Mixed generalized quasi-Einstein manifolds with applications to Relativity

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Abstract. The present paper aims to study and analyse mixed generalized quasi-Einstein manifolds. Some geometric properties of $MG(QE)_n$ had been discussed. We had also outlined the behaviour of $MG(QE)_4$ space-time with space-matter tensor and discussed some of its related properties. Finally, we constructed examples of mixed generalized quasi-Einstein manifolds.

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1 Introduction

An n-dimensional semi-Riemannian or Riemannian manifold (M^n, g) $(n > 2)$, is said to be an Einstein manifold if its Ricci tensor S satisfies the condition

$$S = \frac{r}{n} g, \tag{1.1}$$

where r denotes the scalar curvature of (M^n, g). In other words, an Einstein manifold is a Riemannian or pseudo Riemannian manifold whose Ricci tensor is proportional to the metric. The notion of quasi-Einstein manifold was introduced by M. C. Chaki and R. K. Maity [6]. A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is a quasi-Einstein manifold if its Ricci tensor S satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) \tag{1.2}$$

and is not identically zero, where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$g(X, U) = A(X),$$

for all vector field X. U being a unit vector field.

Here a and b are called the associated scalars, A is called the associated 1-form and U is called the generator of the vector field of the manifold. Such an n-dimensional manifold is denoted by $(QE)_n$.

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The notion of a generalized quasi-Einstein manifold was introduced by U. C. De and G. C. Ghosh [8]. According to them, a non-flat Riemannian manifold is called a generalized quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y), \tag{1.3}$$

where a, b, c are certain non-zero scalars and A, B are two non-zero 1-forms such that for two unit vector fields U and V corresponding to the 1-forms A and B respectively, defined as

$$g(X,U) = A(X),\ g(X,V) = B(X) \text{ and } g(U,V) = 0.$$

In such a case a, b, c are called the associated scalars, A, B respectively are called the associated main and auxiliary 1-forms and U, V respectively are called the main and auxiliary generators of the vector fields of the manifold. This type of manifold is denoted by $G(QE)_n$.

In [4, 11], A. Bhattacharyya, T. De and S. Dey introduced the notion of mixed generalized quasi-Einstein manifold. A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is called mixed generalized quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) = {} & ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ & + e[A(X)B(Y) + A(Y)B(X)], \end{aligned} \tag{1.4}$$

where a, b, c, e are scalars of which $b \neq 0$, $c \neq 0$, $e \neq 0$ and A, B are two non-zero 1-forms such that

$$g(X,U) = A(X),\ g(X,V) = B(X) \text{ and } g(U,V) = 0,$$

where U, V are unit vector fields. In this case a, b, c, e are called associated scalars. A, B are called the associated 1-forms and U, V are called the generators of the vector fields of the manifold. If $e = 0$, then the manifold becomes to $G(QE)_n$. This type of manifold is denoted by $MG(QE)_n$.

In [9], the authors introduce the notion of a manifold of generalized quasi-constant curvature.

A Riemannian manifold is said to be a manifold of generalized quasi-constant curvature if the curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies the following condition

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = {} & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + q[g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W) \\ & + g(Y,Z)A(X)A(W) - g(Y,W)A(X)A(Z)] \\ & + s[g(X,W)B(Y)B(Z) - g(X,Z)B(Y)B(W) \\ & + g(Y,Z)B(X)B(W) - g(Y,W)B(X)B(Z)], \end{aligned} \tag{1.5}$$

where p, q, s are scalars, A and B are non-zero 1-forms. U and V are unit orthogonal vector fields such that

$$g(X,U) = A(X),\ g(X,V) = B(X) \text{ and } g(U,V) = 0.$$

A Riemannian manifold is said to be a manifold of mixed generalized quasi-constant curvature if the curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = {} & p\left[g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)\right] \\ & + q\left[g(X,W)\,A(Y)\,A(Z) - g(Y,W)\,A(X)\,A(Z)\right. \\ & \left. + g(Y,Z)\,A(X)\,A(W) - g(X,Z)\,A(Y)\,A(W)\right] \\ & + s\left[g(X,W)\,B(Y)\,B(Z) - g(Y,W)\,B(X)\,B(Z)\right. \\ & \left. + g(Y,Z)\,B(X)\,B(W) - g(X,Z)\,B(Y)\,B(W)\right] \\ & + t\left[\{A(Y)\,B(Z) + B(Y)\,A(Z)\}\,g(X,W)\right. \\ & - \{A(X)\,B(Z) + B(X)\,A(Z)\}\,g(Y,W) \\ & + \{A(X)\,B(W) + B(X)\,A(W)\}\,g(Y,Z) \\ & \left. - \{A(Y)\,B(W) + B(Y)\,A(W)\}\,g(X,Z)\right], \end{aligned} \tag{1.6}$$

where p, q, s, t are scalars. A, B are non-zero 1-forms. U and V are orthonormal unit vectors corresponding to A and B such that

$$g(X,U) = A(X),\ g(X,V) = B(X) \text{ and } g(U,V) = 0.$$

The notion of quasi-conformal curvature tensor was introduced by Yano and Sawaki [18] and they defined it as:

$$\begin{aligned}C^*(X,Y)\,Z = {} & a_1 R(X,Y)\,Z + b_1\left[S(Y,Z)\,X - S(X,Z)\,Y + g(Y,Z)\,QX\right. \\ & \left. - g(X,Z)\,QY\right] - \frac{r}{n}\left[\frac{a_1}{n-1} + 2b_1\right]\left[g(Y,Z)\,X - g(X,Z)\,Y\right], \end{aligned} \tag{1.7}$$

where a_1 and b_1 are constants, R is the curvature tensor of type $(1,3)$, S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator and r is the scalar curvature of the manifold.
If $a_1 = 1$ and $b_1 = -\frac{1}{n-2}$, then (1.7) reduces to the conformal curvature tensor C. Thus the conformal curvature tensor C is a particular case of the tensor C^*. For this reason C^* is called the quasi-conformal curvature tensor. A Riemannian or a semi-Riemannian manifold is called quasi-conformally flat if $C^* = 0$ for $n > 3$.

In a smooth manifold (M^n, g) Petrov [17] introduced a tensor $\tilde{P}$ of the type $(0,4)$ and defined it by

$$\tilde{P} = \tilde{R} + \frac{\kappa}{2} g \wedge T - \sigma G, \tag{1.8}$$

where $\tilde{R}$ is the curvature tensor of type $(0,4)$, T is the energy momentum tensor of type $(0,2)$, κ is the gravitational constant, σ is the energy density, G is a tensor of type $(0,4)$ given by

$$G(X,Y,Z,W) = g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W), \tag{1.9}$$

for all $X, Y, Z, W \in \chi(M)$ and Kulkarni-Nomizu product $E \wedge F$ of two $(0,2)$ tensors E and F is defined by

$$\begin{aligned}(E \wedge F)(X,Y,Z,W) = {} & E(Y,Z)\,F(X,W) + E(X,W)\,F(Y,Z) \\ & - E(X,Z)\,F(Y,W) - E(Y,W)\,F(X,Z), \end{aligned} \tag{1.10}$$

where $X, Y, Z, W \in \chi(M)$. The tensor $\tilde{P}$ is called the space-matter tensor of type $(0,4)$ of the manifold M. The space-matter tensor have been studied by Ahsan, Ali and Siddiqui [1, 2, 3] and many others.

After studying and analyzing various papers [7, 10, 12, 13, 14], we got motivated to work in this area. We have tried to develop a new concept. This paper is organized as follows:

After introduction in Section 2, we have studied $MG(QE)_n$ with divergence free quasi-conformal curvature tensor. In Section 3, we have studied sectional curvatures at a point of a quasi-conformally flat $MG(QE)_n$. In the next two sections, we have studied $MG(QE)_4$ spacetime with vanishing space-matter tensor and divergence free space-matter tensor. In section 6, we have studied perfect fluid $MG(QE)_4$ spacetime. Finally, we have given two examples of $MG(QE)_n$.

2 $MG(QE)_n$ $(n > 3)$ with divergence free quasi-conformal curvature tensor

In this section we look for a sufficient condition in order that a $MG(QE)_n$ $(n > 3)$ may be quasi-conformally conservative.

Theorem 2.1. *If in a $MG(QE)_n$ the associated scalars are constants and the generators U and V of the vector fields of the manifold are parallel vector fields, then the manifold is quasi-conformally conservative.*

Proof. Quasi-conformal curvature tensor is said to be conservative if the divergence of C^* vanishes, i.e., $\operatorname{div}(C^*) = 0$.

In a $MG(QE)_n$ if the associated scalars a, b, c, e are constants, then contracting (1.4) we get

$$r = an + b + c,$$

which implies that the scalar curvature r is constant, i.e., $dr = 0$.
Using $dr = 0$ we obtain from (1.7) that

$$\begin{aligned}(\nabla_W C^*)(X,Y,Z) = a_1 (\nabla_W R)(X,Y)Z + b_1 [(\nabla_W S)(Y,Z)X \\ - (\nabla_W S)(X,Z)Y + g(Y,Z)(\nabla_W Q)(X) \\ -g(X,Z)(\nabla_W Q)(Y)]. \end{aligned} \tag{2.1}$$

We know $(\operatorname{div} R)(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)$ and from (1.4) we obtain

$$\begin{aligned}(\nabla_X S)(Y,Z) = b[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z)] \\ + c[(\nabla_X B)(Y)B(Z) + B(Y)(\nabla_X B)(Z)] \\ + e[(\nabla_X A)(Y)B(Z) + A(Y)(\nabla_X B)(Z) \\ + (\nabla_X A)(Z)B(Y) + A(Z)(\nabla_X B)(Y)], \end{aligned} \tag{2.2}$$

since a, b, c and e are constants.

Contracting (2.1) and using (2.2) we obtain

$$
\begin{aligned}
(\mathrm{div}C^*)(X,Y,Z) = (a_1+b_1)\,[&b\{(\nabla_X A)(Y)A(Z)+A(Y)(\nabla_X A)(Z)\\
&-(\nabla_Y A)(X)A(Z)-A(X)(\nabla_Y A)(Z)\}\\
&+c\{(\nabla_X B)(Y)B(Z)+B(Y)(\nabla_X B)(Z)\\
&-(\nabla_Y B)(X)B(Z)-B(X)(\nabla_Y B)(Z)\}\\
&+e\{(\nabla_X A)(Y)B(Z)+A(Y)(\nabla_X B)(Z)\\
&+(\nabla_X A)(Z)B(Y)+A(Z)(\nabla_X B)(Y)\\
&-(\nabla_Y A)(X)B(Z)-A(X)(\nabla_Y B)(Z)\\
&-(\nabla_Y A)(Z)B(X)-A(Z)(\nabla_Y B)(X)\}]\,. \qquad (2.3)
\end{aligned}
$$

Using the condition that the generators U and V of the vector fields of the manifold are parallel vector fields which gives $\nabla_X U=0$ and $\nabla_X V=0$. Hence

$$g(\nabla_X U,Y)=0,\ \text{i.e.,}\ (\nabla_X A)(Y)=0$$

and

$$g(\nabla_X V,Y)=0,\ \text{i.e.,}\ (\nabla_X B)(Y)=0.$$

Therefore from (2.3) we get

$$(\mathrm{div}C^*)(X,Y,Z)=0.$$

Thus the manifold is quasi-conformally conservative. □

3 Sectional curvatures at a point of a quasi-conformally flat $MG(QE)_n$

Let us consider $U^{\perp}$ and $V^{\perp}$ as $(n-1)$-dimensional distribution in a quasi-conformally flat $MG(QE)_n$ $(n>3)$ orthogonal to U and V respectively. Then for any $X\in U^{\perp}$ and $X\in V^{\perp}$, $g(X,U)=0$ and $g(X,V)=0$, i.e., $A(X)=0$ and $B(X)=0$. In this section we will determine sectional curvature K at the plane determined by the vectors $X,Y\in U^{\perp}$ and $X,Y\in V^{\perp}$ or by X, U and X, V.

Theorem 3.1. *In a quasi-conformally flat $MG(QE)_n$ $(n>3)$ the sectional curvature of the plane determined by two vectors $X,Y\in U^{\perp}$ and $X,Y\in V^{\perp}$ is*

$$\frac{a_1(an+b+c)+2b_1(n-1)(b+c)}{n(n-1)a_1},$$

while the sectional curvature of the plane determined by two vectors X, U is

$$\frac{a_1(an+b+c)-b_1(n-1)\{b(n-2)-2c\}}{n(n-1)a_1}$$

and the sectional curvature of the plane determined by two vectors X, V is

$$\frac{a_1(an+b+c)-b_1(n-1)\{c(n-2)-2b\}}{n(n-1)a_1}.$$

Proof. In [5], A. Bhattacharyya, T. De and D. Debnath proved that every quasi-conformally flat $MG\,(QE)_n$ $(n>3)$ is a manifold of mixed generalized quasi-constant curvature, i.e.,

$$\begin{aligned}\tilde{R}\,(X,Y,Z,W) = {} & \left\{\frac{a_1 r + 2b_1\,(n-1)\,(r-an)}{n\,(n-1)\,a_1}\right\}[g\,(Y,Z)\,g\,(X,W) \\ & -g\,(X,Z)\,g\,(Y,W)] + \left(-\frac{bb_1}{a_1}\right)[g\,(X,W)\,A\,(Y)\,A\,(Z) \\ & -g\,(Y,W)\,A\,(X)\,A\,(Z) + g\,(Y,Z)\,A\,(X)\,A\,(W) \\ & -g\,(X,Z)\,A\,(Y)\,A\,(W)] + \left(-\frac{cb_1}{a_1}\right)[g\,(X,W)\,B\,(Y)\,B\,(Z) \\ & -g\,(Y,W)\,B\,(X)\,B\,(Z) + g\,(Y,Z)\,B\,(X)\,B\,(W) \\ & -g\,(X,Z)\,B\,(Y)\,B\,(W)] + \left(-\frac{eb_1}{a_1}\right)[\{A\,(Y)\,B\,(Z) \\ & +B\,(Y)\,A\,(Z)\}\,g\,(X,W) - \{A\,(X)\,B\,(Z) \\ & +B\,(X)\,A\,(Z)\}\,g\,(Y,W) + \{A\,(X)\,B\,(W) \\ & +B\,(X)\,A\,(W)\}\,g\,(Y,Z) - \{A\,(Y)\,B\,(W) \\ & +B\,(Y)\,A\,(W)\}\,g\,(X,Z)]\,. \end{aligned} \tag{3.1}$$

Putting $Z=Y$ and $W=X$ in (3.1) we have

$$\tilde{R}\,(X,Y,Y,X) = \frac{a_1 r + 2b_1\,(n-1)\,(r-an)}{n\,(n-1)\,a_1}\left[g\,(X,X)\,g\,(Y,Y) - \{g\,(X,Y)\}^2\right]. \tag{3.2}$$

Putting $Y=Z=U$ and $W=X$ in (3.1) we have

$$\tilde{R}\,(X,U,U,X) = \left\{\frac{a_1 r + 2b_1\,(n-1)\,(r-an)}{n\,(n-1)\,a_1} - \frac{bb_1}{a_1}\right\} g\,(X,X)\,. \tag{3.3}$$

Putting $Y=Z=V$ and $W=X$ in (3.1) we get

$$\tilde{R}\,(X,V,V,X) = \left\{\frac{a_1 r + 2b_1\,(n-1)\,(r-an)}{n\,(n-1)\,a_1} - \frac{cb_1}{a_1}\right\} g\,(X,X)\,. \tag{3.4}$$

Now contracting (1.4) over X and Y we have

$$r = an + b + c. \tag{3.5}$$

Using (3.2), (3.5), (3.3) and (3.4) we obtain

$$K\,(X,Y) = \frac{\tilde{R}\,(X,Y,Y,X)}{g\,(X,X)\,g\,(Y,Y) - \{g\,(X,Y)\}^2} = \frac{a_1\,(an+b+c) + 2b_1\,(n-1)\,(b+c)}{n\,(n-1)\,a_1},$$

$$K\,(X,U) = \frac{\tilde{R}\,(X,U,U,X)}{g\,(X,X)\,g\,(U,U) - \{g\,(X,U)\}^2} = \frac{a_1\,(an+b+c) - b_1\,(n-1)\,\{b\,(n-2) - 2c\}}{n\,(n-1)\,a_1}$$

and

$$K\,(X,V) = \frac{\tilde{R}\,(X,V,V,X)}{g\,(X,X)\,g\,(V,V) - \{g\,(X,V)\}^2} = \frac{a_1\,(an+b+c) - b_1\,(n-1)\,\{c\,(n-2) - 2b\}}{n\,(n-1)\,a_1}.$$

Thus the proof of theorem is completed. □

4 $MG\,(QE)_4$ spacetime with vanishing space-matter tensor

In this section we study $MG\,(QE)_4$ spacetime with vanishing space-matter tensor.

Theorem 4.1. *A $MG\,(QE)_4$ spacetime satisfying Einstein's field equation and with vanishing space-matter tensor is a spacetime of mixed generalized quasi-constant curvature.*

Proof. The equation (1.8) can be written as

$$\begin{aligned}\tilde{P}(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) &+ \frac{\kappa}{2}\left[g(Y,Z)\,T(X,W) + g(X,W)\,T(Y,Z)\right.\\ &\left.-g(X,Z)\,T(Y,W) - g(Y,W)\,T(X,Z)\right]\\ &-\sigma\left[g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)\right].\end{aligned}\tag{4.1}$$

If $\tilde{P}=0$, then (4.1) becomes

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = &-\frac{\kappa}{2}\left[g(Y,Z)\,T(X,W) + g(X,W)\,T(Y,Z)\right.\\ &\left.-g(X,Z)\,T(Y,W) - g(Y,W)\,T(X,Z)\right]\\ &+\sigma\left[g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)\right].\end{aligned}\tag{4.2}$$

The Einstein's field equation without cosmological constant is given by [15, 16]

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y),\tag{4.3}$$

where κ is the gravitational constant and r is the scalar curvature of the spacetime. Using (1.4) and Einstein's field equation (4.3) in (4.2) we have

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = &\left(\sigma - a + \frac{r}{2}\right)\left[g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)\right]\\ &-\frac{b}{2}\left[g(X,W)\,A(Y)\,A(Z) - g(Y,W)\,A(X)\,A(Z)\right.\\ &\left.+g(Y,Z)\,A(X)\,A(W) - g(X,Z)\,A(Y)\,A(W)\right]\\ &-\frac{c}{2}\left[g(X,W)\,B(Y)\,B(Z) - g(Y,W)\,B(X)\,B(Z)\right.\\ &\left.+g(Y,Z)\,B(X)\,B(W) - g(X,Z)\,B(Y)\,B(W)\right]\\ &-\frac{e}{2}\left[\{A(Y)\,B(Z) + B(Y)\,A(Z)\}\,g(X,W)\right.\\ &-\{A(X)\,B(Z) + B(X)\,A(Z)\}\,g(Y,W)\\ &+\{A(X)\,B(W) + B(X)\,A(W)\}\,g(Y,Z)\\ &\left.-\{A(Y)\,B(W) + B(Y)\,A(W)\}\,g(X,Z)\right].\end{aligned}\tag{4.4}$$

Comparing (1.6) and (4.4) we can say that the manifold under consideration is a manifold of mixed generalized quasi-constant curvature. □

5 $MG\,(QE)_4$ spacetime with divergence free space-matter tensor

In this section we look for a sufficient condition in order that a $MG\,(QE)_4$ may be of divergence free space-matter tensor.

Theorem 5.1. *In a $MG\,(QE)_4$ spacetime satisfying Einstein's field equation with divergence free space-matter tensor the energy density is constant.*

Proof. In a $MG\,(QE)_n$ if the associated scalars a, b, c and e are constants, then contracting (1.4) we get

$$r = an + b + c,$$

which implies that the scalar curvature r is constant, i.e., $dr = 0$.
Using (4.3), we obtain from (4.1) that

$$(\mathrm{div}P)\,(X,Y,Z) = \mathrm{div}R\,(X,Y)\,Z + \frac{1}{2}\left[(\nabla_X S)\,(Y,Z) - (\nabla_Y S)\,(X,Z)\right] - g\,(Y,Z)\left[d\sigma\,(X) + \frac{1}{4}dr\,(X)\right] + g\,(X,Z)\left[d\sigma\,(Y) + \frac{1}{4}dr\,(Y)\right]. \tag{5.1}$$

We know that in a semi-Riemannian manifold

$$(\mathrm{div}R)\,(X,Y,Z) = (\nabla_X S)\,(Y,Z) - (\nabla_Y S)\,(X,Z)\,. \tag{5.2}$$

From (5.1) and (5.2) we have

$$(\mathrm{div}P)\,(X,Y,Z) = \frac{3}{2}\left[(\nabla_X S)\,(Y,Z) - (\nabla_Y S)\,(X,Z)\right] - g\,(Y,Z)\left[d\sigma\,(X) + \frac{1}{4}dr\,(X)\right] + g\,(X,Z)\left[d\sigma\,(Y) + \frac{1}{4}dr\,(Y)\right]. \tag{5.3}$$

By assuming $(\mathrm{div}P)\,(X,Y,Z) = 0$ and then contracting (5.3) over Y and Z, we have

$$d\sigma\,(X) = 0.$$

Thus the energy density is constant. □

Theorem 5.2. *If in a $MG\,(QE)_4$ spacetime satisfying Einstein's field equation the associated scalars and the energy density σ are constants, then the divergence of the space-matter tensor vanishes.*

Proof. Using (1.4), equation (5.3) can be written as

$$\begin{aligned}
(\mathrm{div}P)(X,Y,Z) = {} & \frac{3}{2}\left[da(X)g(Y,Z) - da(Y)g(X,Z)\right] + \frac{3}{2}\left[db(X)A(Y)A(Z)\right. \\
& \left. -db(Y)A(X)A(Z)\right] + \frac{3}{2}\left[dc(X)B(Y)B(Z)\right. \\
& \left. -dc(Y)B(X)B(Z)\right] + \frac{3}{2}\left[de(X)\{A(Y)B(Z)\right. \\
& \left. +A(Z)B(Y)\} - de(Y)\{A(X)B(Z) + A(Z)B(X)\}\right] \\
& + \frac{3b}{2}\left[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z) - (\nabla_Y A)(X)A(Z)\right. \\
& \left. -A(X)(\nabla_Y A)(Z)\right] + \frac{3c}{2}\left[(\nabla_X B)(Y)B(Z) + B(Y)(\nabla_X B)(Z)\right. \\
& \left. -(\nabla_Y B)(X)B(Z) - B(X)(\nabla_Y B)(Z)\right] + \frac{3e}{2}\left[(\nabla_X A)(Y)B(Z)\right. \\
& + A(Y)(\nabla_X B)(Z) + (\nabla_X A)(Z)B(Y) + A(Z)(\nabla_X B)(Y) \\
& - (\nabla_Y A)(X)B(Z) - A(X)(\nabla_Y B)(Z) - (\nabla_Y A)(Z)B(X) \\
& \left. -A(Z)(\nabla_Y B)(X)\right] - g(Y,Z)\left[d\sigma(X) + \frac{1}{4}dr(X)\right] \\
& + g(X,Z)\left[d\sigma(Y) + \frac{1}{4}dr(Y)\right].
\end{aligned} \tag{5.4}$$

Using the conditions that the associated scalars and the energy density σ are constants and the generators U and V of the vector fields of the manifold are parallel vector fields which gives $\nabla_X U = 0$ and $\nabla_X V = 0$. Hence $dr(X) = 0$, $d\sigma(X) = 0$, for all X. Also

$$g(\nabla_X U, Y) = 0, \text{ i.e., } (\nabla_X A)(Y) = 0$$

and

$$g(\nabla_X V, Y) = 0, \text{ i.e., } (\nabla_X B)(Y) = 0.$$

Therefore from (5.4) we get

$$(\mathrm{div}P)(X,Y,Z) = 0.$$

Thus the divergence of the space-matter tensor vanishes. □

6 Perfect fluid $MG(QE)_4$ spacetime

Theorem 6.1. *If a perfect fluid $MG(QE)_4$ spacetime admits Einstein's field equation without cosmological constant, then in this case isotropic pressure is $\frac{-6a+b-c}{6\kappa}$ and energy density is $\frac{2a+3b+c}{2\kappa}$.*

Proof. In a perfect fluid spacetime, the energy momentum tensor T of type $(0,2)$ is of the form:

$$T(X,Y) = pg(X,Y) + (\sigma + p)A(X)A(Y), \tag{6.1}$$

where σ and p are the energy density and the isotropic pressure, respectively. Then in the general relativistic spacetime whose matter content is perfect fluid satisfying the Einstein's field equation, the Ricci tensor holds the following equation

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y). \tag{6.2}$$

From (6.1) and (6.2), we get

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa\left[pg(X,Y) + (\sigma + p)A(X)A(Y)\right]. \tag{6.3}$$

Taking a frame field and contracting (6.3) over X and Y, we have

$$r = \kappa(\sigma - 3p). \tag{6.4}$$

Here, if we consider the general relativistic perfect fluid $MG(QE)_4$ spacetime with unit timelike velocity vector field U, then we have

$$g(U,U) = -1. \tag{6.5}$$

Now putting $X = Y = U$ in (6.3) and then using (6.4), we have

$$S(U,U) = \frac{\kappa}{2}(\sigma + 3p). \tag{6.6}$$

In the case of $MG(QE)_4$ spacetime, contracting (1.4) over X and Y, we have

$$r = 4a + b + c. \tag{6.7}$$

From (6.4) and (6.7) we have

$$4a + b + c = \kappa(\sigma - 3p). \tag{6.8}$$

Again from (1.4), we get

$$S(U,U) = -a + b. \tag{6.9}$$

From (6.6) and (6.9), we obtain

$$b - a = \frac{\kappa}{2}(\sigma + 3p). \tag{6.10}$$

Solving equations (6.8) and (6.10), we get

$$p = \frac{-6a + b - c}{6\kappa}, \quad \sigma = \frac{2a + 3b + c}{2\kappa}.$$

This completes the proof of the theorem. □

7 Examples of $MG\left(QE\right)_4$

In this section, we show the existence of $MG\left(QE\right)_4$ by constructing two non-trivial concrete examples.

Example 7.1. Let $\left(x^1, x^2, ..., x^n\right) \in \mathbb{R}^n$, where $\mathbb{R}^n$ an n-dimensional real number space. We consider a Riemannian metric g on $\mathbb{R}^4 = \left(x^1, x^2, x^3, x^4\right)$, by [10]

$$ds^2 = g_{ij}dx^i dx^j = \left(dx^1\right)^2 + \left(x^1\right)^2 \left(dx^2\right)^2 + \left(x^2\right)^2 \left(dx^3\right)^2 + \left(dx^4\right)^2, \tag{7.1}$$

where $i, j = 1, 2, 3, 4$. Using (7.1), we see the non-vanishing components of Riemannian metric are

$$g_{11} = 1, \;\; g_{22} = \left(x^1\right)^2, \;\; g_{33} = \left(x^2\right)^2, \;\; g_{44} = 1 \tag{7.2}$$

and its associated components are

$$g^{11} = 1, \;\; g^{22} = \frac{1}{\left(x^1\right)^2}, \;\; g^{33} = \frac{1}{\left(x^2\right)^2}, \;\; g^{44} = 1. \tag{7.3}$$

With the help of (7.2) and (7.3), it can be calculated that the non-vanishing components of Christoffel symbols, curvature tensor and Ricci tensor are given by

$$\Gamma^1_{22} = -x^1, \;\; \Gamma^2_{33} = -\frac{x^2}{\left(x^1\right)^2}, \;\; \Gamma^2_{12} = \frac{1}{x^1}, \;\; \Gamma^3_{23} = \frac{1}{x^2},$$

$$R_{1332} = -\frac{x^2}{x^1}, \;\; S_{12} = -\frac{1}{x^1 x^2}$$

and the other components are obtained by the symmetric properties. It can be easily shown that the scalar curvature r of the resulting manifold $\left(\mathbb{R}^4, g\right)$ is zero. We shall show that $\left(\mathbb{R}^4, g\right)$ is a $MG\left(QE\right)_4$.
Let us consider the associated scalars as follows:

$$a = \frac{1}{x^1 \left(x^2\right)^2}, \;\; b = -\frac{1}{\left(x^2\right)^3}, \;\; c = \frac{1}{\left(x^2\right)^4}, \;\; e = -\frac{2}{\left(x^1\right)^2 x^2}. \tag{7.4}$$

We choose the 1-form as follows:

$$A_i\left(x\right) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{when } i = 1 \\ \frac{x^2}{\sqrt{2}}, & \text{when } i = 3 \\ 0, & \text{otherwise} \end{cases} \tag{7.5}$$

and

$$B_i\left(x\right) = \begin{cases} \frac{x^1}{\sqrt{2}}, & \text{when } i = 2 \\ \frac{1}{\sqrt{2}}, & \text{when } i = 4 \\ 0, & \text{otherwise} \end{cases} \tag{7.6}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.4) reduces to the equation

$$S_{12} = ag_{12} + bA_1A_2 + cB_1B_2 + e\left(A_1B_2 + A_2B_1\right), \tag{7.7}$$

since, for the other cases (1.4) holds trivially.
From the equations (7.4), (7.5), (7.6) and (7.7) we get

$$\begin{aligned}
\text{Right hand side of (7.7)} &= ag_{12} + bA_1A_2 + cB_1B_2 + e\left(A_1B_2 + A_2B_1\right)\\
&= \frac{1}{x^1\left(x^2\right)^2}\cdot 0 + \left(-\frac{1}{\left(x^2\right)^3}\right)\cdot\frac{1}{\sqrt{2}}\cdot 0 + \frac{1}{\left(x^2\right)^4}\cdot 0\cdot\frac{x^1}{\sqrt{2}}\\
&\quad + \left(-\frac{2}{\left(x^1\right)^2 x^2}\right)\left(\frac{1}{\sqrt{2}}\cdot\frac{x^1}{\sqrt{2}} + 0\right)\\
&= -\frac{1}{x^1x^2} = S_{12}.
\end{aligned}$$

We shall now show that the associated vectors A_i and B_i are unit and also they are orthogonal.
Here,

$$g^{ij}A_iA_j = 1, \;\; g^{ij}B_iB_j = 1, \;\; g^{ij}A_iB_j = 0.$$

So, $\left(\mathbb{R}^4, g\right)$ is a $MG\left(QE\right)_4$.

Example 7.2. Let $\left(x^1, x^2, ..., x^n\right) \in \mathbb{R}^n$, where $\mathbb{R}^n$ denotes n-dimensional real number space. We consider a Lorentzian metric g on $\mathbb{R}^4 = \left(x^1, x^2, x^3, x^4; x^1 \neq \frac{(1+2p)\,\pi}{4}, p \in \mathbb{Z}\right)$, ($\mathbb{Z}$ is the set of positive integer), by [7]

$$ds^2 = g_{ij}dx^idx^j = \left\{\sin\left(x^1\right) - \cos\left(x^1\right)\right\}\left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right] - \left(dx^4\right)^2, \tag{7.8}$$

where $i, j = 1, 2, 3, 4$. Using (7.8), we see the non-vanishing components of the Lorentzian metric are

$$g_{11} = g_{22} = g_{33} = \sin\left(x^1\right) - \cos\left(x^1\right), \;\; g_{44} = -1 \tag{7.9}$$

and its associated components are

$$g^{11} = g^{22} = g^{33} = \frac{1}{\sin\left(x^1\right) - \cos\left(x^1\right)}, \;\; g^{44} = -1. \tag{7.10}$$

With the help of (7.9) and (7.10), it can be found that the non-vanishing components of Christoffel symbols, curvature tensor, Ricci tensor and scalar curvature are given by

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \frac{\sin\left(x^1\right) + \cos\left(x^1\right)}{2\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)}, \;\; \Gamma^1_{22} = \Gamma^1_{33} = \frac{\sin\left(x^1\right) + \cos\left(x^1\right)}{2\left(\cos\left(x^1\right) - \sin\left(x^1\right)\right)},$$

$$R_{1331} = \frac{1}{\cos\left(x^1\right) - \sin\left(x^1\right)}, \;\; S_{33} = \frac{-3 + \sin 2\left(x^1\right)}{4\left(1 - \sin 2\left(x^1\right)\right)},$$

$$r = \frac{-3 + \sin 2\left(x^1\right)}{4\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)^3} \,(\neq 0) \tag{7.11}$$

and the other components are obtained by the symmetric properties. From (7.11), it is clear that the manifold $\left(\mathbb{R}^4, g\right)$ is a Lorentzian manifold. Now, we are to prove that $\left(\mathbb{R}^4, g\right)$ is a $MG\left(QE\right)_4$.
Let us consider the associated scalars as follows:

(7.12)

$$a = \frac{\sin 2\left(x^1\right)}{4\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)^3}, \quad b = \frac{1}{\sin\left(x^1\right) - \cos\left(x^1\right)}, \quad c = -\frac{3}{4\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)^3},$$
$$e = -\frac{1}{2\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)^2}.$$

We choose the 1-form as follows:

$$A_i\left(x\right) = \begin{cases} \sqrt{\sin\left(x^1\right) - \cos\left(x^1\right)}, & \text{when } i = 1 \\ 0, & \text{otherwise} \end{cases} \tag{7.13}$$

and

$$B_i\left(x\right) = \begin{cases} \sqrt{\sin\left(x^1\right) - \cos\left(x^1\right)}, & \text{when } i = 3 \\ 0, & \text{otherwise} \end{cases} \tag{7.14}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.4) reduces to the equation

$$S_{33} = ag_{33} + bA_3A_3 + cB_3B_3 + 2eA_3B_3, \tag{7.15}$$

since, for the other cases (1.4) holds trivially.
From the equations (7.12), (7.13), (7.14) and (7.15) we get

$$\begin{aligned} \text{Right hand side of (7.15)} &= ag_{33} + bA_3A_3 + cB_3B_3 + 2eA_3B_3 \\ &= \frac{\sin 2\left(x^1\right)}{4\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)^2} + 0 - \frac{3}{4\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)^2} - 0 \\ &= \frac{-3 + \sin 2\left(x^1\right)}{4\left(1 - \sin 2\left(x^1\right)\right)} = S_{33}. \end{aligned}$$

We shall now show that the 1-forms are unit and orthogonal.
Here,

$$g^{ij}A_iA_j = 1, \quad g^{ij}B_iB_j = 1, \quad g^{ij}A_iB_j = 0.$$

So, $\left(\mathbb{R}^4, g\right)$ is a $MG\left(QE\right)_4$.

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