Ricci solitons and non-existence of a parallel 2-form on Lorentzian alpha-r-Sasakian manifold with a coefficient α

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Abstract. The geometry of Sasakian manifold with pseudo Riemannian metric was developed and studied by Toshio [15] and Matsumoto [12]. In 1926, Levy [11] had proved that a second order parallel non singular tensor on a space of constant curvature is a constant multiple of metric tensor. Sharma [14] has proved that a second order parallel tensor in a Kaehler space of constant holomorphic sectional curvature is a linear combination with constant coefficient of the Kaehler metric and the fundamental 2-form. In this paper, we have shown that a second order symmetric parallel tensor on Lorentzian alpha-r-Sasakian manifold (briefly Lα-r Sasakian) with a coefficient α (non zero scalar function) is a constant multiple of the associated metric tensor and we have also proved that there is no non zero skew symmetric second order parallel tensor on a Lα-r Sasakian manifolds with a coefficient α.

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1 Introduction

In 1923, Eisenhart [9] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of metric is reducible. In 1926 Levy [11] obtained the necessary and sufficient conditions for the existence of such tensors. Sharma [14] has generalized Levy's result by showing that a second order parallel (not necesarily symmetric and non-singular) tensor on an n-dimensional ($n > 2$) space of constant curvature is a constant multiple of the metric tensor. Recently the author [5] has proved that on a Para r-Sasakian manifold with a coefficient α, a second order symmetric parallel tensor is a constant multple of the associated positive definite Riemannian metric tensor. In this paper, we have extended these ideas further and have defined Lorentzian alpha r-Sasakian (briefly Lα-r-Sasakian) manifold with a coefficient α (non-zero scalar function) and have proved the following two theorems:

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Theorem 1.1. *On a regular Lorentzian α-r-Sasakian manifold with a coefficient α, a second order symmetric parallel tensor is a constant multiple of the associated metric tensor.*

Theorem 1.2. *On a regular Lorentzian α-r-Sasakian manifold with a coefficient α, there is no non zero parallel 2-forms.*

Inspired by the works of Hamilton [10] towards the solution of the Poincare conjecture about the characterization of 3-sphere, many geometers have engaged themselves in providing the solutions of solitons of the Ricci flow.

The notion of a soliton structure on the Riemannian manifold (M, g) is the choice of a smooth vector field V on M and a real constant λ satisfying the structural requirement.

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor of the metric g and $\pounds_V g$ is the Lie Derivative of this latter in the direction of V and λ is referred to as the solition constant. The Ricci soliton is called expanding, steady or shinking if $\lambda > 0, \lambda = 0$ or $\lambda < 0$ respectively. In this paper, we prove that the tensor field $\pounds_V g + 2S$ on a Lorentzian α-r-Sasakian manifold with a coefficient α is parallel then (g, V, λ) is a Ricci soliton.

2 Preliminaries

Let M^{2n+r} be an $(2n+r)$-dimensional differentiable manifold equipped with the ring of real valued differentiable functions $\mathcal{F}(M)$ and the module of derivations $\mathfrak{X}(M)$ and a (1,1) tensor field ϕ as a linear map such that $\phi : \mathfrak{X}(M) \to \mathfrak{X}(M)$. Let there be $r(C^\infty)$, 1- forms $A_1,\ A_2,\ A_3,\ ...,\ A_n$ and $r(C^\infty)$ contravariant vector fields $T^1, T^2...T^n$ satisfying the following conditions

$$A_p(T^p) = -1, \quad \text{where } p = 1, 2...n \tag{2.1a}$$

$$A_p(T^q) = \delta^p_q, \quad p, q = 1, 2...n \tag{2.1b}$$

where

$$\delta^p_q = \begin{cases} 1, \text{if } p = q \\ 0, \text{if } p \neq q \end{cases}$$

$$\phi(T^p) = 0 \quad \text{for } p = 1, 2...n \tag{2.2}$$

$$A_p(\phi X) = 0 \quad \text{for } p = 1, 2...n \tag{2.3}$$

$$\phi^2 X = X + A_p(X) T^p \quad \text{for } p = 1, 2...n, \tag{2.4}$$

for any vector field $X \in \mathfrak{X}(M)$. Here the summation convention is employed on repeated indices, where $p = 1, 2...n$. If moreover M^{2n+r} admits a Lorentzian metric g such that

$$A_p(X) = g(X, T^p) \quad \text{for} \quad X \in \mathfrak{X}(M) \tag{2.5}$$

$$g(\phi X, \phi Y) = g(X, Y) + A_p(X) A_p(Y), \tag{2.6}$$

for any vector field X and $Y \in \mathfrak{X}(M)$.

(2.7a) $$\phi(X,Y) = g(X,\phi Y) = g(Y,\phi X)$$

(2.7b) $$\phi(X,T^q) = 0$$

Definition 2.1. If in a Lorentzian alpha r-Sasakian manifold ,the following relations

(2.8a) $$\alpha\phi X = -\nabla_X T^p$$

(2.8b) $$\nabla_X A_p(Y) = -\alpha g(\phi X,Y) = -\alpha\phi(X,Y)$$

(2.9a) $$\alpha X = -\nabla_X\alpha = g(X,\overline{\alpha})$$

(2.9b) $$(\nabla_X\phi)(Y,Z) = \alpha[\{g(X,Y) + A_p(X)A_p(Y)\}A_p(Z) + \{g(X,Z) + A_p(X)A_p(Z)\}A_p(Y)]$$

hold for arbitrary smooth vector fields $X,Y,Z \in \mathfrak{X}(M)$, where ∇ denotes the Riemannian connection of the metric tensor g then M^{2n+r} satisfying conditions (2.1a) - (2.9b) is called a Lorentzian Alpha r-Sasakian manifold with a coefficient α. For the sake of simplicity we will denote M^{2n+r} by M, in what follows.

3 Proofs of Theorems 1.1 and 1.2

For proving theorems 1.1 and 1.2, we need the following theorems.

Theorem 3.1. *On a Lorentzian α-r-Sasakian manifold the following holds*

(3.1) $$A_p(R(X,Y)Z) = \alpha^2[g(Y,Z)A_p(X) - g(X,Z)A_p(Y)] - [\alpha(X)\phi(Y,Z) - \alpha(Y)\phi(X,Z)]$$

Proof. In view of (2.8a), (2.8b) and (2.9b) the proof follows easily. □

Theorem 3.2. *For a Lorentzian α-r-Sasakian manifold with coefficient α, we have*

(3.2) $$R(T^p,X)Y = \alpha^2[g(X,Y)T^p - A_p(Y)X] + \alpha(Y)\phi X - \overline{\alpha}\phi(X,Y),$$

where $g(X,\overline{\alpha}) = \alpha(X)$.

Proof. The proof follows in an obvious manner after making use of (2.9a) and (3.1). □

Theorem 3.3. *For a Lorentzian α-r-Sasakian manifold with coefficient α, the following holds*

$$R(T^p,X)T^p = \beta\phi X + \alpha^2[X + A_p(X)T^p], \text{ for } p = 1,2,...,n$$

where $\alpha(T^p) = \beta$.

Proof. In view of equation (3.2), the proof follows in an obvious manner. □

4 Second order parallel symmetric tensors and Ricci solitons

Proof of Theorem 1.1. Let r denote a $(0,2)$-tensor field on a Lorentzian α-r-Sasakian manifold M with coefficient α such that

$$h(R(W,X)Y,Z) + h(Y,R(W,X)Z) = 0, \tag{4.1}$$

for arbitrary vector fields X, Y, Z, W on M. Substituting $W = Y = Z = T^q$ in (4.1), we get

$$g(R(T^q,X)T^q,T^q) + g(T^q,R(T^q,X)T^q) = 0. \tag{4.2}$$

In view of Theorem 3.3, the above equation becomes

$$2\beta h(\phi X,T^q) + 2\alpha^2 h(X,T^q) + 2\alpha^2 g(X,T^q)h(T^q,T^q) = 0. \tag{4.3}$$

On simplifying (4.3), we get

$$g(X,T^q)h(T^q,T^q) + h(X,T^q) + \frac{\beta}{\alpha^2}h(\phi X,T) = 0 \tag{4.4}$$

Replacing X by ϕXin (4.4), we get

$$h(\phi Y,T^q) = -\frac{\alpha^2}{\beta}[A_p(Y)h(T^q,T^q) + h(Y,T^q)]. \tag{4.5}$$

Using (4.4) and (4.5), we get

$$h(T^q,T^q)A_p(Y) + h(Y,T^q) = 0 \quad \text{if} \quad \alpha^4 - \beta^2 \neq 0. \tag{4.6}$$

Differentiating (4.6) covariantly with respect to Y, we get

$$h(T^q,T^q)g(X,\phi Y) + 2g(X,T^q)h(\phi Y,T^q) + h(X,\phi Y) = 0. \tag{4.7}$$

From the above equation and (2.8a), we obtain

$$h(T^q,T^q)g(X,\phi Y) = -h(X,Y). \tag{4.8}$$

In view of the fact that $h(T^q,T^q)$ is constant along any vector on M, we have proved the theorem unless $\alpha^4 - \beta^2 \neq 0$.

Suppose that the $(0,2)$ type symmetric tensor field $\pounds_V g + 2S$ is parallel for any vector field V on a Lorentzian α-r Sasakian manifold with coefficient α. Then by theorem 1.1 it follows that $\pounds_V g + 2S$ is a constant multiple of the metric tensor g since $\pounds_V g + 2S = -2\lambda g$ for all X, Y on M, where λ is a constant. Hence (1.1) holds. This shows that (g, V, λ) yields a Ricci Soliton. Now we can state the theorem.

Theorem 4.1. *If the tensor field $\pounds_V g + 2S$ on a Lorentzian α-r Sasakian manifold with a coefficient α, is parallel for any vector field V then (g, V, λ) is a Ricci Soliton.*

Corollary 4.2. *If the tensor field $\pounds_V g + 2S$ on a Lorentzian α-Sasakian manifold with coefficient α is parallel for any vector field then (g, V, λ) is Ricci Soliton.*

Let (g, T^p, λ) be a Ricci Soliton on a Lorentzian α-r Sasakian manifold with a coefficient α. Then, we have

$$(\pounds_{T^p} g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) = 0, \tag{4.9}$$

where $\pounds_{T^p}$ is the Lie Derivative along the vector field T^p on M. From (2.8a), it follows that

$$\begin{aligned}(\pounds_{T^p} g)(Y, Z) &= g(\nabla_Y T^p, Z) + g(Y, \nabla_Z T^p) \\ &= -\alpha[g(\phi Y, Z) + g(Y, \phi Z)] \\ &= -2\alpha\phi(Y, Z)\end{aligned} \tag{4.10}$$

Using (4.10) in (4.9) we get

$$S(Y, Z) = \alpha\phi(Y, Z) - \lambda g(Y, Z), \tag{4.11}$$

where α and λ are non zero scalars. This shows that the manifold under consideration is nearly quasi-Einstein manifold [8].

Consequently, we have the following theorem,

Theorem 4.3. *If (g, T^ϕ, λ) is Ricci Soliton on a Lorentzian α-r- Sasakian manifold M with a coefficient α, then M is nearly quasi-Einstein manifold.*

Corollary 4.4. *If (g, T, λ) is a Ricci Soliton on a Lorentzian α-Sasakian manifold M then M is nearly quasi-Einstein manifold.*

Proof of Theorem 1.2. Let h be a parallel 2-form on a Lorentzian α-r Sasakian manifold M with a coefficient α. Then putting $W = Y = T^q$ in (4.1) and using theorem (3.3) and equations (2.1a)-(2.9b), we get

$$\begin{aligned}&\beta h(Z, \phi X) + \alpha^2 h(X, Z) - \alpha^2 h(T^q, Z)A_p(X) + \alpha^2 h(T^q, X)A_p(Z) \\ &\quad + h(T^q, \phi X)\alpha(Z) + h(\overline{\alpha}, T^q)\phi(X, Z) = 0\end{aligned} \tag{4.12}$$

Let ϕ^* be a (2,0) tensor field metrically equivalent to ϕ then contracting (4.12) with ϕ^* and using antisymmetric propert of h and the symmetry property of ϕ^* , we obtain, in view of equations (2.3)-(2.6) and after simplifying, we get

$$h(\overline{\alpha}, T^q) = 0. \tag{4.13}$$

Substituting (4.13) in (4.12) we get

$$\begin{aligned}&\beta h(\phi X, Z) + \alpha^2[h(X, Z) - h(T^q, Z)A_p(X) + h(T^q, X)A_p(Z)] \\ &\quad + h(T^q, \phi X)\alpha(Z) = 0.\end{aligned} \tag{4.14}$$

On simplifying (4.14) we get

$$\begin{aligned}&\beta h(\phi Z, X) - \alpha^2[h(Z, X) + h(T^q, X)A_p(Z) - h(T^q, Z)A_p(X)] \\ &\quad + h(T^q, \phi Z)\alpha(X) = 0.\end{aligned} \tag{4.15}$$

On simplifying (4.14) and (4.15) we get

$$\beta[h(Z,\phi X)+h(X,\phi Z)]+\alpha(X)h(\phi Z,T^q)+\alpha(Z)h(\phi X,T^q)=0. \tag{4.16}$$

On replacing X by ϕY in (4.16), we get

$$\beta[h(Z,\phi^2 Y)+h(\phi Y,\phi Z)]+\alpha(\phi Y)h(\phi Z,T^q)+\alpha(Z)h(\phi^2 Y,T^q)=0. \tag{4.17}$$

On making use of (2.4) in (4.17), we get

$$\begin{aligned}\beta[h(Z,Y)+h(Z,T^q)A_p(Y)+h(\phi Y,\phi Z)]+\alpha(Z)h(Y,T^q)\\ +\alpha(\phi Y)h(\phi Z,T^q)=0.\end{aligned} \tag{4.18}$$

On simplifying (4.18)

$$\begin{aligned}\beta[h(Y,Z)+h(Y,T^q)A_p(Z)+h(\phi Z,\phi Y)]+\alpha(Y)h(Z,T^q)\\ +\alpha(\phi Z)h(\phi Y,T^q)=0.\end{aligned} \tag{4.19}$$

In view of (4.18) and (4.19) on simplifying we obtain

$$\begin{aligned}\beta[h(T^q,Z)A_p(Y)+h(T^q,Y)A_p(Y)]-\alpha(Z)h(T^q,Y)\\ -h(T^q,\phi Z)\alpha(\phi Y)-\alpha(Y)h(Z,T^q)-\alpha(\phi Z)h(T^q,\phi Y)=0.\end{aligned} \tag{4.20}$$

Putting $Y=\overline{\alpha}$ in (4.20) and using (4.13), we get

$$\beta[h(T^q,Z)A_p(\overline{\alpha})-h(T^q,\phi Z)\alpha(\phi\overline{\alpha})-\alpha(\overline{\alpha})h(Z,T^q)=0 \tag{4.21}$$

Let us put $\alpha\overline{\alpha}=\hat{\alpha}$ and $\alpha(\phi\overline{\alpha})=\hat{\beta}$ in (4.21), we get

$$h(Z,T^q)[\beta A(\overline{\alpha})-\alpha(\overline{\alpha})]=h(T^q,\phi Z)\hat{\beta} \tag{4.22}$$

Replacing Z by ϕZ in (4.22), we get

$$h(\phi Z,T^q)[\beta^2-\overline{\alpha}]=\hat{\beta}h(T^q,Z). \tag{4.23}$$

On simplifying (4.23) and replacing Z by ϕZ, we obtain

$$h(\phi^2 Z,T^q)=\frac{\hat{\beta}}{\overline{\alpha}-\beta^2}h(\phi Z,T^q). \tag{4.24}$$

On making use of (2.4) in (4.24), we get

$$\frac{\overline{\alpha}-\beta^2}{\hat{\beta}}h(Z,T^q)=\frac{\hat{\beta}}{\overline{\alpha}-\beta^2}h(Z,T^q). \tag{4.25}$$

From (4.25), it follows immediately that

$$h(Z,T^q)=0 \qquad \text{unless } (\overline{\alpha}-\beta^2)^2-(\hat{\beta})^2\neq 0. \tag{4.26}$$

Using (4.26) in (4.14), we get

$$\beta h(Z,\phi X)+\alpha^2 h(Z,X)=0. \tag{4.27}$$

Differentiating (4.26) covariantly along Y and using the fact that $\nabla h=0$, we get

$$h(Z,\phi Y)=0. \tag{4.28}$$

In view of (4.28) and (4.27), we see that $h(Y,Z)=0$, which completes the proof. □

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