Particular solutions to the Tzitzeica curve equation

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Abstract. The Tzitzeica curve equation is a nonlinear ordinary differential equation arising in differential geometry whose solutions define a space curve for which the ratio of its torsion and the square of the distance from the origin to its osculating plane at an arbitrary point of the curve is constant. These curves have been introduced by the Romanian geometer Gheorghe Tzitzeica in 1911 in his study of affine invariants. Nowadays, there are known only a few examples of Tzitzeica curves. The aim of this paper is to analyze a special case in which the Tzitzeica curve equation may be reduced to an auxiliary third order homogeneous linear ordinary differential equation with constant coefficients whose characteristic equation is the depressed cubic equation, along with a linear equation for the curve's constant. Consequently, it is shown that any three linearly independent solutions of the auxiliary differential equation define a Tzitzeica curve. This new technique allows us to introduce three classes of transcendental Tzitzeica curves defined in parametric form and classify them according to the sign of the determinant to the depressed cubic equation. Previous and new results in the context of theory of Tzitzeica curves are obtained.

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1 Introduction

One of the most famous mathematicians who is recognized today as the founder of the Romanian school of differential geometry, Gheorghe Tzitzeica (1873-1939), has dedicated part of his research work to the study of geometrical objects invariant under affine transformations. In fact, the curves and surfaces that he introduced and analyzed are early examples of affine invariants. In 1911, as a result of his work on a particular class of surfaces that he originally called S-surfaces [7], Tzitzeica introduced a type of spatial curves that satisfy the property that the ratio of the curve's torsion τ and the square of the distance d from the origin to the osculating plane at an arbitrary

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point of the curve is constant, i.e.,

$$\frac{\tau}{d^2} = \alpha, \tag{1.1}$$

where $\alpha \neq 0$ is a real constant [8]. The curves satisfying the condition (1.1) are called nowadays *Tzitzeica curves*. A *Tzitzeica surface* is a surface for which the ratio of its Gaussian curvature and the fourth power of the distance from the origin to the tangent plane at any arbitrary point of the surface is constant. It may be shown that the asymptotic lines of a Tzitzeica surface with negative Gaussian curvature are Tzitzeica curves (see, e.g., [1]).

Interestingly, although the Tzitzeica curves have occurred occasionally in the mathematics literature, their related ordinary differential equation (ODE) resulting from the condition (1.1) has not been studied in detail so far. At the moment, there are known only a few examples of algebraic, transcendental, and integral Tzitzeica curves (see [1], [4], and [10]). Agnew et al [1] have used the `Mathematica` software to determine and visualize the asymptotic curves of the Tzitzeica surface of revolution $z(x^2 + y^2) = 1$ and expressed them in terms of logarithmic and exponential functions in cylindrical coordinates. In [4], Crâşmăreanu has shown that there exist integral Tzitzeica curves derived from a particular forced harmonic oscillator equation. In [10], specific Tzitzeica curves expressed in terms of polynomial, rational, and logarithmic functions have been determined explicitly.

This current work is motivated by the aim of finding new closed-form solutions to the nonlinear ODE (2.3) that will be called the *Tzitzeica curve equation*. We explore an interesting case in which the defining functions of a Tzitzeica curve are linearly independent solutions of a third order homogeneous linear ODE with constant coefficients (3.8) whose characteristic equation is the depressed cubic equation (3.9). The main result of the paper given in Theorem 3.1 states that the nonlinear ODE (2.3) reduces to the relation (3.6) and the linear ODE (3.8). In fact, (3.6) may be regarded as a linear equation for the curve's constant α once the solutions of (3.8) are known. In other words, by using this technique, we obtain three classes of transcendental Tzitzeica curves that may be classified according to the determinant D in (3.10) related to the depressed cubic equation (3.9) which is the characteristic equation of the ODE (3.8). These classes of curves are given in parametric form in terms of exponential and trigonometric functions in (3.11), (3.13), and (3.15). The Figures 1–3 show the `GeoGebra` plots of a few resulting Tzitzeica curves.

The structure of the paper is the following. The Tzitzeica curve equation is derived in Section 2. In Section 3, a specific reduction case is analyzed in detail along with the new transcendental solutions of the Tziteica curve equation. Section 4 is reserved for conclusions.

2 The Tzitzeica curve equation

In this section we determine the second order nonlinear ODE corresponding to the Tzitzeica curves by using concepts from the theory of space curves. Consider a regular curve defined by

$$\mathbf{r}(t) = (x(t), y(t), z(t)), \tag{2.1}$$

where $t \in I$ is the curve parameter, and $I \subset \mathbf{R}$ is an interval. Assume that the curve's curvature k is nonzero at any point of the curve. Recall that the curve's torsion is defined as follows

$$\tau(t) = \frac{\langle \mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t) \rangle}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2},$$

where the primes denote the derivatives with respect to t, the vector $\mathbf{r}' \times \mathbf{r}''$ is the cross product of the tangent vector $\mathbf{r}'$ and the acceleration vector $\mathbf{r}''$, $||\mathbf{r}'(t) \times \mathbf{r}''(t)||$ is the magnitude of $\mathbf{r}' \times \mathbf{r}''$, and

$$\langle \mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t) \rangle = \begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix}$$

is the mixed product of vectors $\mathbf{r}'$, $\mathbf{r}''$, and $\mathbf{r}'''$ (see, e.g., [6], p. 48). We assume that (2.1) has nonzero torsion at each point on the curve. Now consider the osculating plane at an arbitrary point of the curve (2.1) given in the determinant form by

$$\begin{vmatrix} x - x(t) & y - y(t) & z - z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = 0.$$

The osculating plane is generated by the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ at each point of the curve or, equivalently, by the tangent vector $\mathbf{r}'(t)$ and the acceleration vector $\mathbf{r}''(t)$. If d is the distance from the origin to the osculating plane of the curve, then

$$d^2 = \frac{1}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2} \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2.$$

The substitution of the torsion τ and the expression for d^2 into the condition (1.1) leads to the following equation

$$\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2, \tag{2.2}$$

which may be written equivalently as

$$az''' - a'z'' + bz' = \alpha \left(cz'' - c'z' + az\right)^2, \tag{2.3}$$

where the t-dependent functions a, b, and c are defined by

$$a = x'y'' - x''y', \quad b = x''y''' - x'''y'', \quad \text{and} \quad c = xy' - x'y.$$

Consequently, we have derived the following result:

Proposition 2.1. *The space curve* (2.1) *is a Tzitzeica curve if and only if its defining functions x, y, and z are solutions to the nonlinear ODE* (2.3).

Definition 2.1. The equation (2.3) is called the *Tzitzeica curve equation.*

Remark 2.2. In (2.3), the function z may be seen as the unknown function, while x and y may be regarded as arbitrary functions. A circular permutation of the functions x, y, and z implies equivalent Tzitzeica curve equations. Another way of looking at this equation is to consider x, y, and z as the unknown functions, and, hence, (2.3) becomes a nonlinear ODE in three unknowns.

3 A particular case of Tzitzeica curve equation

The nonlinearity encapsulated in the Tzitzeica curve equation is explored in this section with the goal of finding particular solutions. We start by noticing that the ODE (2.3) that is derived from (2.2) is rewritten in terms of Wronskians as follows

$$W(x', y', z')(t) = \alpha \left[W(x, y, z)(t)\right]^2, \tag{3.1}$$

where the Wronskian $W(x, y, z)(t)$ of the functions x, y, and z is expressed as

$$W(x, y, z)(t) = \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}.$$

The relation (3.1) shows that the Wronskian $W(x', y', z')(t)$ and the square of the Wronskian $W(x, y, z)(t)$ are directly proportional, where $\alpha \neq 0$ is the constant of proportionality. On the other hand, since a determinant is invariant under a cyclic permutation of its rows, the left-hand side of (2.2) may be rewritten as

$$\begin{vmatrix} x'''(t) & y'''(t) & z'''(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2. \tag{3.2}$$

Let us suppose that the functions x, y, and z satisfy the auxiliary equations

$$\begin{aligned} x''' + \beta x'' + \gamma x' + \delta x &= 0, \\ y''' + \beta y'' + \gamma y' + \delta y &= 0, \\ z''' + \beta z'' + \gamma z' + \delta z &= 0, \end{aligned} \tag{3.3}$$

where β, γ, and δ are arbitrary real numbers. Since the curve's torsion is assumed to be nonzero, here $\delta \neq 0$. The above equations (3.3) are equivalent to the condition that the defining curve's functions x, y, and z are solutions of the third order homogeneous linear ODE with constant coefficients

$$u''' + \beta u'' + \gamma u' + \delta u = 0, \tag{3.4}$$

in the unknown function $u = u(t)$, where β, γ, and δ real numbers with $\delta \neq 0$. Since the curve's torsion is nonzero at any point of the curve, the solutions x, y, and z are chosen to be linearly independent.

After substituting the third derivatives x''', y''', and z''' from the equations (3.3) into (3.2), the latter equation becomes

$$-\delta \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2,$$

and this may be rewritten equivalently in Wronskians terms as below

$$-\delta W(x,y,z)(t) = \alpha \left[W(x,y,z)(t)\right]^2.$$

Since the functions x, y, and z are assumed to be linearly independent, the Wronskian $W(x,y,z)(t)$ is nonzero on the interval I. Therefore, the above equation implies

$$W(x,y,z)(t) = -\frac{\delta}{\alpha}. \tag{3.5}$$

So far, the Tzitzeica curve equation has been reduced to (3.4) and (3.5). The latter equation implies that the Wronskian of the curve's defining functions is a nonzero constant. Once the Wronskian of the functions x, y, and z is determined, the relation (3.5) may be seen as a linear equation whose solution is

$$\alpha = -\frac{\delta}{W(x,y,z)(t)}. \tag{3.6}$$

On the other hand, according to Abel's differential identity ([3], p. 225), the Wronskian of three independent solutions of (3.4) satisfies the first-order ODE

$$W'(x,y,z)(t) = -\beta W(x,y,z)(t). \tag{3.7}$$

After replacing (3.5) into (3.7), and taking into account that $W(x,y,z)(t)$ is nonzero, we obtain $\beta = 0$. Therefore, the linear ODE (3.4) becomes

$$u''' + \gamma u' + \delta u = 0, \tag{3.8}$$

where γ and $\delta \neq 0$ are arbitrary real numbers.

In what follows, we discuss all possible cases of the ODE (3.8) in the context of the Tzitzeica curves. By seeking solutions of the form $u(t) = e^{vt}$ for (3.8), we find that its characteristic equation

$$v^3 + \gamma v + \delta = 0 \tag{3.9}$$

is the *depressed cubic equation*. The solutions v_i, $i = 1,2,3$, of (3.9) satisfy the following Viète's relations

$$\begin{cases} v_1 + v_2 + v_3 = 0, \\ v_1v_2 + v_1v_3 + v_2v_3 = \gamma, \\ v_1v_2v_3 = -\delta. \end{cases}$$

Since $\delta \neq 0$, the last relation above implies that $v_i \neq 0$, for all $i = 1,2,3$. We notice that the case $v_1 = v_2 = v_3$ does not lead to a Tzitzeica curve because the first Viète's

relation would imply $v_1 = 0$ which contradicts the condition $\delta \neq 0$. The character of the solutions of the equation (3.9) is given by the sign of its associated determinant

$$D = -4\gamma^3 - 27\delta^2. \tag{3.10}$$

In discussing the solutions of (3.8), it is convenient to consider the sign of D to classify the Tzitzeica curves that occur in each of these special cases.

Case 1. If $D > 0$, i.e., $4\gamma^3 + 27\delta^2 < 0$, the equation has three real distinct solutions $v_i \neq 0$, $i = 1, 2, 3$. Since the sum of the solution is zero and $v_3 \neq v_i$, with $i = 1, 2$, we obtain that $v_2 \neq -2v_1$ and $v_2 \neq -v_1/2$. In this case, the general solution of the auxiliary ODE (3.8) is

$$u(t) = C_1 \exp(v_1 t) + C_2 \exp(v_2 t) + C_3 \exp[-(v_1 + v_2)t], \quad t \in \mathbf{R},$$

where C_i are arbitrary real constants. Since a Tzitzeica curve is invariant under a centro-affine transformation, without losing the generality, we may choose the set of fundamental solutions of the ODE (3.8) to be the curve's defining functions. Therefore, we have obtained the class of transcendental Tzitzeica curves in parametric form

$$x(t) = \exp(v_1 t), \quad y(t) = \exp(v_2 t), \quad z(t) = \exp[-(v_1 + v_2)t], \quad t \in \mathbf{R}. \tag{3.11}$$

In this situation, the Wronskian of the curve's defining functions is

$$W(x, y, z)(t) = (v_2 - v_1)(2v_1 + v_2)(v_1 + 2v_2),$$

and the curve's constant α becomes

$$\alpha = -\frac{v_1 v_2 (v_1 + v_2)}{(v_2 - v_1)(2v_1 + v_2)(v_1 + 2v_2)}.$$

The reduction case presented above is shown in the following example.

Example 1. If $\gamma = -7$ and $\delta = 6$, then $D = 400$. The ODE (3.8) becomes $u''' - 7u' + 6u = 0$, and its characteristic equation $v^3 - 7v + 6 = 0$ has the solutions $v_1 = 1$, $v_2 = 2$, and $v_3 = -3$. The corresponding Tzitzeica curve (3.11) has the parametric equations

$$x(t) = \exp(t), \quad y(t) = \exp(2t), \quad z(t) = \exp(-3t), \quad t \in \mathbf{R}. \tag{3.12}$$

The Wronskian of the above functions is $W(x, y, z)(t) = 20$, and the curve's constant is given by $\alpha = -3/10$. The curve (3.12) is plotted in Figure 1 by using `GeoGebra`. This is the most common Tzitzeica curve occurring in the mathematics literature.

Case 2. If $D = 0$, that is, $4\gamma^3 + 27\delta^2 = 0$, the equation (3.9) has three real solutions out of which two are equal, i.e., $v_1 = v_2 \neq 0$ and $v_3 = -2v_1$. By integrating (3.8), we obtain its general solution given by

$$u(t) = C_1 \exp(v_1 t) + C_2 t \exp(v_1 t) + C_3 \exp(-2v_1 t),$$

where C_i are arbitrary real constants. Consequently,

$$x(t) = \exp(v_1 t), \quad y(t) = t \exp(v_1 t), \quad z(t) = \exp(-2v_1 t), \quad t \in \mathbf{R}, \tag{3.13}$$

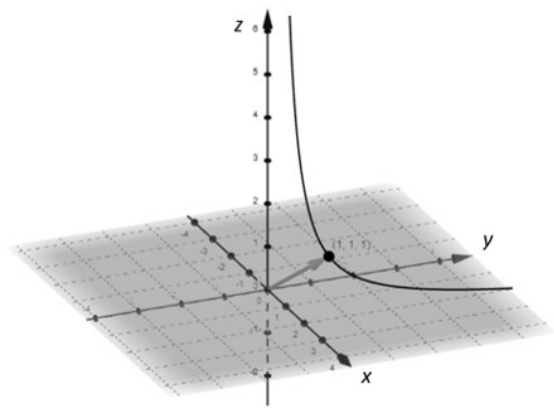

Figure 1: *The Tzitzeica curve* (3.12) *graphed for* $t \in [-10, 13]$.

defines a transcendental Tzizteica curve. The Wronskian related to the above functions is $W(x, y, z)(t) = 9v_1^2$ while the equation's constant is $\alpha = -2v_1/9$. To the best of our knowledge, (3.13) is a new Tzitzeica curve that we could not find in the mathematics literature. The following example illustrates this case.

Example 2. For $\gamma = -3$ and $\delta = 2$, the discriminant D is zero. The equation (3.8) turns into $u''' - 3u' + 2u = 0$, and its characteristic equation is $v^3 - 3v + 2 = 0$. By using the solutions $v_1 = v_2 = 1$ and $v_2 = -2$, we construct the Tzitzeica curve

$$x(t) = \exp(t), \quad y(t) = t\exp(t), \quad z(t) = \exp(-2t) \quad t \in \mathbf{R}. \tag{3.14}$$

The Wronskian of the curve's defining functions is $W(x, y, z)(t) = 9$, and the curve's constant is $\alpha = -2/9$. The graph of (3.14) is shown in Figure 2.

Case 3. For $D < 0$, or equivalently, $4\gamma^3 + 27\delta^2 > 0$, the equation (3.9) has two nonreal complex conjugate solutions, $v_{1,2} = m \pm in$ and one real nonzero solution $v_3 = -2m$, where $m, n \neq 0$ are real numbers. The integration of (3.8) leads to its general solution

$$u(t) = C_1 \exp(mt)\cos(nt) + C_2 \exp(mt)\sin(nt) + C_1 \exp(-2mt),$$

where C_i are arbitrary real constants. Therefore, the functions

$$x(t) = \exp(mt)\cos(nt),\ y(t) = \exp(mt)\sin(nt),\ z(t) = \exp(-2mt) \tag{3.15}$$

define a transcendental Tzitzeica curve in parametric form. In this situation, the Wronskian of the above functions is

$$W(x, y, z)(t) = n(9m^2 + n^2),$$

and the curve's constant is given by

$$\alpha = -\frac{2m(m^2 + n^2)}{n(9m^2 + n^2)}.$$

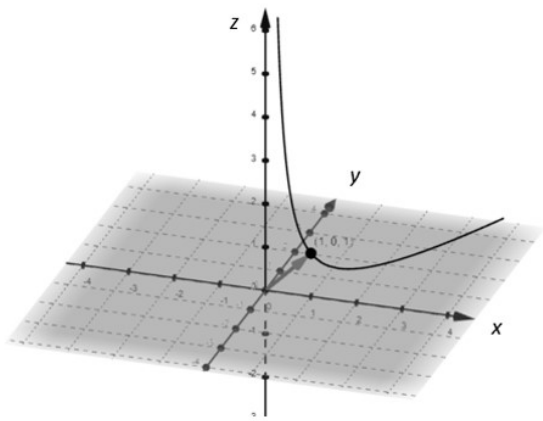

Figure 2: *The Tzitzeica curve defined by* (3.14) *with* $t \in [-10, 13]$.

A particular case of a Tzitzeica curve of the from (3.15) but expressed in a more complicated form in cylindrical coordinates may be found in the paper [1]. To illustrate the third case, we consider the example below.

Example 3. If $\gamma = 0$ and $\delta = -1$, we have $D = 4$. The auxiliary ODE (3.8) is $u''' - u = 0$, and its characteristic equation $v^3 - 1 = 0$ has the solutions $v_1 = 1$ and $v_{2,3} = -1/2 \pm i\sqrt{3}/2$. The corresponding Tzitzeica curve is defined by

$$x(t) = \exp\left(-\frac{t}{2}\right)\cos\left(\frac{\sqrt{3}}{2}t\right),\ y(t) = \exp\left(-\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}t\right),\ z(t) = \exp(t), \tag{3.16}$$

with $t \in \mathbf{R}$. The Wronskian and the curve's constant are given, respectively, by $W(x, y, z)(t) = 3\sqrt{3}/2$ and $\alpha = 2\sqrt{3}/9$. The curve (3.16) is plotted in Figure 3.

In conclusion, we have shown that there are three classes of transcendental Tzitzeica curves that are classified in terms of the sign of the determinant D of the depressed equation (3.9). These curves are given in parametric form by (3.11), (3.13), and (3.15) (here the constants v_i occurring in the curve's functions satisfy the depressed cubic equation (3.9)).

Theorem 3.1. *Any three linearly independent solutions of the auxiliary third order homogeneous linear ODE with constant coefficients* (3.8) *define a Tzitzeica curve. The curve's constant* α *is given by the relation* (3.6).

4 Conclusions

The Tzitzeica curve equation has been written in the particular form (2.3) in the paper [10]. In this nonlinear equation, one of the functions, say z, may be regarded as the unknown function while x and y are seen as arbitrary functions. Alternatively, x, y, and z are all considered unknown functions. In both cases, it is still difficult to

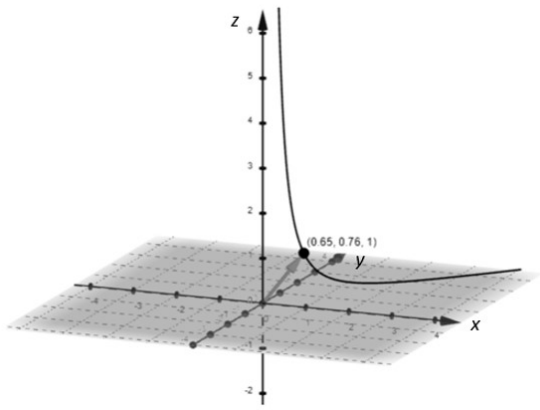

Figure 3: *The Tzitzeica curves defined in* (3.16) *for* $t \in [-10, 13]$.

find closed-form solutions, even with computer assistance. In this paper, we turned our attention to a special side condition given by an auxiliary equation that followed from the observation that the Tzitzeica curve equation may be expressed in terms of Wronskians. This key remark led us to explore the assumption that the defining functions of a Tzitzeica curve satisfy a third order homogeneous linear ODE with constant coefficients (3.8). In Section 3, we have shown that the characteristic equation of the auxiliary ODE is the depressed cubic equation (3.9) and, consequently, there is a correspondence between the set of its solutions v_i and specific Tzitzeica curves. The determinant D in (3.10) of the cubic equation may be used to classify the resulting Tzitzeica curves. By using this technique, Tzitzeica curve equation (2.3) reduces to a linear equation (3.6) for α. Interestingly, we have obtained that the nonlinear Tzitzeica curve equation admits closed-form transcendental solutions originated by solving a linear ODE. Since of a Tzitzeica curve is invariant under a centro-affine transformations, according to the symmetry group theory (see Olver [5]), one can use that property to construct new Tzitzeica curves from the known ones. More restrictively, by using central equi-affine transformations in generating new Tzitzeica curves, it may be shown that the curve's constant α is preserved under these transformations [10]. Having in mind that there is no general theory for solving nonlinear ODEs, these results indicate that it will be exciting to further investigate other possible side conditions augmented to the Tzitzeica curve equation that might help us reveal meaningful information about the nonlinearity encapsulated in this intriguing equation.

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References

[1] A. F. Agnew, A. Bobe, W. G. Boskoff, B. D. Suceava, *Tzitzeica curves and surfaces*, The Mathematica Journal, 12 (2010), 1-18.

[2] N. Bîlă, *Symmetry groups and conservation laws of certain partial differential equations*, Balkan J. Geom. Appl. 5, 1 (2000), 39-56.

[3] W. E. Boyce, R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 4th edition, Wiley, New York, 1986.

[4] M. Crâşmăreanu, *Cylindrical Tzitzeica curves implies forced harmonic oscillators*, Balkan J. Geom. Appl. 7, 1 (2002), 37-42.

[5] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics, vol. 107, Springer-Verlag, New York, 1986.

[6] A. Pressley, *Elementary Differential Geometry*, Springer Undergraduate Mathematics Series, Springer-Verlag London Ltd., 2012.

[7] G. Tzitzeica, *Sur une nouvelle classes de surfaces*, Comptes Rendus, Acad. Sci. Paris, Paris, 144 (1907), 1257-1259.

[8] G. Tzitzeica, *Sur certaines courbes gauches*, Ann. de l'Ec. Normale Sup., 28 (1911), 9-32.

[9] C. Udrişte, N. Bîlă, *Symmetry group of Tzitzeica surfaces PDE*, Balkan J. Geom. Appl., 4, 2 (1999), 123-140.

[10] L. R. Williams, *On the Tzitzeica curve equation*, Explorations: The Undergraduate Research and Creative Activities Journal for the State of North Carolina, Vol. VIII (2013), 105-115.

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