Geometry of tangent bundles with the horizontal Sasaki gradient metric

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Abstract. In this paper, we introduce the horizontal Sasaki gradient metric on the tangent bundle TM over an m-dimensional Riemannian manifold (M, g) as a new metric with respect to g non-rigid on TM. First, we investigate the Levi-Civita connection and we characterize the sectional curvature, the scalar curvature for the horizontal Sasaki gradient metric.

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1 Introduction

We recall some basic facts about the geometry of the tangent bundle. In the present paper, we denote by $\Gamma(TM)$ the space of all vector fields of a Riemannian manifold (M, g).
Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle.

A local chart $(U, x^i)_{i=1\dots n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1\dots n}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.
We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by

$$\mathcal{V}_{(x,u)} = \ker(d\pi_{(x,u)}) = \Big\{ a^i \frac{\partial}{\partial y^i}|_{(x,u)}; a^i \in \mathbb{R} \Big\}$$
$$\mathcal{H}_{(x,u)} = \Big\{ a^i \frac{\partial}{\partial x^i}_{|(x,u)} - a^i y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}_{|(x,u)}; a^i \in \mathbb{R} \Big\}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.
Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}$$

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$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \Big\{ \frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \Big\} \tag{1.1}$$

For consequences, we have $\Big(\frac{\partial}{\partial x^i}\Big)^H = \frac{\delta}{\delta x^i}$ and $\Big(\frac{\partial}{\partial x^i}\Big)^V = \frac{\partial}{\partial y^i}$; then $\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\Big)_{i=1..n}$ is a local adapted frame in TTM.

The geometry of the tangent bundle of a Riemannian manifold (M, g) is very important in many areas of mathematics and physics. In recent years, a lot of studies about their local or global geometric properties have been published in the literature. When the authors studied this topic, they used different metrics which are called natural metrics on the tangent bundle. Firstly, the geometry of a tangent bundle has been studied by using a new metric g^s, which is called Sasaki metric, with the aid of a Riemannian metric g on a differential manifold M in 1958 by Sasaki [17]. It is uniquely determined by

$$\begin{array}{rcl} g^s(X^H, Y^H) & = & g(X, Y) \circ \pi \\ g^s(X^H, Y^V) & = & 0 \\ g^s(X^V, Y^V) & = & g(X, Y) \circ \pi \end{array} \tag{1.2}$$

for all vector fields X and Y on M. More intuitively, the metric g^s is constructed in such a way that the vertical and horizontal sub bundles are orthogonal and the bundle map $\pi : (TM, g^s) \longrightarrow (M, g)$ is a Riemannian submersion.

After that, the tangent bundle could be split to its horizontal and vertical subbundles with the aid of the Levi Civita connection ∇ on (M, g). Later, the Lie bracket of the tangent bundle TM, the Levi Civita connection ∇^s on TM and its Riemannian curvature tensor R^s have been obtained in [8] and [12]. Furthermore, were derived explicit formulas of another natural metric g_{CG}, which is called Cheeger-Gromoll metric on the tangent bundle TM of a Riemannian manifold (M, g) and which is uniquely determined by

$$\begin{aligned} g_{CG}(X^H, Y^H) &= g(X, Y) \circ \pi \\ g_{CG}(X^H, Y^V) &= 0 \\ g_{CG}(X^V, Y^V) &= \frac{1}{\alpha}\{g(X, Y) + g(X, u)g(Y, u)\} \circ \pi, \end{aligned} \tag{1.3}$$

where $X, Y \in \Gamma(TM)$, $(x, u) \in TM$, $\alpha = 1 + g_x(u, u)$. This metric has been given by Musso and Tricerri in [13], using Cheeger and Gromoll's study [7]. The Levi Civita connection ∇^{CG} and the Riemannian curvature tensor R^{CG} of (TM, g_{CG}) have been obtained in [18] and [11], respectively. The sectional curvatures and the scalar curvature of this metric were derived as well. These results were completed in 2002 by S. Gudmundson and E. Kappos in [11]. They have also shown that the scalar curvature of the Cheeger Gromoll metric is never constant if the metric on the base manifold has constant sectional curvature. Furthermore, in [2] M.T.K. Abbassi, M. Sarih have proved that TM with the Cheeger Gromoll metric is never a space of constant sectional curvature. A more general metric was given by M. Anastasiei in [1], which generalizes both metrics mentioned above in the following sense: it preserves the orthogonality of the two distributions, on the horizontal distribution it coincides with the one from the base manifold, and finally the Sasaki and the Cheeger-Gromoll metric can be obtained as particular cases of this metric. A compatible almost complex structure is also introduced and hence TM becomes an locally conformal almost Käherian manifold. V. Oproiu and his collaborators constructed a family of Riemannian metrics on the tangent bundles of Riemannian manifolds which possess interesting geometric properties (see [15], [16]). In particular, the scalar curvature of TM can be constant also for a non-flat base manifold with constant sectional curvature. Then M.T.K. Abbassi and M.Sarih proved in [3] that the considered metrics by Oproiu form a particular subclass of the so-called g-natural metrics on the tangent bundle. Recently, the geometry of the tangent bundles with Cheeger-Gromoll metric has been studied by many mathematicians (see [2], [14], [19], etc).

Zayatuev in [24] introduced a Riemannian metric on TM given by

$$\begin{aligned} g_f^s(X^H, Y^H) &= f(p)g_p(X,Y) \\ g_f^s(X^H, Y^V) &= 0 \\ g_f^s(X^V, Y^V) &= g_p(X,Y) \end{aligned} \tag{1.4}$$

for all vector fields X and Y on (M, g), where f is strictly positive smooth function on (M, g). In [21] J. Wang, Y. Wang called g_f^s the rescaled Sasaki metric and studied the geometry of TM endowed with g_f^s.

H. M. Dida, F. Hathout in [9], defined a new class of naturally metric on TM given by

$$\begin{aligned} G^f_{(p,u)}(X^H, Y^H) &= g_p(X,Y) \\ G^f_{(p,u)}(X^H, Y^V) &= 0 \\ G^f_{(p,u)}(X^V, Y^V) &= f(p)g_p(X,Y) \end{aligned} \tag{1.5}$$

for some strictly positive smooth function f in (M, g) and any vector fields X and Y on M. We call G^f a vertical rescaled metric.

L. Belarbi, H. El Hendi in [4], defined a new class of natural metrics on TM, given by

$$\begin{aligned} G^{f,h}_{(p,u)}(X^H, Y^H) &= f(p)g_p(X,Y) \\ G^{f,h}_{(p,u)}(X^V, Y^H) &= 0 \\ G^{f,h}_{(p,u)}(X^V, Y^V) &= h(p)g_p(X,Y) \end{aligned} \tag{1.6}$$

where f, h are strictly positive smooth functions on M and X, Y are vector fields on M. For $h = 1$, the metric $G^{f,h}$ is exactly the rescaled Sasaki metric. If $f = 1$, then the metric $G^{f,h}$ is exactly the vertical rescaled metric. We call $G^{f,h}$ the twisted Sasaki metric.

L. Belarbi, H. El Hendi in [5], defined a new class of natural metrics on TM, given by

$$
\begin{aligned}
g^f(X^H, Y^H)_{(x,u)} &= g_x(X,Y) \\
g^f(X^V, Y^H)_{(x,u)} &= 0 \qquad\qquad (1.7)\\
g^f(X^V, Y^V)_{(x,u)} &= g_x(X,Y) + X_x(f)Y_x(f),
\end{aligned}
$$

where f is a strictly positive smooth function on M and X, Y are vector fields on M. We call g^f the gradient Sasaki metric. If f is constant, then the metric g^f is exactly the Sasaki metric.

The main idea in this note consists in the modification of the gradient Sasaki metric. Firstly, we introduce a new metric called the horizontal Sasaki gradient metric on the tangent bundle TM; then we establish the Levi-Civita connection of this metric (Theorem 2.3 and Proposition 2.4) we investigate the Riemannian curvature (Theorem 3.1) and characterize the sectional curvature (Theorem 3.4 and Proposition 3.5), and also the scalar curvature (Theorem 3.8 and Proposition 3.9).

1.1 Basic notions and definition on TM

Let (M, g) be an m-dimensional Riemannian manifold and let (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=\overline{1,n}}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=\overline{1,n}}$ on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by:

$$
\begin{aligned}
\mathcal{V}_{(x,u)} &= Ker(d\pi_{(x,u)}) = \{a^i \frac{\partial}{\partial y^i}|_{(x,u)},\ a^i \in \mathbb{R}\}, \\
\mathcal{H}_{(x,u)} &= \{a^i \frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}|_{(x,u)},\ a^i \in \mathbb{R}\},
\end{aligned}
$$

where $(x,u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \tag{1.8}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{\frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}\}, \tag{1.9}$$

As consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=\overline{1,n}}$ forming a local adapted frame on TTM.

If $w = w^i \frac{\partial}{\partial x^i} + \overline{w}^j \frac{\partial}{\partial x^j} \in T_{(x,u)}TM$, then its horizontal and vertical parts are defined by

$$w^h = w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)}, \tag{1.10}$$

$$w^v = (\overline{w}^k + w^i u^j \Gamma_{ij}^k)\frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}. \tag{1.11}$$

Proposition 1.1. *[10] Let (M, g) be a Riemannian manifold and R its tensor curvature; then for all vector fields $X, Y \in \Gamma(TM)$ we have:*

1. $[X^H, Y^H]_p = [X, Y]_p^H - (R_x(X, Y)u)^V$,
2. $[X^H, Y^V]_p = (\nabla_X Y)_p^V$,
3. $[X^V, Y^V]_p = 0$,

where $p = (x, u) \in TM$.

Definition 1.1. Let (M, g) be a Riemannian manifold and $K : TM \longrightarrow TM$ be a smooth bundle endomorphism of TM. Then the vertical and horizontal vector fields VK and HK are defined on TM by

$$\begin{array}{rcl} VK : TM & \rightarrow & TTM \\ (x, u) & \mapsto & (K(u))^V, \end{array}$$

$$\begin{array}{rcl} HK : TM & \rightarrow & TTM \\ (x, u) & \mapsto & (K(u))^H, \end{array}$$

locally we have

$$VK = y^i K_i^j \frac{\partial}{\partial y^j} = y^i (K(\frac{\partial}{\partial x^i}))^V, \tag{1.12}$$

$$HK = y^i K_i^j \frac{\partial}{\partial x^j} - y^i y^k K_i^j \Gamma_{jk}^s \frac{\partial}{\partial y^s} = y^i (K(\frac{\partial}{\partial x^i}))^H. \tag{1.13}$$

2 Horizontal Sasaki gradient metric

Definition 2.1. Let (M, g) be a Riemannian manifold and $f : M \rightarrow]0, +\infty[$. On the tangent bundle TM, we define a horizontal Sasaki gradient metric noted g_f^H by

1. $g_f^H(X^H, Y^H)_{(x,u)} = g_x(X, Y) + X_x(f)Y_x(f)$,
2. $g_f^H(X^H, Y^V)_{(x,u)} = 0$,
3. $g_f^H(X^V, Y^V)_{(x,u)} = g_x(X, Y)$,

where $X, Y \in \Gamma(TM)$, $(x, u) \in TM$.

Remark 2.2.
1. If f is constant, then g_f^H is the Sasaki metric [22],
2. $g_f^H(X^V, (grad\, f)^V) = g(X, grad\, f) = X(f)$,
3. $g_f^H(X^H, (grad\, f)^H) = (1 + \|grad\, f\|^2)X(f) = \alpha X(f)$,

where $X, Y \in \Gamma(TM)$ and $\alpha = 1 + \|grad\, f\|^2$, $\|.\|$ denotes the norm with respect to (M, g).

In the following, we consider $\alpha = 1 + \|grad\, f\|^2$.

Lemma 2.1. *Let (M, g) be a Riemannian manifold; then we have the following*

1. $X^V g_f^H(Y^V, Z^V) = 0$,
2. $X^V g_f^H(Y^H, Z^H) = 0$,
3. $X^H g_f^H(Y^V, Z^V) = Xg(Y, Z)$,
4. $X^H g_f^H(Y^H, Z^H) = Xg(Y, Z) + X(Y(f)).Z(f) + Y(f).X(Z(f))$,

where $X, Y, Z \in \Gamma(TM)$.

Proof. Then the claim follows from Definition 2.1. □

2.1 The Levi-Civita connection

We shall calculate the Levi-Civita connection ∇^f of TM endowed with the horizontal Sasaki gradient metric g_f^H. This connection is characterized by the Koszul formula:

$$\begin{aligned}2g_f^H(\nabla^f_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}) \;\; = \;\; & \widetilde{X}g_f^H(\widetilde{Y}, \widetilde{Z}) + \widetilde{Y}g_f^H(\widetilde{Z}, \widetilde{X}) - \widetilde{Z}g_f^H(\widetilde{X}, \widetilde{Y}) \\ & + g_f^H(\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]) + g_f^H(\widetilde{Y}, [\widetilde{Z}, \widetilde{X}]) - g_f^H(\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]).\end{aligned} \tag{2.1}$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(TM)$.

Lemma 2.2. *Let (M, g) be a Riemannian manifold and let (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. If ∇ (resp. ∇^f) denote the Levi-Civita connection of (M, g) (resp. (TM, g_f^H)), then we have:*

$$\begin{aligned}
1)\; g_f^H(\nabla^f_{X^H}Y^H, Z^H) =& g_f^H\big((\nabla_X Y)^H + \frac{1}{\alpha}g(Y, \nabla_X grad\, f)(grad\, f)^H, Z^H\big),\\
2)\; g_f^H(\nabla^f_{X^H}Y^H, Z^V) =& -\frac{1}{2}g_f^H\big((R(X, Y)u)^V, Z^V\big),\\
3)\; g_f^H(\nabla^f_{X^H}Y^V, Z^H) =& -\frac{1}{\alpha}g(R(u, Y)X, grad\, f)g_f^H\big((grad\, f)^H, Z^H\big)\\
&+\frac{1}{2}g_f^H\big((R(u, Y)X)^H, Z^H\big),\\
4)\; g_f^H(\nabla^f_{X^H}Y^V, Z^V) =& g_f^H\big((\nabla_X Y)^V, Z^V\big),\\
5)\; g_f^H(\nabla^f_{X^V}Y^H, Z^H) =& -\frac{1}{\alpha}g(R(u, X)Y, grad\, f)g_f^H\big((grad\, f)^H, Z^H\big)\\
&+\frac{1}{2}g_f^H\big((R(u, X)Y)^H, Z^H\big),\\
6)\; g_f^H(\nabla^f_{X^V}Y^H, Z^V) =& 0,\\
7)\; g_f^H(\nabla^f_{X^V}Y^V, Z^H) =& 0,\\
8)\; g_f^H(\nabla^f_{X^V}Y^V, Z^V) =& 0.
\end{aligned}$$

for all vector fields $X, Y \in \Gamma(TM)$, where R denotes the curvature tensor of (M, g).

Proof. The proof follows directly from the Kozul formula (2.1), Proposition 1.1, Definition 2.1 and Lemma 2.1.

1) The statement is obtained as follows.

$$
\begin{aligned}
2g_f^H(\nabla^f_{X^H}Y^H, Z^H) =& X^H g_f^H(Y^H, Z^H) + Y^H g_f^H(Z^H, X^H) - Z^H g_f^H(X^H, Y^H) \\
&+ g_f^H(Z^H, [X^H, Y^H]) + g_f^H(Y^H, [Z^H, X^H]) \\
&- g_f^H(X^H, [Y^H, Z^H]) \\
=& Xg(Y,Z) + X(Y(f)).Z(f) + Y(f).X(Z(f)) + Yg(Z,X) \\
&+ Y(Z(f)).X(f) + Z(f).Y(X(f)) - Zg(X,Y) \\
&- Z(X(f)).Y(f) - X(f).Z(Y(f)) + g_f^H(Z^H, [X,Y]^H) \\
&+ g_f^H(Y^H, [Z,X]^H) - g_f^H(X^H, [Y,Z]^H) \\
=& Xg(Y,Z) + X(Y(f)).Z(f) + Yg(Z,X) + Z(f).Y(X(f)) \\
&- Zg(X,Y) + g(Z, [X,Y]) + Z(f).[X,Y](f) \\
&+ g(Y, [Z,X]) - g(X, [Y,Z]) \\
=& 2g(\nabla_X Y, Z) + 2X(Y(f)).Z(f) \\
=& 2[g_f^H((\nabla_X Y)^H, Z^H) - \nabla_X Y(f).Z(f) + X(Y(f)).Z(f)]
\end{aligned}
$$

which implies that

$$
\begin{aligned}
2g_f^H(\nabla^f_{X^H}Y^H, Z^H) =& 2[g_f^H((\nabla_X Y)^H, Z^H) + \frac{1}{\alpha} g(Y, \nabla_X grad\, f) g_f^H((grad\, f)^H, Z^H)] \\
=& 2g_f^H\big((\nabla_X Y)^H + \frac{1}{\alpha} g(Y, \nabla_X grad\, f)(grad\, f)^H, Z^H\big).
\end{aligned}
$$

2) Direct calculations give

$$
\begin{aligned}
2g_f^H(\nabla^f_{X^H}Y^H, Z^V) =& X^H g_f^H(Y^H, Z^V) + Y^H g_f^H(Z^V, X^H) - Z^V g_f^H(X^H, Y^H) \\
&+ g_f^H(Z^V, [X^H, Y^H]) + g_f^H(Y^H, [Z^V, X^H]) \\
&- g_f^H(X^H, [Y^H, Z^V]) \\
=& g_f^H(Z^V, [X^H, Y^H]) \\
=& -g_f^H((R(X,Y)u)^V, Z^V)
\end{aligned}
$$

The other formulas are obtained by a similar calculation. □

As a direct consequence of Lemma 2.2, we get the following theorem.

Theorem 2.3. *Let (M, g) be a Riemannian manifold and (TM, g_f^H) its tangent bundle equipped with the horizontal Sasaki gradient metric. If ∇ (resp ∇^f) denote the*

Levi-Civita connection of (M,g) (resp. (TM, g_f^H)), then we have:

$$\begin{aligned}
1.\ (\nabla^f_{X^H}Y^H)_p &= (\nabla_X Y)^H_p + \frac{1}{\alpha} g_x(Y, \nabla_X grad\, f)(grad\, f)^H_P \\
&\quad -\frac{1}{2}(R_x(X,Y)u)^V, \\
2.\ (\nabla^f_{X^H}Y^V)_p &= (\nabla_X Y)^V_p - \frac{1}{2\alpha} g_x(R(u,Y)X, grad\, f)(grad\, f)^H_p \\
&\quad +\frac{1}{2}(R_x(u,Y)X)^H, \\
3.\ (\nabla^f_{X^V}Y^H)_p &= \frac{1}{2}(R_x(u,X)Y)^H - \frac{1}{2\alpha} g_x(R(u,X)Y, grad\, f)(grad\, f)^H_p, \\
4.\ (\nabla^f_{X^V}Y^V)_p &= 0,
\end{aligned}$$

for all vector fields $X, Y \in \Gamma(TM)$ and $p = (x,u) \in TM$, where R denote the curvature tensor of (M,g).

From Definition 1.1 and Theorem 2.3, we have:

Proposition 2.4. *Let (M,g) be a Riemannian manifold and let (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. If ∇ (resp. ∇^f) denote the Levi-Civita connection of (M,g) (resp. (TM, g_f^H)) and K is a tensor field of type $(1,1)$ on M, then:*

$$\begin{aligned}
1)\ (\nabla^f_{X^H}HK)_p &= H(\nabla_X K)_p + \frac{1}{\alpha} g_x(K(u), \nabla_X grad\, f)(grad\, f)^H_P \\
&\quad -\frac{1}{2}\big(R_x(X, K(u))u\big)^V,
\end{aligned}$$

$$\begin{aligned}
2)\ (\nabla^f_{X^H}VK)_p &= V(\nabla_X K)_p - \frac{1}{2\alpha} g_x(R(u,K(u))X, grad\, f)(grad\, f)^H_p \\
&\quad +\frac{1}{2}(R_x(u,K(u))X)^H,
\end{aligned}$$

$$\begin{aligned}
3)\ (\nabla^f_{X^V}HK)_p &= (K(X))^H_p - \frac{1}{2\alpha} g_x(R(u,X)K(u), grad\, f)(grad\, f)^H_p \\
&\quad +\frac{1}{2}\big(R_x(u,X)K(u)\big)^H, \\
4)\ (\nabla^f_{X^V}VK)_p &= (K(X))^V_p,
\end{aligned}$$

where $p = (x,u) \in TM$, $X \in \Gamma(TM)$, and R denotes the curvature tensor of (M,g).

Proof. Let $p = (x,u) \in TM$, $u = u^i \frac{\partial}{\partial x^i}$ and let $U = u^i \frac{\partial}{\partial x^i}$ be a constant vector field.

By Definition 2.1 and Theorem 2.3, we have:

$$\begin{aligned}
1)\,(\nabla^f_{X^H} HK)_p &= \Big[\nabla^f_{X^H} y^k (K(\frac{\partial}{\partial x^k}))^H\Big]_p \\
&= \Big[X^H(y^k)(K(\frac{\partial}{\partial x^k}))^H + y^k \nabla^f_{X^H} K(\frac{\partial}{\partial_k})^H\Big]_p \\
&= -X^i u^j \Gamma^k_{ij} (K(\frac{\partial}{\partial x^k}))^H_p + u^k(\nabla_X K(\frac{\partial}{\partial_k}))^H_p \\
&\quad + \frac{u^k}{\alpha} g_x(K(\frac{\partial}{\partial x^k}), \nabla_X grad\, f)(grad\, f)^H_P \\
&\quad - \frac{u^k}{2}\big(R_x(X, K(\frac{\partial}{\partial x^k}))u\big)^V \\
&= -(K(\nabla_X U))^H_p + (\nabla_X K(U))^H_p - \frac{1}{2}\big(R_x(X, K(u))u\big)^V \\
&\quad + \frac{1}{\alpha} g_x(K(u), \nabla_X grad\, f)(grad\, f)^H_P \\
&= ((\nabla_X K)(U))^H_p + \frac{1}{\alpha} g_x(K(u), \nabla_X grad\, f)(grad\, f)^H_P \\
&\quad - \frac{1}{2}\big(R_x(X, K(u))u\big)^V \\
&= (H(\nabla_X K))_p + \frac{1}{\alpha} g_x(K(u), \nabla_X grad\, f)(grad\, f)^H_P \\
&\quad - \frac{1}{2}\big(R_x(X, K(u))u\big)^V.
\end{aligned}$$

$$\begin{aligned}
2)\,(\nabla^f_{X^H} VK)_p &= \Big[\nabla^f_{X^H} y^k (K(\frac{\partial}{\partial x^k}))^V\Big]_p \\
&= \Big[X^H(y^k)(K(\frac{\partial}{\partial x^k}))^V + y^k \nabla^f_{X^H} K(\frac{\partial}{\partial_k})^V\Big]_p \\
&= -X^i u^j \Gamma^k_{ij} (K(\frac{\partial}{\partial x^k}))^V_p + u^k(\nabla_X K(\frac{\partial}{\partial x^k}))^V \\
&\quad - \frac{u^k}{2\alpha} g_x(R(u, K(\frac{\partial}{\partial x^k}))X, grad\, f)(grad\, f)^H_p \\
&\quad + \frac{u^k}{2}(R_x(u, K(\frac{\partial}{\partial x^k}))X)^H
\end{aligned}$$

$$\begin{aligned}
&= -(K(\nabla_X U))^V_p + (\nabla_X K(U))^V_p + \frac{1}{2}(R_x(u, K(u))X)^H \\
&\quad -\frac{1}{2\alpha} g_x(R(u, K(u))X, grad\, f)(grad\, f)^H_p \\
&= ((\nabla_X K)(U))^V_p + \frac{1}{2}(R_x(u, K(u))X)^H \\
&\quad -\frac{1}{2\alpha} g_x(R(u, K(u))X, grad\, f)(grad\, f)^H_p \\
&= V(\nabla_X K)_p + \frac{1}{2}(R_x(u, K(u))X)^H \\
&\quad -\frac{1}{2\alpha} g_x(R(u, K(u))X, grad\, f)(grad\, f)^H_p .
\end{aligned}$$

The other formulas are obtained by a similar calculation. □

3 Curvatures of horizontal Sasaki gradient metric

3.1 The Riemannian curvature

We shall calculate the Riemannian curvature tensor of TM with the horizontal Sasaki gradient metric g^H_f.

Theorem 3.1. *Let (M, g) be a Riemannian manifold and (TM, g^H_f) its tangent bundle equipped with the horizontal Sasaki gradient metric. If R (resp. R^f) denote the Riemann curvature tensor of (M, g) (resp. (TM, g^H_f)), then we have the following formula:*

$$R^f_p(X^V, Y^V)Z^V = 0. \tag{3.1}$$

$$\begin{aligned}
R^f_p(X^H, Y^V)Z^V = \Big[&\frac{1}{4\alpha} g(R(u,Y)R(u,Z)X, grad\, f)grad\, f \\
&+ \frac{1}{4\alpha} g(R(u,Z)X, grad\, f)R(u,Y)grad\, f \\
&- \frac{1}{2}R(Y,Z)X - \frac{1}{4}R(u,Y)R(u,Z)X\Big]^H_x .
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
R^f_p(X^V, Y^V)Z^H = \Big[&R(X,Y)Z + \frac{1}{4}R(u,X)R(u,Y)Z - \frac{1}{4}R(u,Y)R(u,X)Z \\
&+ \frac{1}{4\alpha} g(R(u,Y)R(u,X)Z, grad\, f)grad\, f \\
&- \frac{1}{4\alpha} g(R(u,X)R(u,Y)Z, grad\, f)grad\, f \\
&+ \frac{1}{4\alpha} g(R(u,X)Z, grad\, f)R(u,Y)grad\, f \\
&- \frac{1}{4\alpha} g(R(u,Y)Z, grad\, f)R(u,X)grad\, f\Big]^H_x .
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
R_p^f(X^H,Y^V)Z^H = \Big[&\frac{1}{2\alpha}g(R(u,Y)grad\,f,Z)\nabla_X grad\,f\\
&-\frac{1}{2\alpha}g(Z,\nabla_X grad\,f)R(u,Y)grad\,f\\
&+\frac{1}{2}(\nabla_X R)(u,Y)Z+\frac{X(\alpha)}{4\alpha^2}g(R(u,Y)Z),grad\,f)grad\,f\\
&-\frac{1}{2\alpha}g((\nabla_X R)(u,Y)Z+R(\nabla_X U,Y)Z,grad\,f)grad\,f\Big]_x^H\\
&+\Big[\frac{1}{2}R(X,Z)Y-\frac{1}{4}R(X,R(u,Y)Z)u\\
&+\frac{1}{4\alpha}g(R(u,Y)Z,grad\,f))R(X,grad\,f)u\Big]_x^V.
\end{aligned}\tag{3.4}$$

$$\begin{aligned}
R_p^f(X^H,Y^H)Z^V = \Big[\big[&\frac{1}{2\alpha}g(R(\nabla_Y U,Z)X-R(\nabla_X U,Z)Y,grad\,f)\\
&+\frac{X(\alpha)}{4\alpha^2}g(R(u,Z)Y,grad\,f)-\frac{Y(\alpha)}{4\alpha^2}g(R(u,Z)X,grad\,f)\\
&+\frac{1}{2\alpha}g((\nabla_Y R)(u,Z)X-(\nabla_X R)(u,Z)Y,grad\,f)\big]grad\,f\\
&+\frac{1}{2}(\nabla_X R)(u,Z)Y-\frac{1}{2\alpha}g(R(u,Z)Y,grad\,f)\nabla_X grad\,f\\
&-\frac{1}{2}(\nabla_Y R)(u,Z)X+\frac{1}{2\alpha}g(R(u,Z)X,grad\,f)\nabla_Y grad\,f\Big]_x^H\\
&+\Big[R(X,Y)Z-\frac{1}{4}(R(X,R(u,Z)Y)u+\frac{1}{4}R(Y,R(u,Z)X)u\\
&+\frac{1}{4\alpha}g(R(u,Z)Y,grad\,f)R(X,grad\,f)u\\
&-\frac{1}{4\alpha}g(R(u,Z)X,grad\,f)R(Y,grad\,f)u\Big]_x^V.
\end{aligned}\tag{3.5}$$

$$\begin{aligned}
R_p^f(X^H,Y^H)Z^H = \Big[\big[&\frac{1}{\alpha}g(R(X,Y)grad\,f,Z)-\frac{1}{2\alpha}g(R(u,R(X,Y)u)Z,grad\,f)\\
&+\frac{1}{4\alpha}g(R(u,R(Y,Z)u)X-R(u,R(X,Z)u)Y,grad\,f)\\
&+\frac{Y(\alpha)}{2\alpha^2}g(X,\nabla_Z grad\,f)-\frac{X(\alpha)}{2\alpha^2}g(Y,\nabla_Z grad\,f)\big]grad\,f\\
&+\frac{1}{\alpha}g(Y,\nabla_Z grad\,f)\nabla_X grad\,f+\frac{1}{4}R(u,R(X,Z)u)Y\\
&-\frac{1}{\alpha}g(X,\nabla_Z grad\,f)\nabla_Y grad\,f-\frac{1}{4}R(u,R(Y,Z)u)X\\
&+R(X,Y)Z+\frac{1}{2}R(u,R(X,Y)u)Z\Big]_x^H\\
&+\Big[\frac{1}{2}(\nabla_Z R)(X,Y)u-\frac{1}{2\alpha}g(Y,\nabla_Z grad\,f)R(X,grad\,f)u\\
&+\frac{1}{2\alpha}g(X,\nabla_Z grad\,f)R(Y,grad\,f)u\Big]_x^V.
\end{aligned}\tag{3.6}$$

for all $(x,u)\in TM,\ X,Y,Z,U\in\Gamma(TM)$ *and* $U_x=u$.

Proof. Let $p=(x,u)\in TM,\ X,Y,Z,U\in\Gamma(TM)$ and $U_x=u$ is a constant vector field. By applying Definition 2.1, Lemma 2.1, Theorem 2.3 and Proposition 2.4 we have:

1) Since $\nabla^f_{X^V}Y^V=0$, Hence

$$R^f(X^V,Y^V)Z^V=\nabla^f_{X^V}\nabla^f_{Y^V}Z^V-\nabla^f_{Y^V}\nabla^f_{X^V}Z^V-\nabla^f_{[X^V,Y^V]}Z^V=0.$$

2)$R^f(X^H,Y^V)Z^V=\nabla^f_{X^H}\nabla^f_{Y^V}Z^V-\nabla^f_{Y^V}\nabla^f_{X^H}Z^V-\nabla^f_{[X^H,Y^V]}Z^V$

$i)$ We have $\nabla^f_{X^H}\nabla^f_{Y^V}Z^V=0$.

$ii)$ Let K be the bundle endomorphism given by

$$\begin{aligned}K:TM&\to TM\\ u&\mapsto R(u,Z)X\end{aligned}$$

From direct calculation we get,

$$\begin{aligned}\nabla^f_{Y^V}\nabla^f_{X^H}Z^V=&\nabla^f_{Y^V}\big(\nabla_XZ\big)^V+\frac{1}{2}\nabla^f_{Y^V}\big[HK\big]\\ &-\nabla^f_{Y^V}\big[\frac{1}{2\alpha}g(R(u,Z)X,grad\,f)(grad\,f)^H\big]\\ &\frac{1}{2}(R(Y,Z)X)^H+\frac{1}{4}(R(u,Y)R(u,Z)X)^H\\ &-\frac{1}{4\alpha}g(R(u,Y)R(u,Z)X,grad\,f)(grad\,f)^H,\\ &+\frac{1}{4\alpha}g(R(u,Z)X,grad\,f)(R(u,Y)grad\,f)^H.\end{aligned}$$

$iii)$ We have

$$\nabla^f_{[X^H,Y^V]}Z^V=\nabla^f_{(\nabla_XY)^V}Z^V=0,$$

which gives,

$$\begin{aligned}R^f(X^H,Y^V)Z^V=&\Big[\frac{1}{4\alpha}g(R(u,Y)R(u,Z)X,grad\,f)grad\,f\\ &+\frac{1}{4\alpha}g(R(u,Z)X,grad\,f)R(u,Y)grad\,f\\ &-\frac{1}{2}R(Y,Z)X-\frac{1}{4}R(u,Y)R(u,Z)X\Big]^H.\end{aligned}$$

for all $,X,Y,Z\in\Gamma(TM)$.

3) Applying formula (3.2) and 1^{st} Bianchi identity.

$$R^f(X^V,Y^V)Z^H=R^f(Z^H,Y^V)X^V-R^f(Z^H,X^V)Y^V,$$

we get

$$\begin{aligned}R^f(Z^H,Y^V)X^V=&\Big[\frac{1}{4\alpha}g(R(u,Y)R(u,X)Z,grad\,f)grad\,f\\ &+\frac{1}{4\alpha}g(R(u,X)Z,grad\,f)R(u,Y)grad\,f\\ &-\frac{1}{2}R(Y,X)Z-\frac{1}{4}R(u,Y)R(u,X)Z\Big]^H,\end{aligned}$$

and

$$\begin{aligned}R^f(Z^H,X^V)Y^V = \Big[&\frac{1}{4\alpha}g(R(u,X)R(u,Y)Z, grad\, f)grad\, f\\ &+\frac{1}{4\alpha}g(R(u,Y)Z, grad\, f)R(u,X)grad\, f\\ &-\frac{1}{2}R(X,Y)Z-\frac{1}{4}R(u,X)R(u,Y)Z\Big]^H,\end{aligned}$$

which gives,

$$\begin{aligned}R^f(X^V,Y^V)Z^H = \Big[&R(X,Y)Z+\frac{1}{4}R(u,X)R(u,Y)Z-\frac{1}{4}R(u,Y)R(u,X)Z\\ &+\frac{1}{4\alpha}g(R(u,Y)R(u,X)Z, grad\, f)grad\, f\\ &-\frac{1}{4\alpha}g(R(u,X)R(u,Y)Z, grad\, f)grad\, f\\ &+\frac{1}{4\alpha}g(R(u,X)Z, grad\, f)R(u,Y)grad\, f\\ &-\frac{1}{4\alpha}g(R(u,Y)Z, grad\, f)R(u,X)grad\, f\Big]^H.\end{aligned}$$

for all , $X,Y,Z\in\Gamma(TM)$.
The other formulas are obtained by a similar calculation. □

3.2 Sectional curvature

In the following let $Q^f(V,W)$ denote the square of the area of the parallelogram with sides V and W for $V,W\in\Gamma(TTM)$, given by

$$Q^f(V,W)=g_f^H(V,V)g_f^H(W,W)-g_f^H(V,W)^2. \tag{3.7}$$

Let G^f be the $(2,0)$-tensor on the tangent bundle TM, for $V,W\in\Gamma(TTM)$ given by

$$G^f(V,W)=g_f^H(R^f(V,W)W,V). \tag{3.8}$$

Let further $p\in TM$, V_p and W_p be linearly independent, and let

$$K^f(V_p,W_p)=\frac{G^f(V_p,W_p)}{Q^f(V_p,W_p)}, \tag{3.9}$$

be the sectional curvature of the plane spanned by V_p and W_p.

Lemma 3.2. *Let (M,g) be a Riemannian manifold and let (TM,g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. Then for any vector fields $X,Y\in\Gamma(TM)$, we have*

$$\begin{aligned}I)\ Q^f(X^H,Y^H) &= Q(X,Y)+\|X\|^2[Y(f)]^2+\|Y\|^2[X(f)]^2\\ &\quad -2g(X,Y)X(f)Y(f),\\ II)\ Q^f(X^H,Y^V) &= [\|X\|^2+[X(f)]^2]\|Y\|^2,\\ III)\ Q^f(X^V,Y^V) &= Q(X,Y),\end{aligned}$$

where $Q(X,Y)=\|X\|^2\|Y\|^2-g(X,Y)^2$.

Proof. Direct calculations give

$$\begin{aligned}
I)\; Q^f(X^H,Y^H) &= g^H_f(X^H,X^H)g^H_f(Y^H,Y^H) - g^H_f(X^H,Y^H)^2 \\
&= \big(g(X,X)+[X(f)]^2\big)\big(g(Y,Y)+[Y(f)]^2\big) \\
&\quad -[g(X,Y)+X(f)Y(f)]^2 \\
&= \big(\|X\|^2+[X(f)]^2\big)\big(\|Y\|^2+[Y(f)]^2\big) \\
&\quad -[g(X,Y)+X(f)Y(f)]^2 \\
&= Q(X,Y)+\|X\|^2[Y(f)]^2+\|Y\|^2[X(f)]^2 \\
&\quad -2g(X,Y)X(f)Y(f).
\end{aligned}$$

$$\begin{aligned}
II)\; Q^f(X^H,Y^V) &= g^H_f(X^H,X^H)g^H_f(Y^V,Y^V) - g^H_f(X^H,Y^V)^2 \\
&= \big(g(X,X)+X(f)X(f)\big)g(Y,Y) \\
&= [\|X\|^2+[X(f)]^2]\|Y\|^2.
\end{aligned}$$

$$\begin{aligned}
III)\; Q^f(X^V,Y^V) &= g^H_f(X^V,X^V)g^H_f(Y^V,Y^V) - g^H_f(X^V,Y^V)^2 \\
&= g(X,X)g(Y,Y)-g(X,Y)^2 \\
&= \|X\|^2\|Y\|^2-g(X,Y)^2 \\
&= Q(X,Y).
\end{aligned}$$

□

Lemma 3.3. *Let (M,g) be a Riemannian manifold and let (TM,g^H_f) be its tangent bundle equipped with the horizontal Sasaki gradient metric. Then for any vector fields $X,Y\in\Gamma(TM)$, we have*

$$\begin{aligned}
i)\; G^f(X^H,Y^H) &= g(R(X,Y)Y,X)-\frac{3}{4}\|R(X,Y)u\|^2-\frac{1}{\alpha}[g(X,\nabla_Y grad\, f)]^2 \\
&\quad +\frac{1}{\alpha}g(X,\nabla_X grad\, f)g(Y,\nabla_Y grad\, f), \\
ii)\; G^f(X^H,Y^V) &= \frac{1}{4}\|R(u,Y)X\|^2-\frac{1}{4\alpha}[g(R(u,Y)X,grad\, f)]^2, \\
iii)\; G^f(X^V,Y^V) &= 0.
\end{aligned}$$

Proof. *i*) From formula (3.6), we have

$$G^f(X^H,Y^H) \;=\; g^H_f(R^f(X^H,Y^H)Y^H,X^H)$$

$$\begin{aligned}
G^f(X^H, Y^H) &= \frac{1}{\alpha} g(R(X,Y)grad\, f, Y)\alpha X(f) \\
&-\frac{1}{2\alpha} g(R(u, R(X,Y)u)Y, grad\, f)\alpha X(f) \\
&+\frac{1}{4\alpha} g(R(u, R(Y,Y)u)X, grad\, f)\alpha X(f) \\
&-\frac{1}{4\alpha} g(R(u, R(X,Y)u)Y, grad\, f)\alpha X(f) \\
&+\frac{Y(\alpha)}{2\alpha^2} g(X, \nabla_Y grad\, f)\alpha X(f) \\
&-\frac{X(\alpha)}{2\alpha^2} g(Y, \nabla_Y grad\, f)\alpha X(f) \\
&+\frac{1}{\alpha} g(Y, \nabla_Y grad\, f) g(\nabla_X grad\, f, X) \\
&+\frac{1}{\alpha} g(Y, \nabla_Y grad\, f) g(\nabla_X grad\, f, grad\, f)X(f) \\
&-\frac{1}{\alpha} g(X, \nabla_Y grad\, f) g(\nabla_Y grad\, f, X) \\
&-\frac{1}{\alpha} g(X, \nabla_Y grad\, f) g(\nabla_Y grad\, f, grad\, f)X(f)] \\
&+g(R(X,Y)Y, X) + g(R(X,Y)Y, grad\, f)X(f) \\
&+\frac{1}{2} g(R(u, R(X,Y)u)Y, X) \\
&+\frac{1}{2} g(R(u, R(X,Y)u)Y, grad\, f)X(f) \\
&+\frac{1}{4} g(R(u, R(X,Y)u)Y, X) \\
&+\frac{1}{4} g(R(u, R(X,Y)u)Y, grad\, f)X(f) \\
&-\frac{1}{4} g(R(u, R(Y,Y)u)X, X) \\
&-\frac{1}{4} g(R(u, R(Y,Y)u)X, grad\, f)X(f),
\end{aligned}$$

which implies that

$$\begin{aligned}
G^f(X^H, Y^H) &= g(R(X,Y)Y, X) - \frac{3}{4}\|R(X,Y)u\|^2 - \frac{1}{\alpha}[g(X, \nabla_Y grad\, f)]^2 \\
&+\frac{1}{\alpha} g(X, \nabla_X grad\, f) g(Y, \nabla_Y grad\, f).
\end{aligned}$$

ii) From formula (3.2), we have

$$G^f(X^H, Y^V) = g_f^H(R^f(X^H, Y^V)Y^V, X^H)$$

$$\begin{aligned} G^f(X^H, Y^V) &= \frac{1}{4\alpha} g(R(u,Y)R(u,Y)X, grad\, f)\alpha X(f) \\ &\quad -\frac{1}{4} g(R(u,Y)R(u,Y)X, X) \\ &\quad -\frac{1}{4} g(R(u,Y)R(u,Y)X, grad\, f)X(f) \\ &\quad +\frac{1}{4\alpha} g(R(u,Y)X, grad\, f)g(R(u,Y)grad\, f, X) \\ &= \frac{1}{4}\|R(u,Y)X\|^2 - \frac{1}{4\alpha}[g(R(u,Y)X, grad\, f)]^2. \end{aligned}$$

$iii)$ The result follows immediately from formula (3.1)

$$G^f(X^V, Y^V) \;=\; g_f^H(R^f(X^V, Y^V)Y^V, X^V) = 0.$$

□

Theorem 3.4. *Let (M, g) be a Riemannian manifold and let (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. If K (resp., K^f) denote the sectional curvature tensor of (M, g) (resp., (TM, g_f^H)), then for any vector fields $X, Y \in \Gamma(TM)$, we have*

$$\begin{aligned} (1)\, K^f(X^H, Y^H) &= \frac{1}{Q^f(X^H, Y^H)}\Big[Q(X,Y)K(X,Y) - \frac{3}{4}\|R(X,Y)u\|^2 \\ &\quad -\frac{1}{\alpha}[g(X, \nabla_Y grad\, f)]^2 \\ &\quad +\frac{1}{\alpha} g(X, \nabla_X grad\, f)g(Y, \nabla_Y grad\, f)\Big], \\ (2)\, K^f(X^H, Y^V) &= \frac{1}{[\|X\|^2 + [X(f)]^2]\|Y\|^2}\Big[\frac{1}{4}\|R(u,Y)X\|^2 \\ &\quad -\frac{1}{4\alpha}[g(R(u,Y)X, grad\, f)]^2\Big], \\ (3)\, K^f(X^V, Y^V) &= 0. \end{aligned}$$

Here we assumed that X and Y are linearly independent in (1).

Proof. The division of $G^f(X^i, Y^j)$ by $Q^f(X^i, Y^j)$ for $i, j \in \{H, V\}$ gives the result. □

Proposition 3.5. *Let (M, g) be a Riemannian manifold of constant sectional curvature λ and let (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. If K^f denotes the sectional curvature tensor of (TM, g_f^H), then for* *any orthonormal vector fields $X, Y \in \Gamma(TM)$, we have*

$$\begin{aligned}
(1)\, K^f(X^H, Y^H) &= \frac{1}{1+[X(f)]^2+[Y(f)]^2}\Big[\lambda - \frac{3\lambda^2}{4}\big[g(X,u)^2+g(Y,u)^2\big] \\
&\quad -\frac{1}{\alpha}[g(X, \nabla_Y grad\, f)]^2 \\
&\quad +\frac{1}{\alpha} g(X, \nabla_X grad\, f) g(Y, \nabla_Y grad\, f)\Big], \\
(2)\, K^f(X^H, Y^V) &= \frac{\lambda^2 g(X,u)^2(1-[Y(f)]^2)}{4\alpha(1+[X(f)]^2)}, \\
(3)\, K^f(X^V, Y^V) &= 0,
\end{aligned}$$

Proof. for any orthonormal vector fields $X, Y \in \Gamma(TM)$,

$$Q(X,Y) = 1\ ,\ Q^f(X^H, Y^H) = 1 + [X(f)]^2 + [Y(f)]^2.$$

Since M has constant curvature $K(X,Y) = \lambda$, then

$$R(X,Y)u = \lambda\big[g(Y,u)X - g(X,u)Y\big],$$

and direct calculations lead to

$$\begin{aligned}
[g(R(u,Y)X, grad\, f)]^2 &= \lambda^2 g(X,u)^2[Y(f)]^2, \\
\|R(X,Y)u\|^2 &= \lambda^2\big[g(X,u)^2 + g(Y,u)^2\big], \\
\|R(u,Y)X\|^2 &= \lambda^2 g(X,u)^2,
\end{aligned}$$

then the result. □

Remark 3.1. Let (M,g) be an m-dimensional Riemannian manifold and let (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. If $(E_i)_{i=\overline{1,m}}$ a local orthonormal frame on M such that $E_1 = \dfrac{grad\, f}{\alpha - 1}$, $\alpha \neq 1$, then, we get an orthonormal frame $(F_a)_{a=\overline{1,2m}}$ on TM, where

$$F_1 = \frac{1}{\sqrt{\alpha}} E_1^H\ \ ,\ \ F_i = E_i^H \ \ and\ \ F_{m+j} = E_j^V\ \ ,\ \ i = \overline{2,m}\ \ ,\ \ j = \overline{1,m}.$$

Then the following Lemma is a direct consequence of Remark 3.1

Lemma 3.6. *Let (M,g) be a m-dimensional Riemannian manifold and let (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. If $(E_i)_{i=\overline{1,m}}$ is a local orthonormal frame on M and $(F_a)_{a=\overline{1,2m}}$ is a local orthonormal frame on TM defined by Remark* 3.1*, then*

1) $Q^f(F_a, F_b) = 1\ \ ,\ \ K^f(F_a, F_b) = G^f(F_a, F_b)\ \ ,\ \ a, b = \overline{1,2m}$ *et* $a \neq b$,

2) $E_1(f) = \sqrt{\alpha - 1}\ \ ,\ \ E_j(f) = 0\ \ ,\ \ j = \overline{2,m}$,

3) $2g(\nabla_{E_i} grad\, f, grad\, f) = E_i(\alpha)\ \ ,\ \ i = \overline{1,m}$.

Lemma 3.7. *Let (M, g) be a m-dimensional Riemannian manifold and let (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. If $(E_i)_{i=\overline{1,m}}$ (resp., $(F_a)_{a=\overline{1,2m}}$) are local orthonormal frames on (M resp., TM), then the sectional curvature K^f satisfies the following equations for $i, j = \overline{2,m}$ and $k, l = \overline{1,m}$*

$$\begin{aligned}
K^f(F_i, F_j) &= K(E_i, E_j) - \frac{3}{4}\|R(E_i, E_j)u\|^2 - \frac{1}{\alpha}[g(E_i, \nabla_{E_j} grad\, f)]^2 \\
&\quad + \frac{1}{\alpha} g(E_i, \nabla_{E_i} grad\, f) g(E_j, \nabla_{E_j} grad\, f), \\
K^f(F_1, F_j) &= \frac{1}{\alpha} K(E_1, E_j) - \frac{3}{4\alpha}\|R(E_1, E_j)u\|^2 - \frac{1}{\alpha^2}[g(E_1, \nabla_{E_j} grad\, f)]^2 \\
&\quad + \frac{1}{\alpha^2} g(E_1, \nabla_{E_1} grad\, f) g(E_j, \nabla_{E_j} grad\, f), \\
K^f(F_i, F_{m+k}) &= \frac{1}{4}\|R(u, E_k)E_i\|^2 - \frac{1}{4\alpha}[g(R(u, E_k)E_i, grad\, f)]^2, \\
K^f(F_1, F_{m+k}) &= \frac{1}{4\alpha}\|R(u, E_k)E_1\|^2, \\
K^f(F_{m+l}, F_{m+k}) &= 0.
\end{aligned}$$

Proof. The statement is a direct consequence of Theorem 3.4 and Lemma 3.6. □

3.3 The scalar curvature

Theorem 3.8. *Let (M, g) be a m-dimensional Riemannian manifold and let (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. If σ (resp., σ^f) denote the scalar curvature of (M, g) (resp., (TM, g_f^H)), then we have*

$$\begin{aligned}
\sigma^f &= \sigma - \frac{1}{4}\sum_{i,j=1}^{m}\|R(E_i, E_j)u\|^2 - \frac{1}{2\alpha}\sum_{i,j=1}^{m}[g(R(E_i, E_j)u, grad\, f)]^2 + \frac{1}{\alpha}\Delta(f)^2 \\
&\quad - \frac{2}{\alpha}\sum_{j=1}^{m} g(R(grad\, f, E_j)E_j, grad\, f) - \frac{\alpha+2}{2\alpha(\alpha-1)}\sum_{j=1}^{m}\|R(u, E_j)grad\, f\|^2 \\
&\quad - \frac{1}{\alpha}\sum_{i,j=2}^{m}[g(\nabla_{E_i} grad\, f, E_j)]^2 - \frac{1}{\alpha^2} g(grad\, f, grad\, \alpha)\Delta(f) + \frac{1}{\alpha^2}\|grad\, \alpha\|^2.
\end{aligned}$$

where $(E_i)_{i=\overline{1,m}}$ is a local orthonormal frame of (M, g) and Δ is the Laplacian of f.

Proof.

$$
\begin{aligned}
\sigma^f &= \sum_{\substack{i,j=1\\ i\neq j}}^{m} K^f(F_i,F_j) + 2\sum_{i,j=1}^{m} K^f(F_i,F_{m+j}) + \sum_{\substack{i,j=1\\ i\neq j}}^{m} K^f(F_{m+i},F_{m+j})\\
&= \sum_{\substack{i,j=2\\ i\neq j}}^{m} K^f(F_i,F_j) + 2\sum_{j=1}^{m} K^f(F_1,F_j) + 2\sum_{j=1}^{m} K^f(F_1,F_{m+j})\\
&\quad + 2\sum_{i=2,j=1}^{m} K^f(F_i,F_{m+j})\\
&= \sum_{\substack{i,j=2\\ i\neq j}}^{m} \Big[K(E_i,E_j) - \frac{3}{4}\|R(E_i,E_j)u\|^2 - \frac{1}{\alpha}[g(E_i,\nabla_{E_j} grad\, f)]^2\\
&\quad + \frac{1}{\alpha} g(E_i,\nabla_{E_i} grad\, f) g(E_j,\nabla_{E_j} grad\, f)\Big]\\
&\quad + 2\sum_{j=1}^{m}\Big[\frac{1}{\alpha}K(E_1,E_j) - \frac{3}{4\alpha}\|R(E_1,E_j)u\|^2 - \frac{1}{\alpha^2}[g(E_1,\nabla_{E_j} grad\, f)]^2\\
&\quad + \frac{1}{\alpha^2} g(E_1,\nabla_{E_1} grad\, f) g(E_j,\nabla_{E_j} grad\, f)\Big] + 2\sum_{j=1}^{m}\frac{1}{4\alpha}\|R(u,E_j)E_1\|^2\\
&\quad + 2\sum_{i=2,j=1}^{m}\Big[\frac{1}{4}\|R(u,E_j)E_i\|^2 - \frac{1}{4\alpha}[g(R(u,E_j)E_i, grad\, f)]^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&=\sigma - 2\sum_{j=1}^{m} K(E_1,E_j) - \frac{3}{4}\sum_{i,j=1}^{m}\|R(E_i,E_j)u\|^2 - \frac{1}{\alpha}\sum_{i,j=1}^{m}[g(E_i,\nabla_{E_j} grad\, f)]^2\\
&\quad + \frac{1}{\alpha}\sum_{j=1}^{m}[g(E_1,\nabla_{E_j} grad\, f)]^2 + \frac{1}{\alpha}\sum_{i,j=1}^{m} g(E_i,\nabla_{E_i} grad\, f) g(E_j,\nabla_{E_j} grad\, f)\\
&\quad - \frac{1}{\alpha}\sum_{j=1}^{m} g(E_1,\nabla_{E_1} grad\, f) g(E_j,\nabla_{E_j} grad\, f) + \frac{2}{\alpha}\sum_{j=1}^{m} K(E_1,E_j)\\
&\quad - \frac{3}{2\alpha}\sum_{j=1}^{m}\|R(E_1,E_j)u\|^2 - \frac{2}{\alpha^2}\sum_{j=1}^{m}[g(E_1,\nabla_{E_j} grad\, f)]^2\\
&\quad + \frac{2}{\alpha^2} g(E_1,\nabla_{E_1} grad\, f)\sum_{j=1}^{m} g(E_j,\nabla_{E_j} grad\, f) + \frac{1}{2\alpha}\sum_{j=1}^{m}\|R(u,E_j)E_1\|^2\\
&\quad + \frac{1}{2}\sum_{i,j=1}^{m}\|R(u,E_j)E_i\|^2 - \frac{1}{2}\sum_{j=1}^{m}\|R(u,E_j)E_1\|^2
\end{aligned}
$$

$$- \frac{1}{2\alpha} \sum_{i,j=1}^{m} [g(R(u, E_j)E_i, grad\, f)]^2$$

In order to simplify this last expression, we put $u = \sum_{j=1}^{m} u_i E_i$ and get

$$\begin{aligned}
\sum_{i,j=1}^{m} \|R(u, E_j)E_i\|^2 &= \sum_{i,j,k,l=1}^{m} u_k u_l g(R(E_k, E_j)E_i, R(E_l, E_j)E_i) \\
&= \sum_{i,j,k,l,s=1}^{m} u_k u_l g(R(E_k, E_j)E_i, E_s) g(R(E_l, E_j)E_i, E_s) \\
&= \sum_{i,j,k,l,s=1}^{m} u_k u_l g(R(E_i, E_s)E_k, E_j) g(R(E_i, E_s)E_l, E_j) \\
&= \sum_{i,j,s=1}^{m} g(R(E_i, E_s)u, g(R(E_i, E_s)u, E_j)E_j) \\
&= \sum_{i,j=1}^{m} \|R(E_i, E_j)u\|^2.
\end{aligned}$$

Then, using the formula

$$\nabla_{grad\, f} grad\, f = \frac{1}{2} grad\, \alpha,$$

we infer

$$\begin{aligned}
\sigma^f &= \sigma - \frac{1}{4} \sum_{i,j=1}^{m} \|R(E_i, E_j)u\|^2 - \frac{1}{2\alpha} \sum_{i,j=1}^{m} [g(R(E_i, E_j)u, grad\, f)]^2 + \frac{1}{\alpha} \Delta(f)^2 \\
&\quad - \frac{2}{\alpha} \sum_{j=1}^{m} g(R(grad\, f, E_j)E_j, grad\, f) - \frac{\alpha + 2}{2\alpha(\alpha - 1)} \sum_{j=1}^{m} \|R(u, E_j) grad\, f\|^2 \\
&\quad - \frac{1}{\alpha} \sum_{i,j=2}^{m} [g(\nabla_{E_i} grad\, f, E_j)]^2 - \frac{1}{\alpha^2} g(grad\, f, grad\, \alpha)\Delta(f) + \frac{1}{\alpha^2} \|grad\, \alpha\|^2.
\end{aligned}$$

□

Proposition 3.9. *Let (M, g) be a m-dimensional Riemannian manifold of constant sectional curvature λ and (TM, g_f^H) its tangent bundle equipped with the horizontal Sasaki gradient metri. Si σ^f denote the scalar curvature of (TM, g_f^H), then, we have*

$$\begin{aligned}
\sigma^f &= \Big[\frac{\alpha(3 - m) + 3 - 2m}{2\alpha(\alpha - 1)} g(u, grad\, f)^2 - \frac{\alpha(m + 3) + 2}{2\alpha} \|u\|^2\Big]\lambda^2 \\
&\quad + \frac{\alpha(m - 2) + 2}{\alpha} (m - 1)\lambda + \frac{1}{\alpha^2} \|grad\, \alpha\|^2 \\
&\quad - \frac{1}{\alpha} \sum_{i,j=2}^{m} [g(\nabla_{E_i} grad\, f, E_j)]^2 + \frac{1}{\alpha} \Delta(f)^2 - \frac{1}{\alpha^2} g(grad\, f, grad\, \alpha)\Delta(f),
\end{aligned}$$

where $(E_i)_{i=\overline{1,m}}$ is a local orthonormal frame of (M, g).

Proof. Using the formulas of curvature and scalar curvature of a Riemannian manifold with constant sectional curvature λ we have, for all vector fields $X, Y, Z \in TM$,

$$R(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)Y]$$

and

$$\sigma = m(m-1)\lambda.$$

Hence

$$\sum_{i,j=1}^{m} \|R(E_i,E_j)u\|^2 = 2(m-1)\lambda^2\|u\|^2,$$

$$\sum_{i,j=1}^{m} [g(R(E_i,E_j)u, grad\, f)]^2 = 2\lambda^2[(\alpha-1)\|u\|^2 - g(u, grad\, f)^2],$$

$$\sum_{j=1}^{m} g(R(grad\, f, E_j)E_j, grad\, f) = (m-1)(\alpha-1)\lambda,$$

$$\sum_{j=1}^{m} \|R(u,E_j)grad\, f\|^2 = 2\lambda^2[(\alpha-1)\|u\|^2 + (m-2)g(u, grad\, f)^2].$$

From Theorem (3.8), we deduce the result. □

3.4 Curvature properties.

We shall now compare the geometries of the manifold (M, g) of and its tangent bundle (TM, g_f^H) equipped with the horizontal Sasaki gradient metric.

Theorem 3.10. *Let (M, g) be a Riemannian manifold, let (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric and let f be a constant. Then TM is flat if and only if M is flat.*

Proof. If f is constant, then from Theorem 3.1, we have that $R = 0$ implies $R^f = 0$. Applying formula (3.6), for all $X, Y, Z \in \Gamma(TM)$ and $x \in M$, we have

$$R^f_{(x,0)}(X^H, Y^H)Z^H = (R_x(X,Y)Z)^H.$$

Then $R^f = 0$ implies $R = 0$. □

Corollary 3.11. *Let (M, g) be a Riemannian manifold, let (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric, and let f be nonconstant. Then TM flat implies M flat.*

Proof. Applying formula (3.3), for all $X, Y, Z \in \Gamma(TM)$ and $x \in M$, we have

$$R^f_{(x,0)}(X^V, Y^V)Z^H = (R_x(X,Y)Z)^H;$$

then $R^f = 0$ implies $R = 0$. $\square$

Corollary 3.12. *Let (M, g) be a Riemannian manifold, let (TM, g^H_f) be its tangent bundle equipped with the horizontal Sasaki gradient metric and let f be constant. Then (TM, g^H_f) has constant scalar curvature if and only if the scalar curvature is zero.*

Proof. The result is an immediate consequence of Theorem 3.8. $\square$

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References

[1] M. Anastasiei, *Locally conformal Kaehler structures on tangent manifold of a space form*, Libertas Math., 19 (1999), 71-76.

[2] M.T.K. Abbassi, Sarih. M,*Killing vector fields on tangent bundles with Cheeger-Gromoll metric*, Tsukuba J. Math., 27 (2) (2003), 295-306.

[3] M. T. K. Abbassi, M. Sarih, *On natural metrics on tangent bundles of Riemannian manifolds*, Archivum Mathematicum (Brno), 41 (2005), 71-92.

[4] L. Belarbi, H. El Hendi, *Geometry of twisted Sasaki metric*, J. Geom. Symmetry Phys., 53 (2019), 1-19.

[5] L. Belarbi, H. El Hendi, *On the geometry of the tangent bundle with gradient Sasaki metric*, Arab Journal Math. Sci., https://www.emerald.com/insight/1319-5166.htm

[6] H. El Hendi, L. Belarbi, *Naturally harmonic maps between tangent bundles*, Balkan Journal of Geometry and Its Applications, 25, 1 (2020), 34-46.

[7] J. Cheeger, D. Gromoll, *On the structure of complete manifolds of non negative curvature*, Ann. of Math., 96 (2) (1972), 413-443.

[8] P. Dombrowski, *On the geometry of the tangent bundle*, J. Reine Angew. Math., 210 (1962), 73-88.

[9] H. M. Dida, F. Hathout and A. Azzouz, *On the geometry of the tangent bundle with vertical rescaled metric*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 68 (1) (2019), 222-235.

[10] S. Gudmundsson, E. Kappos, *On the Geometry of the tangent bundles*, Expo. Math., 20 (1) (2002), 1-41.

[11] S. Gudmundsson, E. Kappos, *On the geometry of the tangent bundle with the Cheeger-Gromoll metric*, Tokyo J. Math., 25(1) (2002), 75-83.

[12] O. Kowalski, *Curvature of the induced Riemannian metric of the tangent bundle of a Riemannian manifold*, J. Reine Angew. Math., 250 (1971), 124-129.

[13] E. Musso, F. Tricerri, *Riemannian metrics on tangent bundles*, Ann. Mat. Pura Appl., 150 (4) (1988), 1-19.

[14] M. I. Munteanu, *Cheeger-Gromoll type metrics on the tangent bundle*, Sci. Ann. Univ. Agric. Sci. Vet. Med., 49 (2)(2006), 257-268.

[15] V. Oproiu, *Some new geometric structures on the tangent bundle*, Publ. Math. Debrecen, 55 (3-4) (1999), 261-281.

[16] V. Oproiu, N. Papaghiuc,*General natural Einstein Kahler structures on tangent bundles*, Differential Geom. Appl., 27 (2009), 384-392.

[17] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tohoku. Math. J., 10(3)(1958), 338-354.

[18] M. Sekizawa, *Curvatures of tangent bundles with Cheeger-Gromoll metric*, Tokyo J. Math., 14 (2)(1991), 407-417.

[19] A. A. Salimov, K. Akbulut, *A note on a paraholomorphic Cheeger-Gromoll metric*, Proc. Indian Acad. Sci. (Math. Sci.), 119 (2) (2009), 187-195.

[20] I. Vaisman, *From generalized Kähler to generalized Sasakian structures*, J.G.S.P., 18 (2010), 63-86.

[21] J. Wang, Y. Wang, *On the geometry of tangent bundles with the rescaled metric*, arXiv:1104.5584v1[math.DG] 29 Apr 2011.

[22] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*, Marcel Dekker Inc., New York, 1973.

[23] B. V. Zayatuev, *On geometry of tangent Hermitian surface*, Webs and Quasigroups, T.S.U, (1995), 139-143.

[24] A. Zagane, H. El Hendi, *Harmonic vector fields on vertical rescaled generalized Cheeger-Gromoll metrics*, Balkan Journal of Geometry and Its Applications, 25, 2 (2020), 140-156.

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