Harmonic vector fields on extended 3-dimensional Riemannian Lie groups

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Abstract. Given two Riemannian manifolds (B, g_B) and (F, g_F), we give harmonicity conditions for vector fields on the Riemannian warped product $B \times_f F$, with $f : B \longrightarrow]0, +\infty[$, using a characteristic variational condition. Then, we apply this to the case $B = I \subset \mathbb{R}$ and F is a three-dimensional connected Riemannian Lie group G equipped with a left-invariant metric, to determine harmonic vector fields on $I \times_f G$. We give examples of harmonic vector fields on G which are not left-invariant and determine harmonic vector fields on $I \times_f G$. We conclude with of vector fields on $I \times_f G$ which are harmonic maps.

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1 Introduction

One of the most studied objects in Differential Geometry is the energy functional of a map $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds of dimensions m and n, respectively, given by

$$E(\varphi) = \int_D e(\varphi) v_g.$$

where D is a compact domain of M, $e(\varphi) : M \to [0, \infty[$ the energy density of φ defined by

$$e(\varphi)(x) = \frac{1}{2}\|d\varphi_x\|^2 = \frac{1}{2}\sum_{i=1}^{m} h(d\varphi_x(e_i), d\varphi_x(e_i)),$$

for $x \in M, \{e_i\}_{i=1}^m$ an orthonormal basis of T_xM and $d\varphi_x$ the differential of the map φ at the point x ([1],[5]).

Denote by $C^\infty(M, N)$ the space of smooth maps from M to N, ∇^φ the connection of the vector bundle $\varphi^{-1}TN$ induced from the Levi-Civita connection $\bar{\nabla}$ of (N, h) and ∇ the Levi-Civita connection of (M, g).

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A map $\varphi : (M, g) \to (N, h)$ is said to be *harmonic* if it is a critical point of the energy functional $E(.; D) : C^\infty(M, N) \to \mathbb{R}$ for any compact domain D. It is well-known ([5]) that the map $\varphi : (M, g) \to (N, h)$ is harmonic if and only if

$$\tau(\varphi) = tr(\nabla d\varphi) = \sum_{i=1}^{m} \left\{ \nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla_{e_i} e_i) \right\} = 0. \tag{1.1}$$

where $\{e_i\}_{i=1}^m$ an orthonormal basis of T_xM. The equation $\tau(\varphi) = 0$ is called the *harmonic map equation*. Denote by (TM, g_S) the tangent bundle of (M, g) equipped with the Sasaki metric g_S (cf. section 2).A vector field X on M is a section of the tangent bundle, and in particular it is a map of M into TM. Its *energy* $E(V, D)$ on a compact domain D is given by

$$E(V, D) = \frac{m}{2} Vol(D) + \frac{1}{2} \int_D \|\nabla V\|^2 v_g \approx Vol(D) + E^v(V).$$

It was shown in [9] and [12] that if M is compact and a vector field X is a harmonic map from (M, g) into (TM, g_S), then X must be parallel. Critical points of the restriction of E to vector fields, with respect to variations through vector fields are called *harmonic vector fields* The corresponding critical point conditions have been determined in [14] and [16]. It should be pointed out that a harmonic vector field determines a harmonic map when an additional condition involving the curvature is satisfied ([6],[8]). The main goal of this paper is to study the harmonicity conditions for vector fields on the Riemannian warped product $M = B \times_f F$. Then we apply this to the case $B = I \subset \mathbb{R}$ and F is a three-dimensional connected Riemannian Lie group G equipped with a left-invariant metric, to determine harmonic vector fields on $I \times_f G$. We give examples of harmonic vector fields on G which are not left-invariant and determine a harmonic vector fields on $I \times_f G$. We also determine the vector fields on $I \times_f G$ which are harmonic maps.

2 Preliminaries

2.1 The tangent bundle and the unit tangent sphere bundle

Let (M, g) be an m-dimensional Riemannian manifold and ∇ the associated Levi-Civita connection. Its Riemann curvature tensor R is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all vector fields X, Y and Z on M. The tangent bundle of (M, g), denoted by TM, consists of pairs (x, u) where x is a point in M and u a tangent vector to M at x. The mapping $\pi : TM \to M : (x, u) \mapsto x$ is the natural projection from TM onto M. The tangent space $T_{(x,u)}TM$ at a point (x, u) in TM is the direct sum of the vertical subspace $\mathcal{V}_{(x,u)} = Ker(d\pi|_{(x,u)})$ and the horizontal subspace $\mathcal{H}_{(x,u)}$, with respect to the Levi-Civita connection ∇ of (M, g):

$$T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}.$$

For any vector $w \in T_xM$, there exists a unique vector $w^h \in \mathcal{H}_{(x,u)}$ at the point $(x, u) \in TM$, called the *horizontal lift* of w to (x, u), such that $d\pi(w^h) = w$ and a

unique vector $w^v \in \mathcal{V}_{(x,u)}$, the *vertical lift* of w to (x,u), such that $w^v(df) = w(f)$ for all functions f on M. Hence, every tangent vector $\bar{w} \in T_{(x,u)}TM$ can be decomposed as $\bar{w} = w_1^h + w_2^v$ for uniquely determined vectors $w_1, w_2 \in T_xM$. The *horizontal* (resp. *vertical*) *lift* of a vector field X on M to TM is the vector field X^h (resp. X^v) on TM whose value at the point (x,u) is the horizontal (respectively, vertical) lift of X_x to (x,u).

The tangent bundle TM of a Riemannian manifold (M,g) can be endowed in a natural way with a Riemannian metric g_S, the *Sasaki metric*, depending only on the Riemannian structure g of the base manifold M. It is uniquely determined by

$$g_S(X^h, Y^h) = g_S(X^v, Y^v) = g(X,Y) \circ \pi, \quad g_S(X^h, Y^v) = 0 \tag{2.1}$$

for all vector fields X and Y on M. More intuitively, the metric g_S is constructed in such a way that the vertical and horizontal subbundles are orthogonal and the bundle map $\pi : (TM, g_S) \mapsto (M,g)$ is a Riemannian submersion. We denote by $\mathfrak{X}(M)$ the set of globally defined vector fields on the base manifold (M,g). In the sequel, we concentrate on the map $V : (M,g) \to (TM, g_S)$. The tension field $\tau(V)$ of $V : (M,g) \to (TM, g_S)$ is given by [6]

$$\tau(V) = (-S(V))^h + (-\bar{\Delta}V)^v, \tag{2.2}$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field of (M,g), $S(V) = \sum_{i=1}^m R(\nabla_{e_i}V, V)e_i$ and $\bar{\Delta}V = \nabla^*\nabla V = \sum_{i=1}^m \{\nabla_{\nabla_{e_i}e_i}V - \nabla_{e_i}\nabla_{e_i}V\}$. Consequently, V defines a harmonic map from (M,g) to (TM, g_s) if and only if

$$tr[R(\nabla_{\cdot}V, V)\cdot] = 0 \quad \text{and} \quad \nabla^*\nabla V = 0. \tag{2.3}$$

A smooth vector field V is said to be a harmonic section if and only if it is a critical point of E^v where E^v is the vertical energy. The corresponding Euler-Lagrange equation is given by

$$\nabla^*\nabla V = 0.$$

2.2 Warped Products

Let (B^m, g_B) and (F^n, g_F) be Riemannian manifolds with $f : B \longrightarrow \ ]0, +\infty[$ a smooth function on B. The warped product $M = B \times_f F$ is the product manifold $B \times F$ equipped with the metric $g = \pi^*(g_B) \oplus (f \circ \pi)^2 \sigma^*(g_F)$; where $\pi : M \longrightarrow B$ and $\sigma : M \longrightarrow F$ are the usual projections. Then (B, g_B) is called the base, (F, g_F) is the fiber and f the warping function of the warped product, $\pi^{-1}(p) = \{p\} \times F$ are the fibers and $\sigma^{-1}(q) = B \times \{q\}$ the leaves. The vectors tangent to leaves are called horizontal and those tangent to the fibers vertical, hence for all $(p,q) \in M$

$$\begin{aligned} T_{(p,q)}(B \times F) &= T_{(p,q)}(\{p\} \times F) \oplus T_{(p,q)}(B \times \{q\}) \\ &= T_{(p,q)}(\{p\} \times F) \oplus T_{(p,q)}(\{p\} \times F)^{\perp} \\ &= T_{(p,q)}(B \times \{q\})^{\perp} \oplus T_{(p,q)}(B \times \{q\}). \end{aligned}$$

A vector fields X on $B \times F$ are horizontal vector if $d\pi_{(p,q)}(X) = X_p$ and $d\sigma_{(p,q)}(X) = 0$ that is $X \in T_{(p,q)}(B \times \{q\})^{\perp}$ and $d\pi_{(p,q)}(X) = X_p$.
A vector fields X on $B \times F$ are vertical vector if $d\pi_{(p,q)}(X) = 0$ and $d\sigma_{(p,q)}(X) = X_q$ that is $X \in T_{(p,q)}(\{p\} \times F)^{\perp}$ and $d\sigma_{(p,q)}(X) = X_p$.

If $X \in T_pB$ and $q \in F$ then the horizontal lift of X to (p,q) is the unique vector X^* in $T_{(p,q)}(B \times F)$ such that $d\pi_{(p,q)}(X^*) = X_p$ and $d\sigma_{(p,q)}(X) = 0$

If $X \in T_p(F)$ and $q \in F$ then the vertical lift of X to (p,q) is the unique vector X^* in $T_{(p,q)}(B \times F)$ such that $d\pi_{(p,q)}(X) = 0$ and $d\sigma_{(p,q)}(X) = X_q$

Lemma 2.1. *[13] Let (B^m, g_B) and (F^n, g_F) be Riemannian manifolds with $f : B \longrightarrow]0, +\infty[$ a smooth function on B. Let X_1, Y_1 be vector fields on B and X_2, Y_2 vector fields on F. Let $\nabla, \nabla^B, \nabla^F$ be the Levi-Civita connections of (M, g), (B, g_B) and (F, g_F) respectively, then*

1. $grad^M(f \circ \pi) = (grad^B(f), 0)$;
2. $\nabla_{(X_1,0)}(Y_1, 0) = (\nabla^B_{X_1} Y_1, 0)$;
3. $$\begin{aligned}\nabla_{(0,X_2)}(0, Y_2) &= (0, \nabla^F_{X_2} Y_2) - f g^F(X_2, Y_2) grad^B(f \circ \pi)\\ &= (0, \nabla^F_{X_2} Y_2) - \frac{1}{f} g((0, X_2), (0, Y_2)) grad^B(f \circ \pi)\end{aligned}$$
4. $\nabla_{(0,X_2)}(Y_1, 0) = \dfrac{Y_1(f)}{f}(0, X_2)$
5. $\nabla_{(X_1,0)}(0, Y_2) = \dfrac{X_1(f)}{f}(0, Y_2)$;
6. $grad^M(h \circ \sigma) = \dfrac{1}{f^2}(0, grad^F h)$, *for* $h : F \longrightarrow \mathbb{R}$.

Lemma 2.2 ([13])**.** *Let (B^m, g_B) and (F^n, g_F) be Riemannian manifolds, with $f : B \longrightarrow \mathbb{R}_+^*$ a smooth function on B, X_1, Y_1, Z_1 vector fields on B, X_2, Y_2, Z_2 vector fields on F, ∇, ∇^B the Levi Civita connections of (M, g), (B, g_B) respectively, and R, R^B, R^F the Riemannian curvature tensors on (M, g), (B, g_B) and (F, g_F) respectively, then*

1. $R((X_1, 0), (Y_1, 0))(Z_1, 0) = \left(R^B(X_1, Y_1)Z_1, 0\right)$
2. $R((X_1, 0), (Y_1, 0))(0, Z_2) = 0$
3. $R((0, X_2), (0, Y_2))(Z_1, 0) = 0$
4. $R((0, X_2), (Y_1, 0))(Z_1, 0) = -\dfrac{H^f(Y_1, Z_1)}{f}(0, X_2)$
5. $R((X_1, 0), (0, Y_2))(0, Z_2) = -f g^F(Y_2, Z_2)\left(\nabla^B_{X_1} grad^B(f), 0\right)$
6. $R\Big((0, X_2), (0, Y_2)\Big)(0, Z_2) = \Big(0, R^F(X_2, Y_2)Z_2\Big) + grad^B f(f)\Big(g^F(X_2, Z_2)(0, Y_2) - g^F(Y_2, Z_2)(0, X_2)\Big)$

where $H^f(U, V) = U(V(f)) - (\nabla^B_U V)(f)$ for $U, V \in \mathfrak{X}(F)$ is the Hessian of f.

3 Harmonic vector fields on warped products

In this section, we determine the harmonicity conditions for vector fields on the warped product $M = B \times_f F$ with $f : B \longrightarrow]0, +\infty[$.
Let $\{e'_i\}_{i=1,\dots,m}$ be an orthonormal basis of (B, g_B) and $\{e''_i\}_{i=1,\dots,n}$ an orthonormal basis of (F, g_F). Then $\{e_i\}_{i=1,,\dots,m+n}$ is an orthonormal basis of (M, g) with $e_i = (e'_i, 0)$ for $i = 1,\dots,m$ and $e_{i+m} = \frac{1}{f}(0, e''_i)$ for $i = 1,\dots,n$. Hence, for $i = 1,\dots,m$ and Using the lemma 2.1, We have

$$\begin{aligned}
\nabla_{e_i} e_i &= (\nabla^B_{e'_i} e'_i, 0) \\
\nabla_{e_i} V &= \nabla_{(e'_i,0)}(V_1, 0) + \nabla_{(e'_i,0)}(0, V_2) \\
&= \left(\nabla^B_{e'_i} V_1, 0\right) + \frac{(e'_i, 0)(f)}{f}(0, V_2).
\end{aligned}$$

Moreover $(e'_i, 0)f = g\Big((grad^B f, 0), (e'_i, 0)\Big) = g_B(grad^B f, e'_i) = e'_i(f)$. Hence, for $i = 1,\dots,m$

$$\nabla_{e_i} V = \left(\nabla^B_{e'_i} V_1, 0\right) + \frac{e'_i(f)}{f}(0, V_2).$$

so

$$\begin{aligned}
\nabla_{e_i}\nabla_{e_i} V &= \nabla_{(e'_i,0)}(\nabla^B_{e'_i} V_1, 0) + \nabla_{(e'_i,0)}\left(\frac{(e'_i, 0)(f)}{f}(0, V_2)\right) \\
&= \left(\nabla^B_{e'_i}\nabla^B_{e'_i} V_1, 0\right) + \frac{e'_i(f)}{f}\nabla_{(e'_i,0)}(0, V_2) + (e'_i, 0)\left(\frac{(e'_i, 0)f}{f}\right)(0, V_2) \\
&= \left(\nabla^B_{e'_i}\nabla^B_{e'_i} V_1, 0\right) + \frac{e'_i(f)^2}{f^2}(0, V_2) + \frac{e'_i e'_i(f)}{f}(0, V_2) - \frac{e'_i(f)^2}{f^2}(0, V_2) \\
&= \left(\nabla^B_{e'_i}\nabla^B_{e'_i} V_1, 0\right) + \frac{e'_i e'_i(f)}{f}(0, V_2) \\
\nabla_{\nabla_{e_i} e_i} V &= \nabla_{\left(\nabla^B_{e'_i} e'_i, 0\right)}(V_1, 0) + \nabla_{\left(\nabla^B_{e'_i} e'_i, 0\right)}(0, V_2) \\
&= \left(\nabla^B_{\nabla^B_{e'_i} e'_i} V_1, 0\right) + \frac{\left(\nabla^B_{e'_i} e'_i\right)(f)}{f}(0, V_2).
\end{aligned}$$

For $i = m+1,\dots,m+n$, we have

$$\begin{aligned}
\nabla_{e_i} e_i &= \frac{1}{f}\nabla_{(0,e''_i)}\frac{1}{f}(0, e''_i) \\
&= \frac{1}{f}\left(\frac{1}{f}\nabla_{(0,e''_i)}(0, e''_i) + (0, e''_i)\big(\frac{1}{f}\big)(0, e''_i)\right) \\
&= \frac{1}{f^2}\left(0, \nabla^F_{e''_i} e''_i\right) - \frac{1}{f}\left(grad^B f, 0\right)
\end{aligned}$$

because $(0, e''_i)(f) = g\left((grad^B f, 0), (0, e''_i)\right) = g_B(grad^B f, 0) + f^2 g_F(0, e''_i) = 0$.

For $i = m+1,\dots,m+n$, we compute

$$\begin{aligned}
\nabla_{e_i} V &= \frac{1}{f}\nabla_{(0,e_i'')}(V_1,V_2)\\
&= \frac{1}{f}\left(\nabla_{(0,e_i'')}(V_1,0)+\nabla_{(0,e_i'')}(0,V_2)\right)\\
&= \frac{1}{f}\left(\left(0,\nabla^F_{e_i''}V_2\right)-fg^F(V_2,e_i'')\left(grad^B f,0\right)+\frac{(V_1,0)(f)}{f}(0,e_i'')\right)\\
&= \frac{1}{f}\left(0,\nabla^F_{e_i''}V_2\right)-g^F(V_2,e_i'')\left(grad^B f,0\right)+\frac{(V_1,0)(f)}{f^2}(0,e_i'')\\
\nabla_{e_i} V &= \frac{1}{f}\left(0,\nabla^F_{e_i''}V_2\right)-g^F(V_2,e_i'')\left(grad^B f,0\right)+\frac{V_1(f)}{f^2}(0,e_i'').
\end{aligned}$$

therefore

$$\begin{aligned}
\nabla_{e_i}\nabla_{e_i} V &= \frac{1}{f}\left[\nabla_{(0,e_i'')}\left(\frac{1}{f}(0,\nabla^F_{e_i''}V_2)\right)-\nabla_{(0,e_i'')}\left(g^F(V_2,e_i'')(grad^B f,0)\right)\right.\\
&\quad\left.+\nabla_{(0,e_i'')}\left(\frac{(V_1,0)f}{f^2}(0,e_i'')\right)\right].
\end{aligned}$$

We now compute the previous terms and sum on $i = m+1,\dots,m+n$:
i.)

$$\begin{aligned}
\nabla_{(0,e_i'')}\left[\frac{1}{f}(0,\nabla^F_{e_i''}V_2)\right] &= \frac{1}{f}\nabla_{(0,e_i'')}\left(0,\nabla^F_{e_i''}V_2\right)+(0,e_i'')\left(\frac{1}{f}\right)\left(0,\nabla^F_{e_i''}V_2\right)\\
&= \frac{1}{f}\nabla_{(0,e_i'')}\left(0,\nabla^F_{e_i''}V_2\right)\\
&= \frac{1}{f}\left\{\left(0,\nabla^F_{e_i''}\nabla^F_{e_i''}V_2\right)-fg^F(e_i'',\nabla^F_{e_i''}V_2)\left(grad^B f,0\right)\right\}\\
&= \frac{1}{f}\left(0,\nabla^F_{e_i''}\nabla^F_{e_i''}V_2\right)-g^F(e_i'',\nabla^F_{e_i''}V_2)\left(grad^B f,0\right).
\end{aligned}$$

ii.)$\nabla_{(0,e_i'')}\left[g^F(V_2,e_i'')(grad^B f,0)\right]$

$$\begin{aligned}
&= g^F(V_2,e_i'')\frac{(grad^B f,0)(f)}{f}(0,e_i'')+(0,e_i'')\left(g^F(V_2,e_i'')\right)\left(grad^B f,0\right)\\
&= g^F(V_2,e_i'')\frac{(grad^B f)(f)}{f}(0,e_i'')+e_i''\left(g^F(V_2,e_i'')\right)\left(grad^B f,0\right)\\
&= \frac{grad^B f(f)}{f}(0,V_2)+e_i''\left(g^F(V_2,e_i'')\right)\left(grad^B f,0\right)
\end{aligned}$$

$$
\begin{aligned}
iii.)\nabla_{(0,e_i'')}\left[\frac{(V_1,0)(f)}{f^2}(0,e_i'')\right] &= \frac{1}{f^2}\left[\nabla_{(0,e_i'')}(V_1,0)(f)\right](0,e_i'') + \\
&\quad (0,e_i'')\left(\frac{1}{f^2}(V_1,0)(f)\right)(0,e_i'') \\
&= \frac{1}{f^2}(V_1,0)(f)\nabla_{(0,e_i'')}(0,e_i'') \\
&= \frac{1}{f^2}V_1(f)\left((0,\nabla^F_{e_i''}e_i'') - f(grad^B f,0)\right) \\
&= \frac{1}{f^2}V_1(f)(0,\nabla^F_{e_i''}e_i'') - \frac{n}{f}V_1(f)(grad^B f,0)
\end{aligned}
$$

Hence, gathering all the terms, and summing on $i = m+1,\dots,m+n$, we obtain

$$
\begin{aligned}
\nabla_{e_i}\nabla_{e_i}V &= \frac{1}{f^2}\left(0,\nabla^F_{e_i''}\nabla^F_{e_i''}V_2\right) - \frac{1}{f}e_i''(g^F(V_2,e_i''))\left(grad^B f,0\right) - \\
&\quad \frac{grad^B f(f)}{f^2}(0,V_2) + \frac{1}{f^3}V_1(f)\left(0,\nabla^F_{e_i''}e_i''\right) - \\
&\quad n\frac{V_1(f)}{f^2}\left(grad^B f,0\right) - \frac{1}{f}g^F(e_i'',\nabla^F_{e_i''}V_2)\left(grad^B f,0\right), \\
\nabla_{\nabla_{e_i}e_i}V &= \nabla\frac{1}{f^2}\left(0,\nabla^F_{e_i''}e_i''\right)^{(V_1,0)} - \nabla\frac{1}{f}(grad^B f,0)^{(V_1,0)} + \\
&\quad \nabla\frac{1}{f^2}\left(0,\nabla^F_{e_i''}e_i''\right)^{(0,V_2)} - \nabla\frac{1}{f}\left(grad^B f,0\right)^{(0,V_2)} \\
&= \frac{1}{f^3}V_1(f)\left(0,\nabla^F_{e_i''}e_i''\right) - \frac{n}{f}\left(\nabla^B_{grad^B f}V_1,0\right) - n\frac{grad^B f(f)}{f^2}(0,V_2) \\
&\quad + \frac{1}{f^2}\left(0,\nabla^F_{\nabla^F_{e_i''}e_i''}V_2\right) - \frac{1}{f}g^F\left(V_2,\nabla^F_{e_i''}e_i''\right)(grad^B f,0).
\end{aligned}
$$

Hence, summing on the index i,

$$
\begin{aligned}
\nabla^*\nabla V &= \nabla_{\nabla_{e_i}e_i}V - \nabla_{e_i}\nabla_{e_i}V \\
&= \frac{1}{f^3}V_1(f)\left(0,\nabla^F_{e_i''}e_i''\right) - \frac{n}{f}\left(\nabla^B_{grad^B f}V_1,0\right) - n\frac{grad^B f(f)}{f^2}(0,V_2) + \\
&\quad \frac{1}{f^2}\left(0,\nabla^F_{\nabla^F_{e_i''}e_i''}V_2\right) - \frac{1}{f}g^F(V_2,\nabla^F_{e_i''}e_i'')(grad^B f,0) + \left(\nabla^B_{\nabla^B_{e_i'}e_i'}V_1,0\right) + \\
&\quad \frac{\left(\nabla^B_{e_i'}e_i'\right)(f)}{f}(0,V_2) - (\nabla^B_{e_i'}\nabla^B_{e_i'}V_1,0) - \frac{e_i'e_i'(f)}{f}(0,V_2) - \frac{1}{f^2}\left(0,\nabla^F_{e_i''}\nabla^F_{e_i''}V_2\right) + \\
&\quad \frac{1}{f}e_i''(g^F(V_2,e_i''))\left(grad^B f,0\right) + \frac{grad^B f(f)}{f^2}(0,V_2) - \frac{1}{f^3}V_1(f)\left(0,\nabla^F_{e_i''}e_i''\right) \\
&\quad + n\frac{V_1(f)}{f^2}\left(grad^B f,0\right) + \frac{1}{f}g^F(e_i'',\nabla^F_{e_i''}V_2)\left(grad^B f,0\right)
\end{aligned}
$$

$$\begin{aligned}\nabla^*\nabla V &= \Bigg(\nabla^*\nabla V_1 - \frac{n}{f}\nabla^B_{grad^B f}V_1 + \frac{2}{f}g^F(e''_i, \nabla^F_{e''_i}V_2)grad^B f + n\frac{V_1(f)}{f^2}grad^B f,\\ &\quad \frac{1}{f^2}\nabla^*\nabla V_2 - \frac{e'_i e'_i(f)}{f}V_2 + \frac{\left(\nabla^B_{e'_i}e'_i\right)(f)}{f}V_2 + (1-n)\frac{gradf(f)}{f^2}(0, V_2)\Bigg).\end{aligned}$$

After these calculations we deduce the following lemma.

Lemma 3.1. *Let (B, g_B) and (F, g_F) be Riemannian manifolds and $f : B \longrightarrow \mathbb{R}^*_+$ a smooth function on B. Let $\{e'_i\}_{i=1,\cdots,m}$ be an orthonormal basis of (B, g_B) and $\{e''_i\}_{i=1,\cdots,n}$ an orthonormal basis of (F, g_F). Then a vector field $V = V_1 + V_2$ on $M = B \times_f F$ is a harmonic vector field if and only if*

$$\begin{cases}\nabla^*\nabla V_1 - \frac{n}{f}\nabla^B_{grad^B f}V_1 + \frac{2}{f}div^F(V_2)grad^B f + n\frac{V_1(f)}{f^2}grad^B f = 0,\\ \frac{1}{f^2}\nabla^*\nabla V_2 + \frac{\Delta^B(f)}{f}V_2 + (1-n)\frac{gradf(f)}{f^2}V_2 = 0,\end{cases} \tag{3.1}$$

where $\Delta^B(f) = -trH^f$.

If G is 2-dimensional Riemannian Lie group, we have

Corollary 3.2. *Let G be a 2-dimensional Riemannian Lie group equipped with a left-invariant metric, $f : I \subset \mathbb{R} \longrightarrow]0, +\infty[$ a smooth function on I, $V_1 = \phi(t)\partial_t$ a vector field on I and V_2 a unit vector field on G. Then $V = \phi(t)\partial_t + V_2$ is a harmonic vector field on the warped product $I \times_f G$ if $f(t) = \sqrt{2\kappa_0 t^2 + c_1 t + c_2}$ on I such that*

1. *$I =]-\infty, -\frac{c_2}{c_1}[$ if $\kappa_0 = 0$ and $c_1 < 0$*
2. *$I =]-\frac{c_2}{c_1}, +\infty[$ if $\kappa_0 = 0$ and $c_1 > 0$*
3. *$I = \mathbb{R}$ if $c_1^2 - 8c_2\kappa_0 < 0$ and $\kappa_0 > 0$*
4. *$I =]t_1, t_2[$ if $c_1^2 - 8c_2\kappa_0 \geq 0$ and $\kappa_0 < 0$*
5. *$I =]-\infty, t_1[\cup]t_2, +\infty[$ if $c_1^2 - 8c_2\kappa_0 \geq 0$ and $\kappa_0 > 0$*

and ϕ is a solution on I of the differential equation

$$(E_0) : x'' + 2\frac{f'}{f}x' - 2\left(\frac{f'}{f}\right)^2 x + 2\kappa_1\frac{f'}{f} = 0 \tag{3.2}$$

where $\kappa_0 = \langle\nabla^*\nabla V_2, V_2\rangle, \kappa_1 = \sum_{i=1}^{2} g^F(e'_i, \nabla_{e'_i}V_2)$, $t_1 = \frac{-c_1 - \sqrt{c_1^2 - 8c_2\kappa_0}}{4\kappa_0}$, $t_2 = \frac{-c_1 + \sqrt{c_1^2 - 8c_2\kappa_0}}{4\kappa_0}$ with $t_1 \leq t_2$ and $\{e_1, e_2\}$ an orthonormal basis on G.

Note that the ODE (E_0) always admits solutions defined on the whole of I.

Take (B, g_B) to be (I, dt^2) and $V = V_1 + V_2$ on $I \times_f F$ where $V_1 = \phi(t)\partial_t$ is a vector field on I and $V_2 = \sum_{i=1}^{n} a_i e_i$ a vector field on F, where the a_i are functions on F. Note

that for an orthonormal basis $\{e_i\}_{i=1,\dots,n}$ of (F,g), the Levi-Civita connection ∇ of (F,g) and $\{\partial_t\}$ the canonical vector field on I, we have:

$$\begin{aligned}
\nabla_{e_i}V_2 &= \sum_{j=1}^{n}\Big(a_j\nabla_{e_i}e_j + e_i(a_j)e_j\Big) \qquad \text{for} \quad i=1,\dots,n,\\
\nabla_{e_i}\nabla_{e_i}V_2 &= \sum_{j=1}^{n}\Big(a_j\nabla_{e_i}\nabla_{e_i}e_j + e_ie_i(a_j)e_j + 2e_i(a_j)\nabla_{e_i}e_j\Big) \qquad \text{for} \quad i=1,\dots,n,\\
\nabla_{\nabla_{e_i}e_i}V_2 &= \sum_{j=1}^{n}\Big(a_j\nabla_{\nabla_{e_i}e_i}e_j + (\nabla_{e_i}e_i)(a_j)e_j\Big) \qquad \text{for} \quad i=1,\dots,n.
\end{aligned}$$

We now compute the previous terms and sum on $i=1,\cdots,n$ to obtain

$$\nabla^*\nabla V_2 = \sum_{i,j=1}^{n}\Big(a_j\nabla_{\nabla_{e_i}e_i}e_j + (\Delta a_j)e_j - 2e_i(a_j)\nabla_{e_i}e_j - a_j\nabla_{e_i}\nabla_{e_i}e_j\Big).$$

We also have, $g(e_i,\nabla_{e_i}V_2) = e_i(a_i) + \sum_{j=1}^{n}\Big(a_jg(e_i,\nabla_{e_i}e_j)\Big)$, and $\nabla^*\nabla V_1 = -\phi''(t)\partial_t$,

$$-\phi''(t)\partial_t, grad^B f = f'(t)\partial_t, \nabla^B_{grad^B f}V_1 = f'(t)\phi'(t)\partial_t; e_i'(e_i'(f)) = \partial_t(\partial_t(f)) = f''(t).$$

Then V is a harmonic vector field if and only if

$$\begin{cases}
\phi''(t) + n\dfrac{f'(t)}{f(t)}\phi'(t) - n\left(\dfrac{f'(t)}{f(t)}\right)^2\phi(t) - \\
2\left(\sum_{i=1}^{n}\left(e_i(a_i) + \sum_{j=1}^{n}a_jg(e_i,\nabla_{e_i}e_j)\right)\right)\dfrac{f'(t)}{f(t)} = 0,\\
\sum_{j=1}^{n}\left((\Delta a_j)e_j + \sum_{i=1}^{n}\big((a_j\nabla_{\nabla_{e_i}e_i}e_j - 2e_i(a_j)\nabla_{e_i}e_j - a_j\nabla_{e_i}\nabla_{e_i}e_j)\big)\right)\\
-\Big(f(t)f''(t) + 2f'(t)^2\Big)\sum_{i=1}^{n}a_je_j = 0.
\end{cases} \tag{3.3}$$

4 Harmonic vector fields on $I\times_f G$: G unimodular Lie group

4.1 Vector fields constructed from unit left-invariant vector fields

In this section, we assume that F is a three-dimensional connected Riemannian Lie group G equipped with a left-invariant metric. We determine harmonic vector fields on $(I\times_f G, g)$ with $f: I \longrightarrow]0;+\infty[$. Let $\{e_1,e_2,e_3\}$ be an orthonormal basis on G. Let $V_2 = ae_1 + be_2 + ce_3$ such that $a^2+b^2+c^2=1$, $V_1 = \phi(t)\partial_t$ a vector field on I and consider the vector field $V = V_1 + V_2$ on $I\times_f G$.

Proposition 4.1. *[11] Let G be a three-dimensional unimodular connected Riemannian Lie group, $\mathfrak{g}$ its Lie algebra and g a left-invariant metric on G. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}$ such that*

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3 \tag{4.1}$$

where $\lambda_1, \lambda_2, \lambda_3$ are constants.

Table 1: *Three-dimensional unimodular Lie groups*

Signs of $\lambda_1, \lambda_2, \lambda_3$	*Associated Lie Groups*
+,+,+	$SU(2)$ *or* $SO(3)$
+,+,-	$SL(2,\mathbb{R})$ *or* $O(1,2)$
+,+,0	$\mathbb{E}(2)$
+,0,-	$\mathbb{E}(1,1)$
+,0,0	$\mathbb{H}^3$
0,0,0	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$

Then the Levi-Civita connection ∇ is giving by:

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = \mu_1 e_3, \quad \nabla_{e_1} e_3 = -\mu_1 e_2; \\ &\nabla_{e_2} e_1 = -\mu_2 e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = \mu_2 e_1; \\ &\nabla_{e_3} e_1 = \mu_3 e_2, \quad \nabla_{e_3} e_2 = -\mu_3 e_1, \quad \nabla_{e_3} e_3 = 0; \end{aligned} \tag{4.2}$$

and its Riemann curvature tensor is given by

$$\begin{aligned} R(e_1, e_2)e_2 &= (\lambda_3\mu_3 - \mu_1\mu_2)e_1; R(e_1, e_3)e_3 = (\lambda_2\mu_2 - \mu_1\mu_3)e_1; \\ R(e_2, e_1)e_1 &= (\lambda_3\mu_3 - \mu_1\mu_2)e_2; R(e_2, e_3)e_3 = (\lambda_1\mu_1 - \mu_2\mu_3)e_2; \\ R(e_3, e_1)e_1 &= (\lambda_2\mu_2 - \mu_1\mu_3)e_3; R(e_3, e_2)e_2 = (\lambda_1\mu_1 - \mu_2\mu_3)e_3; \end{aligned} \tag{4.3}$$

and the other components are zero, where $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i, \ i = 1, 2, 3.$

From this 4.2, we have

$$\nabla_{\nabla_{e_i} e_i} e_j = \nabla_{e_i} \nabla_{e_i} e_i = g(e_i, \nabla_{e_i} e_j) = 0$$

$$\nabla_{e_i} \nabla_{e_i} e_j = -\mu_i^2 e_j, \quad i, j = 1, 2, 3.$$

Hence, Using equation 3.3, $V = V_1 + V_2$ is a harmonic vector field if and only if on $I \times_f G$,

$$\begin{cases} \phi''(t) + 3\dfrac{f'(t}{f(t)}\phi'(t) - 3\left(\dfrac{f'(t)}{f(t)}\right)^2 \phi(t) = 0, \\ a\left(\mu_2^2 + \mu_3^2 - f(t)f''(t) - 2f'(t)^2\right) = 0, \\ b\left(\mu_1^2 + \mu_3^2 - f(t)f''(t) - 2f'(t)^2\right) = 0, \\ c\left(\mu_2^2 + \mu_1^2 - f(t)f''(t) - 2f'(t)^2\right) = 0 \\ t \in I. \end{cases} . \tag{4.4}$$

The first equation of (4.4) always admits a non-trivial solution. For the other equations

$$\begin{cases} a\left(\mu_2^2+\mu_3^2-f(t)f''(t)-2f'(t)^2\right)=0,\\ b\left(\mu_1^2+\mu_3^2-f(t)f''(t)-2f'(t)^2\right)=0,\\ c\left(\mu_2^2+\mu_1^2-f(t)f''(t)-2f'(t)^2\right)=0,\\ t\in I \end{cases}. \tag{4.5}$$

According to the classification of three-dimensional connected unimodular Riemannian Lie groups given in Table 1 where the domain I of the solutions of the non-linear differential equation $yy''+2y'^2=\epsilon, \epsilon\in\mathbb{R}$ $\Bigg($for example an obvious solution is $y(t)=\pm\sqrt{\frac{\epsilon}{2}}\ t\Bigg)$, we obtain

Proposition 4.2. *Let G be a three-dimensional connected unimodular Riemannian Lie group equipped with a left-invariant metric and $V=V_1+V_2$ a vector field on the warped product $I\times_f G$, $f: I\longrightarrow]0,+\infty[$ with $V_1=\phi(t)\partial_t$ a vector field on I and $V_2=ae_1+be_2+ce_3$ a unit left-invariant vector field on G. Let $\{e_1,e_2,e_3\}$ be an orthonormal basis of the Lie algebra satisfying (4.1) and $\lambda_1,\lambda_2,\lambda_3$ the structure constants. Then V is a harmonic vector field if and only if f is a solution on $I\subset\mathbb{R}$ of the ODE $(E_1): xx''+2x'^2=\epsilon,\quad \epsilon\in\mathbb{R}$, ϕ is a solution, on I, of the ODE*

$$(E_0): x''+3\frac{f'}{f}x'-3\left(\frac{f'}{f}\right)^2x=0 \tag{4.6}$$

and one of the following cases occurs:

1. $\lambda_1=\lambda_2=\lambda_3=0(G=\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R})$, $f(t)=(3e^{c_1}t+c_2)^{\frac{1}{3}}$, $I=]-\frac{c_2}{3e^{c_1}},+\infty[$, *and* $V_2=ae_1+be_2+ce_3$ *for any* $a,b,c\in\mathbb{R}, a^2+b^2+c^2=1$,
2. $\lambda_1>0,\lambda_2=\lambda_3=0(G=\mathbb{H}^3),\epsilon=2\mu_1^2$ *and* $V_2=ae_1+be_2+ce_3$ *for any* $a,b,c\in\mathbb{R}$,
   $a^2+b^2+c^2=1$,
3. $\lambda_1>0,\lambda_2=0,\lambda_3<0(G=\mathbb{E}(1,1))$,
   (a) $\epsilon=\mu_1^2+\mu_2^2$ *and* $V_2=ae_1+ce_3$ *for any* $a,c\in\mathbb{R}, a^2+c^2=1$,
   (b) $\epsilon=2\mu_1^2$ *and* $V_2=\pm e_2$,
4. $\lambda_1>0,\lambda_2>0,\lambda_3=0(G=\mathbb{E}(2))$,
   (a) $\epsilon=\mu_1^2+\mu_3^2$ *and* $V_2=ae_1+be_2$ *for any* $a,b\in\mathbb{R}, a^2+b^2=1$,
   (b) $\epsilon=2\mu_1^2$ *and* $V_2=\pm e_3$,
5. $\lambda_1=\lambda_2>0>\lambda_3$ $(G=SL(2,\mathbb{R}),O(1,2))$
   (a) $\epsilon=\mu_1^2+\mu_3^2$ *and* $V_2=ae_1+be_2$ *for any* $a,b\in\mathbb{R}, a^2+b^2=1$,

(b) $\epsilon = 2\mu_1^2$ *and* $V_2 = \pm e_3$,

6. $\lambda_1 > \lambda_2 > 0 > \lambda_3$ $(G = SL(2,\mathbb{R}), O(1,2))$, $\lambda_1 > \lambda_2 > \lambda_3 (G = SU(2))$, $\epsilon = \mu_i^2 + \mu_j^2$ *and* $V_2 = \pm e_k$ $i \neq j \neq k$,

7. $\lambda_1 = \lambda_2 = \lambda_3 > 0 (G = SU(2), SO(3)), \epsilon = 2\mu_1^2$ *and* $V_2 = ae_1 + be_2 + ce_3$ *for any* $a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1$,

8. $\lambda_1 > \lambda_2 = \lambda_3 (G = SU(2), SO(3))$,

   (a) $\epsilon = \mu_1^2 + \mu_2^2$ *and* $V_2 = be_2 + ce_3$ *for any* $b, c \in \mathbb{R}, c^2 + b^2 = 1$,

   (b) $\epsilon = 2\mu_2^2$ *and* $V_2 = \pm e_1$,

9. $\lambda_1 = \lambda_2 > \lambda_3 > 0 (G = SU(2), SO(3))$,

   (a) $\epsilon = \mu_1^2 + \mu_3^2$ *and* $V_2 = ae_1 + be_2$ *for any* $a, b \in \mathbb{R}, a^2 + b^2 = 1$

   (b) $\epsilon = 2\mu_1^2$ *and* $V_2 = \pm e_3$.

Remark 4.1. Note that if f is a non-zero positive constant on $\mathbb{R}$ and G is a three-dimensional connected unimodular Lie group equipped with a left-invariant metric, then $V = V_1 + V_2$ is harmonic vector field on $\mathbb{R} \times_f G$ if and only if ϕ is an affine function, $\mu_j = \mu_i \neq \mu_k$ and $V_2 = \pm e_k$, $i, j, k \in \{1, 2, 3\}$ or $\mu_1 = \mu_2 = \mu_3 = 0$ and V_2 any vector field on G.

4.2 Vector fields constructed from non left-invariant vector fields

In this subsection, we construct new examples of harmonic vector fields from non left-invariant vector fields on a three-dimensional unimodular Lie group.

We first consider the Lie group $G = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ and consider $V_2 = a(x, y, z)\dfrac{\partial}{\partial x}$. Using Relation (3.3), a vector field $V = \varphi(t)\partial_t + V_2$ is a harmonic vector field on $I \times_f \big(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}\big)$ if and only if

$$\begin{cases} \phi''(t) + 3\dfrac{f'(t)}{f(t)}\phi'(t) - 3\left(\dfrac{f'(t)}{f(t)}\right)^2 \phi(t) = 2k\dfrac{f'(t)}{f(t)}, \\ a_x = k, \quad k \in \mathbb{R}, \\ f(t)f''(t) + 2f'(t)^2 = \epsilon \\ \Delta a = \epsilon a, \quad t \in I. \end{cases}$$

This implies that:

Case 1: $\epsilon > 0$, then $\begin{cases} a(x, y, z) = \cos(v_1 y + v_2 z) + \sin(v_1 y + v_2 z), \quad v_1^2 + v_2^2 = \epsilon, \\ f(t)f''(t) + 2f'(t)^2 = \epsilon, \quad t \in I \subset \mathbb{R}, \\ \phi''(t) + 3\dfrac{f'(t)}{f(t)}\phi'(t) - 3\left(\dfrac{f'(t)}{f(t)}\right)^2 \phi(t) = 2k\dfrac{f'(t)}{f(t)}. \end{cases}$

Case 2: $\epsilon < 0$, then $\begin{cases} a(x, y, z) = \exp(v_1 y + v_2 z), \quad v_1^2 + v_2^2 = -\epsilon, \\ f(t)f''(t) + 2f'(t)^2 = \epsilon, \quad t \in I \subset \mathbb{R}, \\ \phi''(t) + 3\dfrac{f'(t)}{f(t)}\phi'(t) - 3\left(\dfrac{f'(t)}{f(t)}\right)^2 \phi(t) = 2k\dfrac{f'(t)}{f(t)}. \end{cases}$

Case 3: $\epsilon = 0$, then $\begin{cases} a(x,y,z) = kx + b(y,z) \quad b \text{ is a harmonic function on } \mathbb{R}^2, \\ f(t) = (3e^{c_1}x + c_2)^{\frac{1}{3}} \quad \text{and} \quad I =]-\frac{c_2}{3e^{c_1}}, +\infty[, \\ \phi''(t) + 3\frac{f'(t)}{f(t)}\phi'(t) - 3\left(\frac{f'(t)}{f(t)}\right)^2\phi(t) = 2k\frac{f'(t)}{f(t)}. \end{cases}$

These computations lead to the construction of vector fields which are not left-invariant but harmonic on the warped product of G and an interval I.

Proposition 4.3. *The vector field $V = \varphi(t)\partial_t + V_2$ is a harmonic vector field on $I \times_f \left(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}\right)$ if and only if*

$$\begin{cases} f(t)f''(t) + 2f'(t)^2 = \epsilon, \quad t \in I \subset \mathbb{R}, \quad \epsilon \neq 0, \\ \phi''(t) + 3\frac{f'(t)}{f(t)}\phi'(t) - 3\left(\frac{f'(t)}{f(t)}\right)^2\phi(t) = 2\kappa\frac{f'(t)}{f(t)}, \end{cases}$$

with $a(x,y,z) = \cos(v_1y + v_2z) + \sin(v_1y + v_2z), v_1^2 + v_2^2 = -\epsilon$ if $\epsilon < 0$ and $a(x,y,z) = \exp(v_1y + v_2z), v_1^2 + v_2^2 = \epsilon$ if $\epsilon > 0$, or

$$\begin{cases} a(x,y,z) = \kappa_1 x + b(y,z), \\ f(t) = (3e^{c_1}t + c_2)^{\frac{1}{3}} \quad \text{and} \quad t \in I =]-\frac{c_2}{3e^{c_1}}, +\infty[, \\ \phi''(t) + 3\frac{f'(t)}{f(t)}\phi'(t) - 3\left(\frac{f'(t)}{f(t)}\right)^2\phi(t) = 2k\frac{f'(t)}{f(t)}, \end{cases}$$

where $b(y,z)$ is a harmonic function on $\mathbb{R}^2$ and $\kappa, \kappa_1 \in \mathbb{R}$.

Secondly, we consider $G = \mathbb{H}^3$ the Heisenberg group of real 3×3 upper-triangular matrices of the form

$$A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.7}$$

endowed with the left-invariant metric given by $dx^2 + \left(dy - xdz\right)^2 + (dz)^2$. We identify $\mathbb{H}^3$ with $\mathbb{R}^3$, endowed with this metric. The left-invariant vector fields $e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z} + x\frac{\partial}{\partial y}$, constitute an orthonormal basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{H}^3$ and the corresponding Levi-Civita connection is determined by

$$\begin{aligned} \nabla_{e_1}e_2 = \nabla_{e_2}e_1 &= -\frac{1}{2}e_3, \\ \nabla_{e_1}e_3 = -\nabla_{e_3}e_1 &= \frac{1}{2}e_2, \\ \nabla_{e_2}e_3 = \nabla_{e_3}e_2 &= \frac{1}{2}e_1, \end{aligned} \tag{4.8}$$

where the remaining covariant derivatives vanish.

By (4.8), we have $\mu_1 = -\frac{1}{2} = -\mu_2 = -\mu_3$. Take $V_2 = a(x,y,z)e_1$, then the vector

field
$V = \varphi(t)\partial_t + V_2$ is a harmonic vector field on $I \times_f \mathbb{H}^3$ if and only if

$$\begin{cases} \phi''(t) + 3\dfrac{f'(t)}{f(t)}\phi'(t) - 3\left(\dfrac{f'(t)}{f(t)}\right)^2 \phi(t) = 2\kappa\dfrac{f'(t)}{f(t)}, \\ a_x = \kappa, \quad \kappa \in \mathbb{R}, a_y = 0, \\ a_z + xa_y = 0, \\ f(t)f''(t) + 2f'(t)^2 = \epsilon, \quad \epsilon \in \mathbb{R}, \\ \Delta a = (\epsilon - \dfrac{1}{2})a, \quad t \in I. \end{cases}$$

Therefore

$$\begin{cases} a(x,y,z) = \kappa x + \kappa' \quad \kappa, \kappa' \in \mathbb{R}, \\ f(t)f''(t) + 2f'(t)^2 = \dfrac{1}{2}, \quad t \in I \subset \mathbb{R}, \\ \phi''(t) + 3\dfrac{f'(t)}{f(t)}\phi'(t) - 3\left(\dfrac{f'(t)}{f(t)}\right)^2 \phi(t) = 2\kappa\dfrac{f'(t)}{f(t)}. \end{cases}$$

and we obtain:

Proposition 4.4. *Let $V_1 = \phi(t)\partial_t$ on I and $V_2 = a(x,y,z)e_1$ on $\mathbb{H}^3$, then $V = V_1 + V_2$ is a harmonic vector field on $I \times_f \mathbb{H}^3$ if and only if*

$$\begin{cases} a(x,y,z) = \kappa x + \kappa' \quad \kappa, \kappa' \in \mathbb{R}, \\ f(t)f''(t) + 2f'(t)^2 = \dfrac{1}{2}, \quad t \in I \subset \mathbb{R}, \\ \phi''(t) + 3\dfrac{f'(t)}{f(t)}\phi'(t) - 3\left(\dfrac{f'(t)}{f(t)}\right)^2 \phi(t) = 2\kappa\dfrac{f'(t)}{f(t)}. \end{cases}$$

5 Harmonic vector fields on $I \times_f G$: G non-unimodular Lie group

5.1 Vector fields constructed from unit left-invariant vector fields

In this subsection, F is now a three-dimensional connected non-unimodular Riemannian Lie group G equipped with a left-invariant metric and we determine harmonic vector fields on $I \times_f G$ with $f : I \longrightarrow]0; +\infty[$. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis on G, $V_2 = ae_1 + be_2 + ce_3$ such that $a^2 + b^2 + c^2 = 1$ and $V_1 = \phi(t)\partial_t$ a vector field on I and consider the vector field $V = V_1 + V_2$ on $I \times_f G$.

Proposition 5.1. *[11] Let G be three-dimensional connected Riemannian non-unimodular Lie group, $\mathfrak{g}$ its Lie algebra and g a left-invariant metric on G. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}$ such that*

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = -\beta e_2 + \delta e_3, \quad [e_2, e_3] = 0. \tag{5.1}$$

where $\alpha + \delta > 0$ and $\alpha \geq \delta$ are constants.

Then the Levi-Civita connection ∇ is determined by [11]

$$\nabla_{e_1} e_1 = 0, \nabla_{e_1} e_2 = \beta e_3, \quad \nabla_{e_1} e_3 = -\beta e_2;$$

$$\nabla_{e_2} e_1 = -\alpha e_2, \quad \nabla_{e_2} e_2 = \alpha e_1, \quad \nabla_{e_2} e_3 = 0; \tag{5.2}$$

$$\nabla_{e_3} e_1 = -\delta e_3, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = \delta e_1;$$

and its Riemann curvature tensor is given by

$$\begin{aligned} R(e_2, e_1)e_1 &= (\alpha\beta - \beta\delta)e_3; R(e_2, e_3)e_1 = 0; R(e_3, e_1)e_1 = (\alpha\beta - \beta\delta)e_2 - \delta^2 e_3; \\ R(e_1, e_2)e_2 &= -\alpha^2 e_1; R(e_3, e_2)e_3 = \alpha\delta e_2; R(e_1, e_3)e_2 = (\alpha\beta - \beta\delta)e_1; \\ R(e_2, e_3)e_2 &= \alpha\delta e_3; R(e_1, e_2)e_3 = (\alpha\beta - \beta\delta)e_1; R(e_1, e_3)e_3 = -\delta^2 e_1; \end{aligned} \tag{5.3}$$

From this 5.1, we have

$$\nabla_{\nabla_{e_1} e_1} e_j = \nabla_{e_2}\nabla_{e_2} e_1 = \nabla_{e_3}\nabla_{e_3} e_1 = 0, \quad \nabla_{e_1}\nabla_{e_1} e_1 = \nabla_{e_3}\nabla_{e_3} e_2 = \nabla_{e_2}\nabla_{e_2} e_3 = 0$$

$$\nabla_{\nabla_{e_3} e_3} e_2 = \delta\beta e_3, \quad \nabla_{e_3}\nabla_{e_3} e_3 = -\delta\beta e_2, \quad \nabla_{\nabla_{e_2} e_2} e_3 = -\alpha\beta e_2, \quad \nabla_{e_2}\nabla_{e_2} e_2 = \alpha\beta e_3,$$

$$\nabla_{e_1}\nabla_{e_1} e_2 = -\beta^2 e_2, \quad \nabla_{e_1}\nabla_{e_1} e_3 = -\beta^2 e_3, \quad \nabla_{e_2}\nabla_{e_2} e_1 = -\alpha^2 e_1, \quad \nabla_{e_2}\nabla_{e_2} e_2 = -\alpha^2 e_2,$$

$$\nabla_{e_3}\nabla_{e_3} e_1 = -\delta^2 e_1, \quad \nabla_{e_3}\nabla_{e_3} e_3 = -\delta^2 e_3, \quad g(e_i, \nabla_{e_i} e_i) = g(e_1, \nabla_{e_1} e_i) = 0,$$

$$g(e_3, \nabla_{e_3} e_2) = g(e_2, \nabla_{e_2} e_3) = 0, \quad g(e_2, \nabla_{e_2} e_1) = -\alpha, \quad g(e_3, \nabla_{e_3} e_1) = -\delta.$$

Hence $V = V_1 + V_2$ is a harmonic vector field on $I \times_f G$ if and only if

$$\begin{cases} \phi''(t) + 3\dfrac{f'(t)}{f(t)}\phi'(t) - 3\dfrac{f'(t)^2}{f(t)^2}\phi(t) = -2a(\delta + \alpha)\dfrac{f'(t)}{f(t)}, \\ a\left(\alpha^2 + \delta^2 - f(t)f''(t) - 2f'(t)^2\right) = 0, \\ b\left(\alpha^2 + \beta^2 - f(t)f''(t) - 2f'(t)^2\right) - \beta(\alpha + \delta)c = 0, \\ c\left(\delta^2 + \beta^2 - f(t)f''(t) - 2f'(t)^2\right) + \beta(\alpha + \delta)b = 0 \\ \quad t \in I. \end{cases} \tag{5.4}$$

The first equation of (5.4) has a solution and the remaining system becomes:

$$\begin{cases} a\left(\alpha^2 + \delta^2 - f(t)f''(t) - 2f'(t)^2\right) = 0, \\ b\left(\alpha^2 + \beta^2 - f(t)f''(t) - 2f'(t)^2\right) - \beta(\alpha + \delta)c = 0, \\ c\left(\delta^2 + \beta^2 - f(t)f''(t) - 2f'(t)^2\right) + \beta(\alpha + \delta)b = 0, \\ \quad t \in I. \end{cases}$$

Using the proposition 5.1, we obtain,

Proposition 5.2. *Let G be a three-dimensional connected Riemannian non-unimodular Lie group equipped with a left-invariant metric and $V = V_1 + V_2$ a vector field on the warped product $I \times_f G$, $f : I \longrightarrow]0, +\infty[$ with $V_1 = \phi(t)\partial_t$ a vector field on I and V_2 a unit left-invariant vector field on G. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of the Lie algebra satisfying equation(5.1) and α, β, δ the structure constants. Then V is a*

harmonic vector field if and only if f satisfies, on $I \subset \mathbb{R}$, the ordinary differential equation $(E_1) : xx'' + 2x'^2 = \epsilon, \quad \epsilon \in \mathbb{R}$, ϕ is a solution, on I, of the equation

$$(E_a) : x'' + 3\frac{f'}{f}x' - 3\left(\frac{f'}{f}\right)^2 x = -2a(\delta+\alpha)\frac{f'}{f} \tag{5.5}$$

and V_2 is determined by one of the following conditions:

1. $\alpha = \delta > 0$
   - (a) $\beta = 0$, $\epsilon = 2\alpha^2$ *and* $V_2 = \pm e_1$,
   - (b) $\beta = 0$, $\epsilon = 2\alpha^2$ *and* $V_2 = \pm e_2$ *or* $V_2 = \pm e_3$,
   - (c) $\beta = 0$, $\epsilon = \alpha^2$ *and* $V_2 = be_2 + ce_3, b^2 + c^2 = 1, b \neq 0, c \neq 0$,
2. $\alpha > \delta > 0$ *or* $\alpha > 0 > \delta$
   - (a) $\epsilon = \alpha^2 + \delta^2$, $V_2 = ae_1 + be_2 + ce_3$,$a \neq 0$, $a^2 + b^2 + c^2 = 1$ *and* $\begin{cases} b(\beta^2 - \delta^2) = \beta(\alpha+\delta)c, \\ c(\beta^2 - \alpha^2) = -\beta(\alpha+\delta)b. \end{cases}$
   - (b) $\beta = 0$, $\epsilon = \delta^2, V_2 = \pm e_3$,
   - (c) $\beta = 0$, $\epsilon = \alpha^2, V_2 = \pm e_2$,
   - (d) $\beta = bc(\alpha - \delta), \epsilon = \beta^2 + b^2\alpha^2 + c^2\delta^2$, $V_2 = be_2 + ce_3, c \neq 0, b \neq 0, b^2 + c^2 = 1$.
3. $\alpha > \delta = 0$
   - (a) $\beta = 0$, *in this case* $f(t) = (3e^{c_1}t + c_2)^{\frac{1}{3}}$, $I =]-\frac{c_2}{3e^{c_1}}, +\infty[$ *and* $V_2 = \pm e_3$,
   - (b) $\beta = 0$, $\epsilon = \alpha^2$ *and* $V_2 = \pm e_2$,
   - (c) $\beta = bc\alpha$, $\epsilon = \beta^2 + b^2\alpha^2$ *and* $V_2 = be_2 + ce_3$, $b \neq 0, c \neq 0, b^2 + c^2 = 1$,
   - (d) $\epsilon = \alpha^2$ *and* $V_2 = \pm e_1$,
   - (e) $\beta = 0$, $\epsilon = \alpha^2$ *and* $V_2 = ae_1 + be_2$, $a \neq 0, a^2 + b^2 = 1$.

Remark 5.1. Note that if f is a non-zero positive constant on $\mathbb{R}$ and G is a three-dimensional connected Riemannian non-unimodular Lie group equipped with a left-invariant metric, then V is harmonic vector field on $\mathbb{R} \times_f G$ if and only if ϕ is an affine function on $\mathbb{R}, \beta = \delta = 0$, $V_2 = be_2 + ce_3$ and $\begin{cases} b(\beta^2 - \delta^2) = \beta(\alpha+\delta)c, \\ c(\beta^2 - \alpha^2) = -\beta(\alpha+\delta)b. \end{cases}$

5.2 Vector fields constructed from non left-invariant vector fields

In this subsection, we give examples of harmonic vector fields on $I \times_f G$ constructed from non left-invariant vector fields on the three-dimensional non-unimodular Lie group G

Consider F to be $(\mathbb{R}\times H^2, g)$ where $H^2=\{(x,y)\in\mathbb{R}^2 : y>0\}$ denotes the Poincaré half-plane with Gaussian curvature equal to $-\alpha$ $(\alpha>0)$ and g the left-invariant metric given by

$$g=\frac{1}{\alpha y^2}\Big(dx^2+dy^2\Big)+dz^2.$$

The left-invariant vector fields $e_1=y\sqrt{\alpha}\dfrac{\partial}{\partial y}, e_2=y\sqrt{\alpha}\dfrac{\partial}{\partial x}, e_3=\dfrac{\partial}{\partial z}$, constitute an orthonormal basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{R}\times H^2$ and $[e_3,e_1]=0, [e_3,e_2]=0$, $[e_1,e_2]=\sqrt{\alpha}\ e_2$.The corresponding Levi-Civita connection is determined by

$$\nabla_{e_2}e_1=-\sqrt{\alpha}e_2,\quad \nabla_{e_2}e_2=\sqrt{\alpha}e_1,$$

where the remaining covariant derivatives vanish. Take $V_2=b(y)e_2+c(y)e_3$ and use (3.3) to see that $V=\varphi(t)\partial_t+V_2$ is a harmonic vector field on $I\times_f(\mathbb{R}\times H^2)$ if and only if

$$\begin{cases}\phi''(t)+3\dfrac{f'(t)}{f(t)}\phi'(t)-3\left(\dfrac{f'(t)}{f(t)}\right)^2\phi(t)=0,\\ \Delta b=\big(f(t)f"(t)+2f'(t)^2-\alpha\big)\,b,\\ \Delta c=(f(t)f"(t)+2f'(t)^2)c,\\ t\in I.\end{cases}$$

This is equivalently to

$$\begin{cases}\phi''(t)+3\dfrac{f'(t)}{f(t)}\phi'(t)-3\left(\dfrac{f'(t)}{f(t)}\right)^2\phi(t)=0,\\ \Delta b=\big(f(t)f"(t)+2f'(t)^2-\alpha\big)\,b,\\ \Delta c=(f(t)f"(t)+2f'(t)^2)c,\\ f(t)f"(t)+2f'(t)^2=\epsilon,\\ t\in I.\end{cases}$$

Hence

$$\begin{cases}\phi''(t)+3\dfrac{f'(t)}{f(t)}\phi'(t)-3\left(\dfrac{f'(t)}{f(t)}\right)^2\phi(t)=0,\\ y^2b''(y)=(\epsilon-\alpha)b(y),\\ y^2c''(y)=\alpha c(y),\\ f(t)f"(t)+2f'(t)^2=\epsilon,\\ t\in I.\end{cases}$$

and we have

$$\begin{cases}c(y)=y^{\frac{1}{2}}\Big(\kappa_1+\kappa_2\ln(y)\Big) \text{ if } 1+4\alpha=0,\\ c(y)=\kappa_1 y^{\frac{1}{2}+\frac{1}{2}\sqrt{1+4\alpha}}+\kappa_2 y^{\frac{1}{2}-\frac{1}{2}\sqrt{1+4\alpha}} \quad \text{if } 1+4\alpha>0,\\ c(y)=y^{\frac{1}{2}}\Big(\kappa_1\cos(y\sqrt{-4\alpha-1})+\kappa_2\sin(y\sqrt{-4\alpha-1})\Big) \quad \text{if } 14\alpha<0,\end{cases}\tag{5.6}$$

and

(5.7)
$$\begin{cases} b(y) = y^{\frac{1}{2}}\Big(\kappa_1 + \kappa_2 \ln(y)\Big) \text{ if } 1+4(\epsilon-\alpha)=0 \\ b(y) = \kappa_1 y^{\frac{1}{2}+\frac{1}{2}\sqrt{1+4(\epsilon-\alpha)}} + \kappa_2 y^{\frac{1}{2}-\frac{1}{2}\sqrt{1+4(\epsilon-\alpha)}} \quad \text{if } 1+4(\epsilon-\alpha)>0, \\ b(y) = y^{\frac{1}{2}}\Big(\kappa_1 \cos(y\sqrt{-1-4(\epsilon-\alpha)}) + \kappa_2 \sin(y\sqrt{-1-4(\epsilon-\alpha)})\Big) \text{ if } 4(\epsilon-\alpha)<-1, \end{cases}$$

with $\kappa_1, \kappa_2 \in \mathbb{R}$. This system of equations lead to new families of harmonic vector fields on the warped product $I \times_f (\mathbb{R} \times H^2)$, which are non left-invariant.

Proposition 5.3. *Let $V_1 = \phi(t)\partial_t$ be a vector field on I and $V_2 = b(y)e_2 + c(y)e_3$ be vector fields on $(\mathbb{R} \times H^2, g)$ where $H^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ denotes the Poincaré half-plane with Gaussian curvature equal to $-\alpha(\alpha > 0)$. Then $V = V_1 + V_2$ is a harmonic vector field on $I \times_f (\mathbb{R} \times H^2)$ if and only if*

$$\begin{cases} \phi''(t) + 3\dfrac{f'(t)}{f(t)}\phi'(t) - 3\left(\dfrac{f'(t)}{f(t)}\right)^2 \phi(t) = 0, \quad t \in I, \\ f(t)f''(t) + 2f'(t)^2 = \epsilon, \ \epsilon \in \mathbb{R}, \end{cases}$$

and $V_2 = b(y)e_2 + c(y)e_3$ with b, c defined by combining Equations (5.6) and (5.7).

6 Harmonic vectors fields which are harmonic maps on warped product $I \times_f G$

In this section, we determine the horizontal part of the tension field on the warped product $B \times_f F$ (cf 2.3) and study the existence of vector fields on $I \times_f G$ which are harmonic maps, where G is a three-dimensional connected Riemannian Lie group equipped with a left-invariant metric. To calculate $S(V)$ where $V = V_1 + V_2$, we write

$$S(V) = S_1(V) + S_2(V)$$

where

$$S_1(V) = \sum_{i=1}^{m} R(\nabla_{e_i} V, V)e_i \quad \text{and} \quad S_2(V) = \sum_{i=m+1}^{m+n} R(\nabla_{e_i} V, V)e_i.$$

and $\{e_i\}_i$ are define on section 3 Then, using the lemma 2.2

$$\begin{aligned}
S_1(V) &= \sum_{i=1}^{m} R\left(\nabla_{(e_i',0)}V,(V_1,V_2)\right)(e_i',0)\\
&= \sum_{i=1}^{m} R\left((\nabla^B_{e_i'}V_1,0),(V_1,0)\right)(e_i',0)+R\left((\nabla^B_{e_i'}V_1,0),(0,V_2)\right)(e_i',0)+\\
&\quad \frac{e_i'(f)}{f}R\left((0,V_2),(V_1,0)\right)(e_i',0)+\frac{e_i'(f)}{f}R\left((0,V_2),(0,V_2)\right)(e_i',0)\\
&= \sum_{i=1}^{m}\left(R^B(\nabla^B_{e_i'}V_1,e_i')V_1,0\right)-\frac{e_i'(f)}{f^2}H^f(V_1,e_i')(0,V_2)+\\
&\quad \frac{1}{f}H^f(\nabla^B_{e_i'}V_1,e_i')(0,V_2)\\
S_2(V) &= \sum_{i=1}^{n}\frac{1}{f}R\left(\nabla_{(0,e_i'')}V,V\right)(0,e_i'')\\
&= \sum_{i=1}^{n}\frac{1}{f^2}R\left((0,\nabla^F_{e_i''}V_2),(V_1,0)\right)(0,e_i'')+\frac{1}{f^2}R\left((0,\nabla^F_{e_i''}V_2),(0,V_2)\right)(0,e_i'')-\\
&\quad \frac{1}{f}g^F(V_2,e_i'')R\left((gradf,0),(V_1,0)\right)(0,e_i'')-\\
&\quad \frac{1}{f}g^F(V_2,e_i'')R\left((gradf,0),(0,V_2)\right)(0,e_i'')\\
&\quad +\frac{V_1(f)}{f^3}R\left((0,e_i''),(0,V_2)\right)(0,e_i'')+\frac{V_1(f)}{f^3}R\left((0,e_i''),(V_1,0)\right)(0,e_i'')
\end{aligned}$$

$$\begin{aligned}
S_2(V) &= \sum_{i=1}^{n}\left(\frac{1}{f^2}\left(0,R^F(\nabla_{e_i''}V_2,V_2)e_i''\right)+\frac{V_1(f)}{f^3}(0,R^F(e_i'',V_2)e_i'')\right)+\\
&\quad \|V_2\|^2(\nabla^B_{gradf}gradf,0)+\frac{gradf(f)}{f^2}\left(\mathrm{div}(V_2)(0,V_2)-\left(0,\nabla^F_{V_2}V_2\right)\right)+\\
&\quad n\frac{V_1(f)}{f^2}\left(\nabla^B_{V_1}gradf,0\right)+\frac{1}{f}\mathrm{div}(V_2)(\nabla^B_{V_1}gradf,0)+\\
&\quad \frac{V_1(f)}{f^3}gradf(f)(n-1)(0,V_2)\\
&= \frac{1}{f^2}(0,S(V_2))+\frac{gradf(f)}{f^2}\left(\mathrm{div}(V_2)(0,V_2)-\left(0,\nabla^F_{V_2}V_2\right)\right)+n\frac{V_1(f)}{f^2}\left(\nabla^B_{V_1}gradf,0\right)\\
&\quad +\frac{V_1(f)}{f^3}\sum_{i=1}^{n}(0,R^F(e_i'',V_2)e_i'')+\|V_2\|^2(\nabla^B_{gradf}gradf,0)+\frac{1}{f}\mathrm{div}(V_2)(\nabla^B_{V_1}gradf,0)\\
&\quad +\frac{V_1(f)}{f^3}gradf(f)(n-1)(0,V_2)
\end{aligned}$$

Hence

$$
\begin{aligned}
S(V) &= S_1(V)+S_2(V)\\
&= \Bigg((S(V_1),0)-\sum_{i=1}^{m}\Big(\frac{e_i'(f)}{f^2}H^f(V_1,e_i')-\frac{1}{f}H^f(\nabla^B_{e_i'}V_1,e_i')\Big)(0,V_2)+\\
&\quad n\frac{V_1(f)}{f^2}\big(\nabla^B_{V_1}gradf,0\big)+\frac{1}{f^2}(0,S(V_2))+\frac{gradf(f)}{f^2}\Big(\mathrm{div}(V_2)(0,V_2)-\big(0,\nabla^F_{V_2}V_2\big)\Big)+\\
&\quad \frac{V_1(f)}{f^3}gradf(f)(n-1)(0,V_2)+\frac{V_1(f)}{f^3}\sum_{i=1}^{n}(0,R^F(e_i'',V_2)e_i'')+\\
&\quad \|V_2\|^2\Big(\nabla^B_{gradf}gradf,0\Big)+\frac{1}{f}\mathrm{div}(V_2)(\nabla^B_{V_1}gradf,0)\Bigg)\\
&= \Bigg(S(V_1)+n\frac{V_1(f)}{f^2}\nabla^B_{V_1}gradf+\|V_2\|^2\nabla^B_{gradf}gradf+\frac{1}{f}\mathrm{div}(V_2)\nabla^B_{V_1}gradf;\frac{1}{f^2}S(V_2)\\
&\quad -\sum_{i=1}^{m}\Big(\frac{e_i'(f)}{f^2}H^f(V_1,e_i')-\frac{1}{f}H^f(\nabla^B_{e_i'}V_1,e_i')\Big)V_2+\frac{gradf(f)}{f^2}\Big(\mathrm{div}(V_2)V_2-\nabla^F_{V_2}V_2\Big)+\\
&\quad \frac{V_1(f)}{f^3}\sum_{i=1}^{n}R^F(e_i'',V_2)e_i''+\frac{V_1(f)}{f^3}gradf(f)(n-1)V_2\Bigg).
\end{aligned}
$$

Take $B=I\subset\mathbb{R}$, $V_1=\phi(t)\partial_t$ on I and V_2 a vector field on (F,g), then

$$
\begin{aligned}
S(V) &= \Bigg(n\frac{\phi^2f'f''}{f^2}\partial_t+\|V_2\|^2f'f''\partial_t+\frac{1}{f}\mathrm{div}(V_2)\phi f''\partial_t;\frac{1}{f^2}S(V_2)-\frac{f'\phi f''}{f^2}V_2+\frac{\phi'f''}{f}V_2+\\
&\quad \frac{f'^2}{f^2}\mathrm{div}(V_2)V_2-\frac{f'^2}{f^2}\nabla_{V_2}V_2+\frac{\phi f'^3}{f^3}(n-1)V_2+\frac{\phi f'}{f^3}\sum_{i=1}^{n}R(e_i'',V_2)e_i''\Bigg).
\end{aligned}
$$

Remark that if f is a non-zero positive constant on $\mathbb{R}$, then a harmonic vector field $V=V_1+V_2$ on the warped product $\mathbb{R}\times_f F$ is a harmonic map if and only if V_2 is a harmonic map on F.
We suppose that G is three dimensional connected unimodular Riemannian Lie group equipped with a left-invariant metric. We have

Proposition 6.1. *Let G be a three-dimensional connected Riemannian unimodular Lie group equipped with a left invariant metric and $V=V_1+V_2$ a harmonic vector field on the warped product $I\times_f G, I\subset\mathbb{R}$, $f:I\longrightarrow]0,+\infty[$ with $V_1=\phi(t)\partial_t$ a vector field on I and $V_2=ae_1+be_2+ce_3$ a unit left-invariant vector field on G. Then V is harmonic map on $I\times_f G$ if and only if one of the following cases occurs:*

1. *$G=\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}:\lambda_1=\lambda_2=\lambda_3=0, f(t)=\beta>0, I=\mathbb{R}, V_2=ae_1+be_2+ce_3$ for any $a,b,c\in\mathbb{R}$ and $\phi(t)=\gamma_1t+\gamma_2,\gamma_1,\gamma_2\in\mathbb{R}$.*

2. *$\lambda_1=\lambda_2=\lambda_3>0\big(G=SU(3)$ or $SO(3)\big), f(t)=\eta\frac{\lambda_1}{2}t+\beta$ $(\eta=\pm1)$,*
   $\beta\in\mathbb{R}, I=\Big]-\frac{2\beta}{\lambda_1},+\infty\Big[$ for $\eta=1$, $I=\Big]-\infty,\frac{2\beta}{\lambda_1}\Big[$ for $\eta=-1$,

$$V_2 = ae_1 + be_2 + ce_3 \text{ and } \phi(t) = c_1\left(\eta\frac{\lambda_1}{2}t + \beta\right) + \frac{c_2}{\left(\eta\frac{\lambda_1}{2}t + \beta\right)^3}, c_1, c_2 \in \mathbb{R}.$$

Proof. Combining Proposition 4.1 and relation 4.3, we get

$$\nabla_{V_2}V_2 = bc(\mu_2 - \mu_3)e_1 + ac(\mu_3 - \mu_1)e_2 + ab(\mu_1 - \mu_2)e_3,$$

$$S(V_2) = A_1bce_1 + A_2ace_2 + A_3abe_3 \quad \text{and} \quad \operatorname{div}(V_2) = 0.$$

$$\sum_{i=1}^{3} R(e_i, V_2)e_i = a(\mu_1\mu_3 - \lambda_2\mu_2 + \mu_1\mu_2 - \lambda_3\mu_3)e_1 + b(\mu_1\mu_2 - \lambda_3\mu_3 + \mu_2\mu_3 - \lambda_1\mu_1)e_2 +$$
$$c(\mu_2\mu_3 - \lambda_1\mu_1 + \mu_1\mu_3 - \lambda_2\mu_2)e_3.$$

where $A_1 = \mu_2^2(\mu_3 - \mu_1) + \mu_3^2(\mu_1 - \mu_2)$, $A_2 = \mu_1^2(\mu_2 - \mu_3) + \mu_3^2(\mu_1 - \mu_2)$, $A_3 = \mu_1^2(\mu_2 - \mu_3) + \mu_2^2(\mu_3 - \mu_1)$. Hence a harmonic vector field $V = V_1 + V_2$ on $I \times_f G$ where G: unimodular Lie group is a harmonic map if and only if

$$\begin{cases} f'(t)f''(t)\left(3\phi(t)^2 + f(t)^2\right) = 0 \\ S(V_2) - f'(t)\phi(t)f''(t)V_2 + \phi'(t)f(t)f''(t)V_2 - f'(t)^2\nabla_{V_2}V_2 + \\ 2\dfrac{\phi(t)f'(t)^3}{f(t)}V_2 + \dfrac{\phi(t)f'(t)}{f(t)}\displaystyle\sum_{i=1}^{3} R(e_i, V_2)e_i = 0 \\ t \in I. \end{cases}$$

This is equivalent to

$$\begin{cases} \forall t \in I, f''(t) = 0 \\ S(V_2) - f'(t)^2\nabla_{V_2}V_2 + 2\dfrac{\phi(t)f'(t)^3}{f(t)}V_2 + \dfrac{\phi(t)f'(t)}{f(t)}\displaystyle\sum_{i=1}^{3} R(e_i, V_2)e_i = 0. \end{cases} \tag{6.1}$$

because of $\forall t \in I, f(t) > 0$ and $f'(t)f''(t) = 0$ we have $f''(t) = 0$.

We will now discuss case by case according to the classification of three-dimensional connected unimodular Lie groups given in Table 1 and combine the result of proposition 4.2, we have

- $G = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$: In this case, $S(V_2) = \nabla_{V_2}V_2 = R(e_i, V_2)e_i = 0$ and we obtain the first case of Proposition.
- $G = \mathbb{H}^3$: In this case we have $S(V_2) = -\frac{1}{4}a\lambda_1^3(ce_2 - be_3), \nabla_{V_2}V_2 = a\lambda_1(ce_2 - be_3)$, $\displaystyle\sum_{i=1}^{3} R(e_i, V_2)e_i = -\frac{1}{2}\lambda_1^2(ae_1 - be_2 - ce_3)$. Using System 6.1 and case 2 of Proposition 4.2, we have

$$\begin{cases} \forall t \in I, f'(t) = \dfrac{1}{4}\lambda_1^2 \\ -\dfrac{1}{4}\lambda_1^3 a(ce_2 - be_3) - \dfrac{1}{4}\lambda_1^3 a(ce_2 - be_3) + \\ \dfrac{f'(t)}{f(t)}\phi(t)\left(\dfrac{1}{4}\lambda_1^2(ae_1 + be_2 + ce_3) - \dfrac{1}{2}\lambda_1^2(ae_1 - be_2 - ce_3)\right) = 0. \end{cases}$$

this is equivalent to $a = 0, \lambda_1 = 0$ that is not possible. we obtain that a harmonic vector field on $I \times_f G$ cannot a harmonic map.

- $G = \mathbb{E}(1,1)$:
  $*$ $b = 0, f(t)f''(t) + 2f'(t)^2 = \mu_3^2 + \mu_2^2$: We have $S(V_2) = -2\mu_3^3 ace_2$, $\nabla_{V_2} V_2 = 2\mu_3 ace_2$ and
  $\sum_{i=1}^{3} R(e_i, V_2)e_i = a(-\mu_3^2 - \mu_2\mu_3 - \lambda_3\mu_3)e_1 + c(-\mu_3^2 + \mu_2\mu_3 + \lambda_1\mu_3)e_3$. Then

$$\begin{cases} \forall t \in I, 2f'(t)^2 = \mu_3^2 + \mu_2^2 \\ \dfrac{f'(t)}{f(t)}\phi(t)\Big(\mu_3^2 + \mu_2^2(ae_1 + ce_3) + a(-\mu_3^2 - \mu_2\mu_3 - \lambda_3\mu_3)e_1 + \\ c(-\mu_3^2 + \mu_2\mu_3 + \lambda_1\mu_3)e_3\Big) - 2\mu_3^3 ace_2 - \dfrac{1}{2}(\mu_3^2 + \mu_2^2)(2\mu_3 ace_2) = 0. \end{cases}$$

  this is equivalently to $a = c = 0$ (not possible because $a^2 + c^2 = 1$).
  $*$ $c = a = 0, \epsilon = 2\mu_1^2$, in this subcase we obtain that a harmonic vector field on $I \times_f G$ cannot a harmonic map.

- $G = \mathbb{E}(2)$, $G = SL(2,\mathbb{R}), O(1,2)$ are similarly to that case $G = \mathbb{E}(1,1)$ and we obtain that a harmonic vector field on $I \times_f G$ cannot a harmonic map.

- $G = SU(3), SO(3)$
  $*$ $\lambda_1 = \lambda_2 = \lambda_3 > 0$: in this subcase $S(V_2) = \nabla_{V_2} V_2 = 0$ and
  $\sum_{i=1}^{3} R(e_i, V_2)e_i = -2\mu V_2$. We have
  $f'(t)^2 = \mu^2 = \frac{1}{4}\lambda_1^2$ and a harmonic vector field is harmonic map
  $*$ For other cases, Similarly to the cases $G = \mathbb{E}(2)$ and the cases $G = SL(2,\mathbb{R}), O(1,2)$, we obtain that a harmonic vector field on $I \times_f G$ cannot be a harmonic map.

□

We now suppose that G is three dimensional connected Riemannian non-unimodular Lie group equipped with a left-invariant metric. We have

Proposition 6.2. *Let G be a three-dimensional connected Riemannian non-unimodular Lie group equipped with a left-invariant metric and $V = V_1 + V_2$ a harmonic vector field on the warped product $I \times_f G, I \subset \mathbb{R}$, $f : I \longrightarrow]0, +\infty[$ with $V_1 = \phi(t)\partial_t$ a vector field on I and $V_2 = ae_1 + be_2 + ce_3$ a unit left-invariant vector field on G. Then V is a harmonic map on $I \times_f G$ if and only if*

1. $\alpha = \delta > 0 = \beta$,$V_2 = ae_1$, *and*

$$\begin{cases} \forall t \in I, f(t)f''(t) + 2f'(t)^2 = 2\alpha^2 \\ f'(t)f''(t)\big(3\phi(t)^2 + f(t)^2\big) = 2a\alpha f(t)f''(t)\phi(t) \\ -2\alpha^3 - af'(t)\phi(t)f''(t) + a\phi'(t)f(t)f''(t) - 2\alpha f'(t)^2 + \\ 2a\dfrac{\phi(t)f'(t)}{f(t)}\left(2f'(t)^2 + \alpha^2\right) = 0 \\ \phi''(t) + 3\dfrac{f'(t)}{f(t)}\phi'(t) - 3\left(\dfrac{f'(t)}{f(t)}\right)^2\phi(t) = -4a\alpha\dfrac{f'(t)}{f(t)}. \end{cases}$$

2. $\alpha > 0, \beta = \delta = 0, f(t) = \mu > 0, I = \mathbb{R}, V_2 = \pm e_3$ *and* $\phi(t) = \gamma_1 t + \gamma_2$ *with* $\mu, \gamma_1, \gamma_2 \in \mathbb{R}$.

3. $\alpha > \delta = 0$,$V_2 = ae_1$, $a = \pm 1$ *and*

$$\begin{cases} f'(t)f''(t)\big(3\phi(t)^2 + f(t)^2\big) = a\alpha f(t)f''(t)\phi(t) \\ -\alpha^3 - f'(t)\phi(t)f''(t)a + \phi'(t)f(t)f''(t)a - \\ \alpha f'(t)^2 + a\dfrac{\phi(t)f'(t)}{f(t)}(2f'(t)^2 + \alpha^2) = 0 \\ f(t)f''(t) + 2f'(t)^2 = \alpha^2 \\ \phi''(t) + 3\dfrac{f'(t)}{f(t)}\phi'(t) - 3\left(\dfrac{f'(t)}{f(t)}\right)^2\phi(t) = -2a\alpha\dfrac{f'(t)}{f(t)} \\ t \in I. \end{cases} \tag{6.2}$$

4. $\alpha > 0 > \delta$ *or* $\alpha > \delta > 0$,

$$\begin{cases} -\alpha^3 b^2 - \delta^3 c^2 + \alpha(\alpha^2 - \delta^2)b^2c^2 - \dfrac{1}{2}(b^2\alpha + c^2\delta)(b^2\alpha^2 + \\ c^2\delta^2 + b^2c^2(\alpha - \delta)^2) = 0 \\ b^2\alpha^2 + c^2\delta^2 + b^2c^2(\alpha - \delta)^2 + \alpha(\alpha + \delta) + c^2(\delta - \alpha) = 0 \\ \alpha + \delta + (\alpha - \delta)(c^2 - b^2) = 0. \\ \epsilon = \beta^2 + b^2\alpha^2 + c^2\delta^2, b \neq 0, c \neq 0. \end{cases}$$

$V_2 = be_2 + ce_3$, $f(t) = \mu\kappa t + \eta$ $(\mu = \pm 1), \eta \in \mathbb{R}$ *and* $I = \left]-\dfrac{\eta}{\kappa}, +\infty\right[$ *for* $\mu = 1$, $I = \left]-\infty, \dfrac{\eta}{\kappa}\right[$ *for* $\mu = -1$
and $\phi(t) = c_1\Big(\mu\kappa t + \eta\Big) + \dfrac{c_2}{\Big(\mu\kappa t + \eta\Big)^3}, c_1, c_2 \in \mathbb{R}$, $\kappa = \sqrt{\dfrac{\epsilon}{2}}$

5. $\alpha > \delta > 0$ *or* $\alpha > 0 > \delta$,$V_2 = ae_1 + be_2 + ce_3$, $a \neq 0$ *and*

(6.3)
$$\begin{cases} b(\beta^2-\delta^2)=\beta(\alpha+\delta)c, \\ c(\beta^2-\alpha^2)=-\beta(\alpha+\delta)b, \\ f(t)f''(t)+2f'(t)=\alpha^2+\delta^2, \\ f''(t)\Big(3f'(t)\phi^2(t)+f^2(t)f'(t)-a(\alpha+\delta)f(t)\phi(t)\Big)=0, \\ -\alpha^3(a^2+b^2)-\delta^3(a^2+c^2)+\beta bc(\alpha^2-\delta^2)-af'(t)f''(t)\phi(t)+af(t)f''(t)\phi'(t)- \\ a^2f'(t)^2(\alpha+\delta)-f'(t)^2(b^2\alpha+c^2\delta)+a\dfrac{\phi(t)f'(t)}{f(t)}\Big(2f'(t)^2+\alpha^2+\delta^2\Big)=0, \\ a\Big(\alpha^2c\beta-b\alpha\delta^2+\alpha\beta^2b-\delta\beta^2b\Big)-bf'(t)\phi(t)f''(t)+f(t)\phi'(t)f''(t)b-abf'(t)^2(\alpha+\delta)+ \\ af'(t)^2(c\beta+b\alpha)+\dfrac{\phi(t)f'(t)}{f(t)}\Big(2bf'(t)^2+b\alpha(\alpha+\delta)+c\beta(\delta-\alpha)\Big)=0, \\ a\Big(-\beta\delta^2b-\alpha^2\delta c-\beta^2(\alpha-\delta)c\Big)-cf'(t)\phi(t)f''(t)+f(t)\phi'(t)f''(t)c- \\ acf'(t)^2(\alpha+\delta)+af'(t)^2(c\delta+b\beta)+\dfrac{\phi(t)f'(t)}{f(t)}\Big(2f'(t)^2+c\delta(\alpha+\delta)+b\beta(\delta-\alpha)\Big)=0, \\ \phi''(t)+3\dfrac{f'(t)}{f(t)}\phi'(t)-3\left(\dfrac{f'(t)}{f(t)}\right)^2\phi(t)=-2a(\alpha+\delta)\dfrac{f'(t)}{f(t)}, t\in I. \end{cases}$$

Proof. Combining Proposition 5.1 and System 5.3, we get $\mathrm{div}(V_2)=-a(\delta+\alpha)$,

$$S(V_2)=\big[-\alpha^3(a^2+b^2)-\delta^3(a^2+c^2)+\beta(\alpha^2-\delta^2)bc\big]e_1+a\big[\alpha^2\beta c-\alpha\delta^2b+\beta^2(\alpha-\delta)b\big]e_2+$$
$$a\big[-\beta\delta^2b-\alpha^2\delta c-\beta^2(\alpha-\delta)c\big]e_3,$$
$$\nabla_{V_2}V_2=\big(b^2\alpha+c^2\delta\big)e_1+\big(-ac\beta-ab\alpha\big)e_2+\big(ab\beta-ac\delta\big)e_3$$

and

$$\sum_{i=1}^{3}R(e_i,V_2)e_i=a(\alpha^2+\delta^2)e_1+(b\alpha(\alpha+\delta)+c\beta(\delta-\alpha))e_2+(c\delta(\alpha+\delta)+b\beta(\delta-\alpha))e_3.$$

Hence a harmonic vector field $V=V_1+V_2$ on $I\times_f G$ where G is a non-unimodular Lie group is a harmonic map if and only if

(6.4)
$$\begin{cases} f'(t)f''(t)\Big(3\phi(t)^2+f(t)^2\Big)+f(t)f''(t)\phi(t)\mathrm{div}(V_2)=0 \\ S(V_2)-f'(t)\phi(t)f''(t)V_2+\phi'(t)f(t)f''(t)V_2-f'(t)^2\nabla_{V_2}V_2+f'(t)^2\mathrm{div}(V_2)V_2+ \\ \dfrac{\phi(t)f'(t)}{f(t)}\Big(2f'(t)^2V_2+\displaystyle\sum_{i=1}^{3}R(e_i,V_2)e_i\Big)=0 \\ t\in I. \end{cases}$$

We will discuss case by case as in Proposition 5.2.

- $\alpha=\delta>0$,
  - $\beta=0,\epsilon=2\alpha^2,b=c=0,f(t)f''(t)+2f'(t)^2=\epsilon$, we have

$\mathrm{div}(V_2) = -2\alpha a, S(V_2) = -2\alpha^3 e_1, \nabla_{V_2} V_2 = 0, \sum_{i=1}^{3} R(e_i, V_2)e_i = 2a\alpha^2 e_1,$

then a vector field V is harmonic map if and only if

$$\begin{cases} \forall t \in I, f(t)f''(t) + 2f'(t)^2 = 2\alpha^2 \\ f'(t)f''(t)\big(3\phi(t)^2 + f(t)^2\big) = 2a\alpha f(t)f''(t)\phi(t) \\ -2\alpha^3 - af'(t)\phi(t)f''(t) + a\phi'(t)f(t)f''(t) - 2\alpha f'(t)^2 + \\ 2a\dfrac{\phi(t)f'(t)}{f(t)}\left(2f'(t)^2 + \alpha^2\right) = 0 \\ \phi''(t) + 3\dfrac{f'(t)}{f(t)}\phi'(t) - 3\left(\dfrac{f'(t)}{f(t)}\right)^2 \phi(t) = -4a\alpha\dfrac{f'(t)}{f(t)}. \end{cases}$$

$*$ $\beta = 0, \epsilon = 2\alpha^2, a = 0$, we have $\mathrm{div}(V_2) = 0, S(V_2) - \alpha^3 e_1,$

$\nabla_{V_2} V_2 = \alpha e_1, \sum_{i=1}^{3} R(e_i, V_2)e_i = 2\alpha^2(be_2 + ce_3)$. From System 6.4 and

$f(t)f''(t) + 2f'(t)^2 = \epsilon$, we have $\forall t \in I, f'(t)^2 = \alpha^2$ and

$$\begin{cases} -\alpha^3 e_1 - f'(t)\phi(t)f''(t)(be_2 + ce_3) + \phi'(t)f(t)f''(t)(be_2 + ce_3) - \alpha^3 e_1 + \\ 2\alpha^2\dfrac{\phi(t)f'(t)}{f(t)}(be_2 + ce_3) = 0, \\ t \in I. \end{cases}$$

This is not possible and we obtain that a harmonic vector field cannot be a harmonic map.

$*$ $\beta = 0, \epsilon = \alpha^2, a = 0$ as in the previous subcase, we obtain that a harmonic vector field cannot be a harmonic map.

- $\alpha > 0, \delta = 0$

  $*$ $\beta = \epsilon = 0, a = b = 0$, we have $\mathrm{div}(V_2) = \nabla_{V_2} V_2 = \sum_{i=1}^{3} R(e_i, V_2)e_i = 0.$

  From System 6.4 and $f(t)f''(t) + 2f'(t)^2 = 0$,we obtain $\forall t \in I = \mathbb{R}, f'(t) = 0$ and a harmonic vector field on $I \times_f G$ is harmonic map in this subcase

  $*$ For the subcases $(b) - (c)$ there is no solution and we obtain that a harmonic vector field on $I \times_f G$ cannot be a harmonic map

  $*$ $b = c = 0, f(t)f''(t) + 2f'(t)^2 = \alpha^2$. We have

  $\mathrm{div}(V_2) = -a\alpha, S(V_2) = -\alpha^3 e_1, \sum_{i=1}^{3} R(e_i, V_2)e_i = a\alpha^2 e_1, \nabla_{V_2} V_2 = 0$ and we obtain the System 6.2

  $*$ The last subcase of this case implies $b = 0$, that is the same result of previous subcase.

- $\alpha > \delta > 0$ or $\alpha > 0 > \delta$

  $*$For the subcase that $a \neq 0, f(t)f''(t) + 2f'(t)^2 = \alpha^2 + \delta^2$, we obtain Equation 6.3

  $*$ $\beta = 0, \epsilon = \alpha^2, a = c = 0$: From this a harmonic vector field cannot a harmonic map

  $*$ $\beta = bc\alpha, \epsilon = b^2\alpha^2(1 + c^2), a = 0$, in this case $S(V_2) = \big[-\alpha^3 b^2 - \delta^3 c^2 + \beta(\alpha^2 -$

$\delta^2)bc\big]e_1$,

$\mathrm{div}(V_2) = 0, \nabla_{V_2}V_2 = (b^2\alpha + c^2\delta)e_1, \sum_{i=1}^{3} R(e_i, V_2)e_i = (b\alpha(\alpha+\delta) + \beta(\delta-\alpha))e_2 + (c\delta(\alpha+\delta) + b\beta(\delta-\alpha))e_3$ and we have

$$\begin{cases} \forall t \in I, 2f'(t)^2 = \beta^2 + b^2\alpha^2 + c^2\delta^2 \\ -\alpha^3 b^2 - \delta^3 c^2 + \beta(\alpha^2 - \delta^2)bc - \dfrac{1}{2}(b^2\alpha + c^2\delta)(b^2\alpha^2 + c^2\delta^2 + b^2c^2(\alpha-\delta)^2) = 0 \\ \dfrac{\phi(t)f'(t)}{f(t)}\Big((b^2\alpha^2 + c^2\delta^2 + b^2c^2(\alpha-\delta)^2)b + b\alpha(\alpha+\delta) + c\beta(\delta-\alpha)\Big) = 0 \\ \dfrac{\phi(t)f'(t)}{f(t)}\Big((b^2\alpha^2 + c^2\delta^2 + b^2c^2(\alpha-\delta)^2)c + c\delta(\alpha+\delta) + b\beta(\delta-\alpha)\Big) = 0 \end{cases}$$

this is equivalent to

$$\begin{cases} \forall t \in I, 2f'(t)^2 = \beta^2 + b^2\alpha^2 + c^2\delta^2 \\ -\alpha^3 b^2 - \delta^3 c^2 + \alpha(\alpha^2 - \delta^2)b^2c^2 - \dfrac{1}{2}(b^2\alpha + c^2\delta)(b^2\alpha^2 + c^2\delta^2 + b^2c^2(\alpha-\delta)^2) = 0 \\ b^2\alpha^2 + c^2\delta^2 + b^2c^2(\alpha-\delta)^2 + \alpha(\alpha+\delta) + c^2(\delta-\alpha) = 0 \\ \alpha + \delta + (\alpha-\delta)(c^2 - b^2) = 0. \end{cases}$$

and we obtain (4) of the Proposition.

□

Using the examples of Propositions 5.1, 4.4 and 5.3 for a non left-invariant vector fields on Lie groups G, we obtain.

Proposition 6.3. *A harmonic vector field* $V = \phi(t)\partial_t + a(x,y,z)\dfrac{\partial}{\partial x}$ *on* $I \times_f (\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R})$ *is a harmonic map if and only if* V_2 *is constant on* $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, ϕ *is an affine function, and* f *is a positive function on* I *and* $I = \mathbb{R}$.

Proposition 6.4. *A harmonic vector field* $V = \phi(t)\partial_t + a(x,y,z)e_1$ *on* $I \times_f \mathbb{H}^3$ *cannot be a harmonic map.*

Proposition 6.5. *A harmonic vector field* $V = \phi(t)\partial_t + b(y)e_2 + c(y)e_3$ *on* $I \times_f (\mathbb{R} \times H^2)$ *cannot be a harmonic map.*

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