since $F$ is Riemannian submersion, using equation (2.14), we derive

$$\begin{aligned} & g_1([X_1, X_2], Z_1) \\ = \; & g_1(\mathcal{V}\nabla_{X_1}BX_2 - \mathcal{V}\nabla_{X_2}BX_1, \phi Z_1) \\ & +g_2(F_*(CX_2), (\nabla F_*)(X_1, \phi Z_1)) - g_2(F_*(CX_1), (\nabla F_*)(X_2, \phi Z_1)). \end{aligned}$$

Next, using equations (2.3), (2.4), (2.5), (3.2), (3.3), (3.8) and lemma 3.2, we get

$$\begin{aligned} & g_1([X_1, X_2], Z_2) \\ = \; & g_1(\phi\nabla_{X_1}X_2, \psi QZ_2) + g_1(\phi\nabla_{X_1}X_2, \omega QZ_2) \\ & -g_1(\phi\nabla_{X_2}X_1, \psi QZ_2) - g_1(\phi\nabla_{X_2}X_1, \omega QZ_2), \\ = \; & \cos^2\theta_1 g_1([X_1, X_2], Z_2) - g_1(\nabla_{X_1}X_2, \omega\psi QZ_2) + g_1(\nabla_{X_2}X_1, \omega\psi QZ_2) \\ & +g_1(\nabla_{X_1}BX_2, \omega QZ_2) + g_1(\nabla_{X_1}CX_2, \omega QZ_2) - g_1(\nabla_{X_2}BX_1, \omega QZ_2) \\ & -g_1(\nabla_{X_2}CX_1, \omega QZ_2) + \eta(X_2)g_1(X_1, \omega QZ_2) - \eta(X_1)g_1(X_2, \omega QZ_2). \end{aligned}$$

Since $F$ is a Riemannian submersion, using equations (2.11) and (2.14), we obtain

$$\begin{aligned} & \sin^2\theta_1 g_1([X_1, X_2], Z_2) \\ = \; & g_1(\mathcal{A}_{X_1}BX_2 - \mathcal{A}_{X_2}BX_1, \omega QZ_2) - g_2((\nabla F_*)(X_1, CX_2), F_*(\omega QZ_2)) \\ & +g_2((\nabla F_*)(X_2, CX_1), F_*(\omega QZ_2)) \\ & +\eta(X_2)g_1(X_1, \omega QZ_2) - \eta(X_1)g_1(X_2, \omega QZ_2). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} & \sin^2\theta_2 g_1([X_1, X_2], Z_3) \\ = \; & g_1(\mathcal{A}_{X_1}BX_2 - \mathcal{A}_{X_2}BX_1, \omega QZ_3) - g_2((\nabla F_*)(X_1, CX_2), F_*(\omega QZ_3)) \\ & +g_2((\nabla F_*)(X_2, CX_1), F_*(\omega QZ_3)) \\ & +\eta(X_2)g_1(X_1, \omega QZ_3) - \eta(X_1)g_1(X_2, \omega QZ_3), \end{aligned}$$

which shows the assertion. □

**Theorem 3.9.** *Let $F$ be a proper quasi bi-slant $\xi^\perp-$Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then the distribution $(\ker F_*)^\perp$ is totally geodesic foliation on $M_1$ if and only if*

$$\begin{aligned} & g_1(\mathcal{V}\nabla_{X_1}\psi BX_2 + \mathcal{A}_{X_1}\omega BX_2, Z_1) \\ = \; & g_2(F_*(CX_2), (\nabla F_*)(X_1, \phi Z_1)), \end{aligned}$$

$$\begin{aligned}
& g_1(\mathcal{A}_{X_1}BX_2,\omega QZ_2) \\
= \ & g_2((\nabla F_*)(X_1,CX_2),F_*(\omega QZ_2)) \\
& -g_2((\nabla F_*)(X_1,X_2),F_*(\omega\psi QZ_2))-\eta(X_2)g_1(X_1,\omega QZ_2),
\end{aligned}$$

$$\begin{aligned}
& g_1(\mathcal{A}_{X_1}BX_2,\omega RZ_3) \\
= \ & g_2((\nabla F_*)(X_1,CX_2),F_*(\omega RZ_3)) \\
& -g_2((\nabla F_*)(X_1,X_2),F_*(\omega\psi RZ_3))-\eta(X_2)g_1(X_1,\omega RZ_3),
\end{aligned}$$

*for all* $X_1,X_2\in\Gamma(\ker F_*)^{\perp}$, $Z_1\in\Gamma(D)$, $Z_2\in\Gamma(D_1)$ *and* $Z_3\in\Gamma(D_2)$.

*Proof.* For all $X_1,X_2\in\Gamma(\ker F_*)^{\perp}$, $Z_1\in\Gamma(D)$, $Z_2\in\Gamma(D_1)$ and $Z_3\in\Gamma(D_2)$, using equations $(2.3),(2.4),(2.5),(2.6),(3.3)$ and $(3.8)$, we have

$$\begin{aligned}
& g_1(\nabla_{X_1}X_2,Z_1) \\
= \ & g_1(\nabla_{X_1}\phi X_2,\phi Z_1) \\
= \ & -g_1(\phi\nabla_{X_1}BX_2,Z_1)-g_1(CX_2,\nabla_{X_1}\phi Z_1) \\
= \ & -g_1(\nabla_{X_1}\psi BX_2,Z_1)-g_1(\nabla_{X_1}\omega BX_2,Z_1)-g_1(CX_2,\nabla_{X_1}\phi Z_1).
\end{aligned}$$

Since $F$ is Riemannian submersion, using equations $(2.11),(2.12)$ and $(2.14)$, we obtain

$$\begin{aligned}
& g_1(\nabla_{X_1}X_2,Z_1) \\
= \ & -g_1(\mathcal{V}\nabla_{X_1}\psi BX_2+\mathcal{A}_{X_1}\omega BX_2,Z_1) \\
& +g_2((\nabla F_*)(X_1,\phi Z_1),F_*(CX_2)).
\end{aligned}$$

Next, using equations $(2.2),(2.3),(2.4),(2.5),(2.6),(3.2),(3.3),(3.8)$ and lemma 3.2, we obtain

$$\begin{aligned}
& g_1(\nabla_{X_1}X_2,Z_2) \\
= \ & g_1(\phi\nabla_{X_1}X_2,\psi QZ_2+\omega QZ_2), \\
= \ & g_1(\nabla_{X_1}BX_2,\omega QZ_2)+g_1(\nabla_{X_1}CX_2,\omega QZ_2) \\
& +\cos^2\theta_1 g_1(\nabla_{X_1}X_2,QZ_2)-g_1(\nabla_{X_1}X_2,\omega\psi QZ_2) \\
& +\eta(X_2)g_1(X_1,\omega QZ_2).
\end{aligned}$$

Since $F$ is a Riemannian submersion, using equations $(2.11)$ and $(2.14)$, we get

$$\begin{aligned}
& \sin^2\theta_1 g_1(\nabla_{X_1}X_2,Z_2) \\
= \ & g_1(\mathcal{A}_{X_1}BX_2,\omega QZ_2)-g_2((\nabla F_*)(X_1,CX_2),F_*(\omega QZ_2)) \\
& +g_2((\nabla F_*)(X_1,X_2),F_*(\omega\psi QZ_2))+\eta(X_2)g_1(X_1,\omega QZ_2).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
& \sin^2\theta_2 g_1(\nabla_{X_1}X_2,Z_3) \\
= \ & g_1(\mathcal{A}_{X_1}BX_2,\omega RZ_3)-g_2((\nabla F_*)(X_1,CX_2),F_*(\omega RZ_3)) \\
& +g_2((\nabla F_*)(X_1,X_2),F_*(\omega\psi RZ_3))+\eta(X_2)g_1(X_1,\omega RZ_3),
\end{aligned}$$

which completes the proof. □

**Theorem 3.10.** *Let $F$ be a proper quasi bi-slant $\xi^\perp-$Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then the distribution $(\ker F_*)$ is totally geodesic foliation on $M_1$ if and only if*

$$C\mathcal{T}_{X_1}\psi X_2 + \omega\mathcal{V}\nabla_{X_1}\psi X_2 + \omega\mathcal{T}_{X_1}\omega X_2 + C\mathcal{H}\nabla_{X_1}\omega X_2 + \eta(X_2)\omega X_1 - g_1(X_2, \phi X_1)\xi = 0,$$

*for all $X_1, X_2 \in \Gamma(\ker F_*)$.*

*Proof.* For all $X_1, X_2 \in \Gamma(\ker F_*)$, using equations $(2.1), (2.3), (2.4), (2.5), (2.6), (2.9), (2.10), (3.3)$ and $(2.8)$, we derive

$$\begin{aligned}
\nabla_{X_1}X_2 &= -(-\nabla_{X_1}X_2),\\
&= -\phi(\nabla_{X_1}\psi X_2 + \nabla_{X_1}\omega X_2) - \eta(X_2)\phi X_1 + g_1(X_2, \phi X_1)\xi,\\
&= -\phi(\mathcal{T}_{X_1}\psi X_2 + \mathcal{V}\nabla_{X_1}\psi X_2 + \mathcal{T}_{X_1}\omega X_2 + \mathcal{H}\nabla_{X_1}\omega X_2)\\
&\quad -\eta(X_2)\phi X_1 + g_1(X_2, \phi X_1)\xi,\\
&= -B\mathcal{T}_{X_1}\psi X_2 - C\mathcal{T}_{X_1}\psi X_2 - \psi\mathcal{V}\nabla_{X_1}\psi X_2 - \omega\mathcal{V}\nabla_{X_1}\psi X_2\\
&\quad -\psi\mathcal{T}_{X_1}\omega X_2 - \omega\mathcal{T}_{X_1}\omega X_2 - B\mathcal{H}\nabla_{X_1}\omega X_2 - C\mathcal{H}\nabla_{X_1}\omega X_2\\
&\quad -\eta(X_2)\psi X_1 - \eta(X_2)\omega X_1 + g_1(X_2, \phi X_1)\xi.
\end{aligned}$$

$\ker F_*$ defines a totally geodesic foliation on $M_1$ if and only if the horizontal part of the last equation vanishes. Which completes the proof. □

**Theorem 3.11.** *Let $F$ be a proper quasi bi-slant $\xi^\perp-$Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then the distribution $D$ is parallel if and only if*

$$g_1(\mathcal{T}_{X_1}\phi PX_2, \omega QZ_1 + \omega RZ_1) = -g_1(\mathcal{V}\nabla_{X_1}\phi PX_2, \psi Z_1 + \psi RZ_1), \tag{3.23}$$

*and*

$$g_1(\mathcal{T}_{X_1}\phi PX_2, CZ_2) = -g_1(\mathcal{V}\nabla_{X_1}\phi PX_2, BZ_2) - \eta(Z_2)g_1(X_2, \phi PX_1), \tag{3.24}$$

*for all $X_1, X_2 \in \Gamma(D)$, $Z_1 \in \Gamma(D_1 \oplus D_2)$ and $Z_2 \in \Gamma(\ker F_*)^\perp$.*

*Proof.* For all $X_1, X_2 \in \Gamma(D)$, $Z_1 \in \Gamma(D_1 \oplus D_2)$ and $Z_2 \in \Gamma(\ker F_*)^\perp$, using equations $(2.2) - (2.6), (2.9), (3.2)$ and $(3.8)$, we get

$$\begin{aligned}
&g_1(\nabla_{X_1}X_2, Z_1)\\
&= g_1(\nabla_{X_1}\phi X_2, \phi Z_1),\\
&= g_1(\nabla_{X_1}\phi PX_2, \phi QZ_1 + \phi RZ_1),\\
&= g_1(\mathcal{T}_{X_1}\phi PX_2, \omega QZ_1 + \omega RZ_1) + g_1(\mathcal{V}\nabla_{X_1}\phi PX_2, \psi QZ_1 + \psi RZ_1).
\end{aligned}$$

Again using equations $(2.2) - (2.6), (2.9), (3.2)$ and $(3.8)$, we derive

$$\begin{aligned}
g_1(\nabla_{X_1}X_2, Z_2) &= g_1(\nabla_{X_1}\phi X_2, \phi Z_2),\\
&= g_1(\nabla_{X_1}\phi PX_2, BZ_2 + CZ_2),\\
&= g_1(\mathcal{V}\nabla_{X_1}\phi PX_2, BZ_2) + g_1(\mathcal{T}_{X_1}\phi PX_2, CZ_2)\\
&\quad +\eta(Z_2)g_1(X_2, \phi PX_1),
\end{aligned}$$

which proves the assertion. □

**Theorem 3.12.** *Let $F$ be a proper quasi bi-slant $\xi^\perp-$Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then the distribution $D_1$ is parallel if and only if*

$$g_1(\mathcal{T}_{Z_1}\omega\psi Z_2, X_1) = g_1(\mathcal{T}_{Z_1}\omega Z_2, \phi PX_1 + \psi RX_1) + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2, \omega RX_1), \tag{3.25}$$

*and*

$$\begin{aligned} & g_1(\mathcal{H}\nabla_{Z_1}\omega\psi Z_2, X_2) \\ = \; & g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2, CX_2) + g_1(\mathcal{T}_{Z_1}\omega Z_2, BX_2) + \eta(X_2)g_1(Z_2, \psi Z_1), \end{aligned} \tag{3.26}$$

*for all $Z_1, Z_2 \in \Gamma(D_1), X_1 \in \Gamma(D \oplus D_2)$ and $X_2 \in \Gamma(\ker F_*)^\perp$.*

*Proof.* For all $Z_1, Z_2 \in \Gamma(D_1)$, $X_1 \in \Gamma(D \oplus D_2)$ and $X_2 \in \Gamma(\ker F_*)^\perp$, using equations (2.3), (2.4), (2.5), (2.6), (2.9), (2.10), (3.2), (3.3) and lemma 3.2, we get

$$\begin{aligned} & g_1(\nabla_{Z_1} Z_2, X_1) \\ = \; & g_1(\nabla_{Z_1}\phi Z_2, \phi X_1) \\ = \; & g_1(\nabla_{Z_1}\psi Z_2, \phi X_1) + g_1(\nabla_{Z_1}\omega Z_2, \phi X_1), \\ = \; & \cos^2\theta_1 g_1(\nabla_{Z_1} Z_2, X_1) - g_1(\mathcal{T}_{Z_1}\omega\psi Z_2, X_1) \\ & + g_1(\mathcal{T}_{Z_1}\omega Z_2, \phi PX_1 + \psi RX_1) + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2, \omega RX_1). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta_1 g_1(\nabla_{Z_1} Z_2, X_1) \\ = \; & -g_1(\mathcal{T}_{Z_1}\omega\psi Z_2, X_1) + g_1(\mathcal{T}_{Z_1}\omega Z_2, \phi PX_1 + \psi RX_1) \\ & + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2, \omega RX_1) \end{aligned}$$

Next, using equations (2.3), (2.4), (2.5), (2.6), (2.10), (3.2), (3.3), (3.8) and lemma 3.3, we obtain

$$\begin{aligned} & g_1(\nabla_{Z_1} Z_2, X_2) \\ = \; & g_1(\nabla_{Z_1}\phi Z_2, \phi X_2), \\ = \; & g_1(\nabla_{Z_1}\psi Z_2, \phi X_2) + g_1(\nabla_{Z_1}\omega Z_2, \phi X_2), \\ = \; & \cos^2\theta_1 g_1(\nabla_{Z_1} Z_2, X_2) - g_1(\mathcal{H}\nabla_{Z_1}\omega\psi Z_2, X_2) \\ & + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2, CX_2) + g_1(\mathcal{T}_{Z_1}\omega Z_2, BX_2) \\ & + \eta(X_2)g_1(Z_2, \psi Z_1). \end{aligned}$$

Thus we have

$$\begin{aligned} & \sin^2\theta_1 g_1(\nabla_{Z_1} Z_2, X_2) \\ = \; & -g_1(\mathcal{H}\nabla_{Z_1}\omega\psi Z_2, X_2) + g_1(\mathcal{H}\nabla_{Z_1}\omega Z_2, CX_2) \\ & + g_1(\mathcal{T}_{Z_1}\omega Z_2, BX_2) + \eta(X_2)g_1(Z_2, \psi Z_1), \end{aligned}$$

which shows our assertion. □

Analogous to the above theorem, we obtain the following theorem:

**Theorem 3.13.** *Let $F$ be a proper quasi bi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then the distribution $D_2$ is parallel if and only if*

$$g_1(\mathcal{T}_{X_1}\omega\psi X_2, Z_1) = g_1(\mathcal{T}_{X_1}\omega X_2, \phi PZ_1 + \phi RZ_1) + g_1(\mathcal{H}\nabla_{X_1}\omega X_2, \omega RZ_1), \tag{3.27}$$

*and*

$$\begin{aligned} & g_1(\mathcal{H}\nabla_{X_1}\omega\psi X_2, Z_2) \\ = \;& g_1(\mathcal{H}\nabla_{X_1}\omega X_2, CZ_2) + g_1(\mathcal{T}_{X_1}\omega X_2, BZ_2) + \eta(Z_2)g_1(X_2, \psi X_1), \end{aligned} \tag{3.28}$$

*for all $X_1, X_2 \in \Gamma(D_2)$, $Z_1 \in \Gamma(D \oplus D_1)$ and $Z_2 \in \Gamma(\ker F_*)^{\perp}$.*

**Theorem 3.14.** *Let $F$ be a proper quasi bi-slant $\xi^{\perp}-$Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(M_2, g_2)$. Then, $F$ is a totally geodesic submersion if and only if*

$$\begin{aligned} & g_1(\mathcal{T}_{V_1}PV_2 + \cos^2\theta_1\mathcal{T}_{V_1}QV_2 + \cos^2\theta_2\mathcal{T}_{V_1}RV_2, Z_1) \\ = \;& g_1(\mathcal{H}\nabla_{V_1}\omega\psi PV_2 + \mathcal{H}\nabla_{V_1}\omega\psi QV_2 + \mathcal{H}\nabla_{V_1}\omega\psi RV_2, Z_1) \\ & -g_1(\mathcal{T}_{V_1}\omega QV_2 + \mathcal{T}_{V_1}\omega RV_2, BZ_1) - g_1(\mathcal{H}\nabla_{V_1}\omega QV_2 + \mathcal{H}\nabla_{V_1}\omega RV_2, CZ_1), \end{aligned}$$

*and*

$$\begin{aligned} & g_1(\mathcal{A}_{Z_1}PV_1 + \cos^2\theta_1\mathcal{A}_{Z_1}QV_1 + \cos^2\theta_2\mathcal{A}_{Z_1}RV_1, Z_2) \\ = \;& g_1(\mathcal{H}\nabla_{Z_1}\omega\psi PV_1 + \mathcal{H}\nabla_{Z_1}\omega\psi QV_1 + \mathcal{H}\nabla_{Z_1}\omega\psi RV_1, Z_2) - \eta(Z_2)g_1(V_1, BZ_1) \\ & -g_1(\mathcal{A}\nabla_{Z_1}\omega QV_1 + \mathcal{A}\nabla_{Z_1}\omega RV_1, BZ_2) - g_1(\mathcal{H}\nabla_{Z_1}\omega QV_1 + \mathcal{H}\nabla_{Z_1}\omega RV_1, CZ_2), \end{aligned}$$

*for all $V_1, V_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}$.*

*Proof.* Since $F$ is a Riemannian submersion, we have

$$(\nabla F_*)(Z_1, Z_2) = 0,$$

for all $Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}$.

For all $V_1, V_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}$, using equations (2.3), (2.4), (2.5), (2.6), (2.9), (2.10), (2.14), (3.2), (3.3) and lemma 3.2, we have

$$\begin{aligned} & g_2((\nabla F_*)(V_1, V_2), F_*Z_1) \\ = \;& -g_1(\nabla_{V_1}V_2, Z_1) \\ = \;& -g_1(\nabla_{V_1}\phi V_2, \phi Z_1) - \eta(Z_1)g_1(V_2, \phi V_1), \\ = \;& -g_1(\nabla_{V_1}\phi PV_2, \phi Z_1) - g_1(\nabla_{V_1}\phi QV_2, \phi Z_1) - g_1(\nabla_{V_1}\phi RV_2, \phi Z_1) \\ & -\eta(Z_1)g_1(V_2, \phi V_1), \\ = \;& -g_1(\nabla_{V_1}PV_2, Z_1) - \cos^2\theta_1 g_1(\nabla_{V_1}QV_2, Z_1) - \cos^2\theta_2 g_1(\nabla_{V_1}RV_2, Z_1) \\ & +g_1(\nabla_{V_1}\omega\psi PV_2, Z_1) + g_1(\nabla_{V_1}\omega\psi QV_2, Z_1) + g_1(\nabla_{V_1}\omega\psi RV_2, Z_1) \\ & -g_1(\nabla_{V_1}\omega QV_2, \phi Z_1) - g_1(\nabla_{V_1}\omega RV_2, \phi Z_1), \\ = \;& -g_1(\mathcal{T}_{V_1}PV_2 + \cos^2\theta_1\mathcal{T}_{V_1}QV_2 + \cos^2\theta_2\mathcal{T}_{V_1}RV_2, Z_1) \\ & +g_1(\mathcal{H}\nabla_{V_1}\omega\psi PV_2 + \mathcal{H}\nabla_{V_1}\omega\psi QV_2 + \mathcal{H}\nabla_{V_1}\omega\psi RV_2, Z_1) \\ & -g_1(\mathcal{T}_{V_1}\omega QV_2 + \mathcal{T}_{V_1}\omega RV_2, BZ_1) - g_1(\mathcal{H}\nabla_{V_1}\omega QV_2 + \mathcal{H}\nabla_{V_1}\omega RV_2, CZ_1). \end{aligned}$$

Next, using equations (2.3), (2.4), (2.5), (2.6), (2.11), (2.12), (2.14), (3.2), (3.3), (3.8) and lemma 3.1, we have

$$\begin{aligned}
& g_2((\nabla F_*)(Z_1,V_1),F_*Z_2)\\
= & -g_1(\nabla_{Z_1}V_1,Z_2),\\
= & -g_1(\nabla_{Z_1}\phi V_1,\phi Z_2)-\eta(Z_2)g_1(V_1,\phi Z_1),\\
= & -g_1(\nabla_{Z_1}\phi PV_1,\phi Z_2)-g_1(\nabla_{Z_1}\phi QV_1,\phi Z_2)-g_1(\nabla_{Z_1}\phi RV_1,\phi Z_2)-\eta(Z_2)g_1(V_1,\phi Z_1),\\
= & -g_1(\nabla_{Z_1}PV_1,Z_2)-\cos^2\theta_1 g_1(\nabla_{Z_1}QV_1,Z_2)-\cos^2\theta_2 g_1(\nabla_{Z_1}RV_1,Z_2)\\
& +g_1(\nabla_{Z_1}\omega\psi PV_1,Z_2)+g_1(\nabla_{Z_1}\omega\psi PV_1,Z_2)+g_1(\nabla_{Z_1}\omega\psi PV_1,Z_2)\\
& -g_1(\nabla_{Z_1}\omega QV_1,\phi Z_2)-g_1(\nabla_{Z_1}\omega RV_1,\phi Z_2)-\eta(Z_2)g_1(V_1,\phi Z_1),\\
= & -g_1(\mathcal{A}_{Z_1}PV_1+\cos^2\theta_1\mathcal{A}_{Z_1}QV_1+\cos^2\theta_2\mathcal{A}_{Z_1}RV_1,Z_2)-\eta(Z_2)g_1(V_1,BZ_1)\\
& +g_1(\mathcal{H}\nabla_{Z_1}\omega\psi PV_1+\mathcal{H}\nabla_{Z_1}\omega\psi QV_1+\mathcal{H}\nabla_{Z_1}\omega\psi RV_1,Z_2)\\
& -g_1(\mathcal{A}\nabla_{Z_1}\omega QV_1+\mathcal{A}\nabla_{Z_1}\omega RV_1,BZ_2)-g_1(\mathcal{H}\nabla_{Z_1}\omega QV_1+\mathcal{H}\nabla_{Z_1}\omega RV_1,CZ_2),
\end{aligned}$$

which proves the assertion. □

# 4 Example

Now, we construct an example of quasi bi-slant $\xi^\perp-$Riemannian submersion from Sasakian manifold.

Let $(R^{2n+1},g_{2n+1},\phi,\xi,\eta)$ denote the manifold $R^{2n+1}$ with Cartesian coordinates $(x_1,x_2,....,x_n,y_1,y_2,....,y_n,z)(i=1,...,n)$ and base field $\{E_i,E_{n+i},\xi\}$, where $E_i=2\frac{\partial}{\partial y_i},E_{n+i}=2(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial z})$ and contravariant vector field $\xi=2\frac{\partial}{\partial z}$. Define the usual Sasakian structure on $R^{2n+1}$ as:

$$\phi(\sum_{i=1}^{n}(X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i})+Z\frac{\partial}{\partial z})=\sum_{i=1}^{n}(Y_i\frac{\partial}{\partial x_i}-X_i\frac{\partial}{\partial y_i})+\sum_{i=1}^{n}Y_i\frac{\partial}{\partial z}$$

$$g_{2n+1}=\eta\otimes\eta+\frac{1}{4}(\sum_{i=1}^{n}[dx_i\otimes dx_i+dy_i\otimes dy_i]),$$

$$\eta=\frac{1}{2}(dz-\sum_{i=1}^{n}dx_i),\xi=2\frac{\partial}{\partial z},$$

where $x_1,...,x_i,y_1....,y_i,z$ are the Cartesian coordinates. It is easy to show that $(R^{2n+1},\phi,\xi,\eta,g_{2n+1})$ is a Sasakian manifold. Throughout this section we will use these notations.

**Example 4.1.** Let $R^{13}$ as well as $R^7$ have got Sasakian structures (as in example 2.1),where the $\phi-$base field $\{E_i,E_{n+i},\xi\}$ defined as $E_i=2\frac{\partial}{\partial y_i},E_{n+i}=2(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial z})$ on $R^{13}$ with structural vector field $\xi=2\frac{\partial}{\partial z}$and another $\phi-$base field $\{E'_i,E'_{n+i},\xi'\}$ defined as $E'_i=2\frac{\partial}{\partial v_i},E'_{n+i}=2(\frac{\partial}{\partial u_i}+\frac{\partial}{\partial w})$on $R^7$ with structural vector field $\xi'=2\frac{\partial}{\partial w}$.

The metric tensor field $g_{R^7}$ is defined by

$$g_{R^7} = \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

on $R^7$.

Let $F : R^{13} \to R^7$ be map defined by

$F(x_1, ..., x_6, y_1, ...., y_6, z) = (2(x_1+x_2), 2(x_3+x_4), 2(x_5+x_6), 2(y_1+y_2), 2\sqrt{2}y_3, 2\sqrt{2}(\cos\alpha y_5+\sin\alpha y_6), 2z),$
which is quasi bi-slant $\xi^{\perp}-$Riemannian submersion map such that

$$\begin{aligned} V_1 &= 2(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}) = E_1 - E_2, V_2 = 2(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}) = E_3 - E_4, \\ V_3 &= 2(\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6}) = E_5 - E_6, V_4 = 2(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}) = E_7 - E_8, \\ V_5 &= 2\sqrt{2}\frac{\partial}{\partial y_4} = \sqrt{2}E_{10}, V_6 = 2\sqrt{2}(\sin\alpha\frac{\partial}{\partial y_5} - \cos\alpha\frac{\partial}{\partial y_6}) = \sqrt{2}(\sin\alpha E_{11} - \cos\alpha E_{12}), \end{aligned}$$

and

$$\ker F_* = D \oplus D_1 \oplus D_2,$$

where

$$D =< V_1, V_4 >, D_1 =< V_2, V_5 >, D_2 =< V_3, V_6 > .$$

Also, we have

$$\begin{aligned} (\ker F_*)^{\perp} &= < H_1 = 2(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial z}) = E_1 + E_2, \\ H_2 &= 2(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + 2\frac{\partial}{\partial z}) = E_3 + E_4, H_3 = 2(\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} + 2\frac{\partial}{\partial z}) = E_5 + E_6, \\ H_4 &= \sqrt{2}(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}) = E_7 + E_8, H_5 = 2\sqrt{2}\frac{\partial}{\partial y_3} = \sqrt{2}E_9, \\ H_6 &= 2\sqrt{2}(\cos\alpha\frac{\partial}{\partial y_5} + \sin\alpha\frac{\partial}{\partial y_6}) = \sqrt{2}(\cos\alpha E_{11} + \sin\alpha E_{12}), H_7 = 2\frac{\partial}{\partial z} = \xi > \end{aligned}$$

$$\begin{aligned} F_*H_1 &= 8\frac{\partial}{\partial u_1} + 8\frac{\partial}{\partial w} = 4E_1', F_*H_2 = 8\frac{\partial}{\partial u_2} + 8\frac{\partial}{\partial w} = 4E_2', \\ F_*H_3 &= 8\frac{\partial}{\partial u_3} + 8\frac{\partial}{\partial w} = 4E_3', F_*H_4 = 8\frac{\partial}{\partial v_1} = 4E_4', F_*H_5 = 8\frac{\partial}{\partial v_2} = 4E_5' \\ F_*H_6 &= 8\frac{\partial}{\partial v_3} = 4E_6', F_*H_7 = 4\frac{\partial}{\partial w} = 2\xi' \end{aligned}$$

with slant angles $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = (\alpha + \frac{\pi}{4})$.

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