An easy sufficient test to prove that a tensor is concise

Edoardo Ballico

Abstract. We give an easy to test sufficient condition to see if a tensor is concise, assuming that the tensor is given by a (not necessarily minimal) rank 1 decomposition. If the test fails, we discuss some useful consequences.

M.S.C. 2010: 5A69, 14N05.
Key words: tensor; rank 1 decomposition; tensor rank, multiprojective space.

1 Introduction

Fix an integer $k \geq 2$ and k finite-dimensional vector spaces $V_1, \dots, V_k$ over the field K, $V_i \neq 0$ for all i. Set $n_i := \dim V_i - 1$. An element $T \in V_1 \otimes \cdots \otimes V_k$ is called a tensor of format $(n_1+1) \times \cdots \times (n_k+1)$. T is said to be concise if the are no linear subspaces $W_i \subseteq V_i$, $1 \leq i \leq k$, such that $T \in W_1 \otimes \cdots \otimes W_k$ and $W_i \neq V_i$ for at least one index i. A rank 1 tensor is a tensor $A = v_1 \otimes \cdots \otimes v_k$ with $v_i \in V_i \setminus \{0\}$. For any tensor $T \neq 0$ a rank 1 decomposition of T is an equality

$$T = \sum_{i=1}^{m} T_i \tag{1.1}$$

with T_i a rank 1 tensor. We do not assume that m is minimal among all rank 1 tensor decompositions of T, because in this case by concision ([3, Proposition 3.2.1.1]) the minimal Segre of T is just described by the m points of the multiprojective space (Remark 1.1).

This is not an " if and only if " criterion, but it is very easy to do the test if the tensor is given as a sum of rank 1 tensors.

For any set $A \subset \mathbb{P}^N$ let $\langle A \rangle$ denote its linear span. To state our result we recall the equivalence between the space of all tensors with format $(n_1+1) \times \cdots \times (n_k+1)$ and the projective space $\mathbb{P}^N$ in which a certain Segre variety lives.

Differential Geometry - Dynamical Systems, Vol.24, 2022, pp. 34-37.

Consider the multiprojective space $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and let $\nu : Y \to \mathbb{P}^N$, $N = -1 + (n_1+1) \times \cdots (n_k+1)$ denote the Segre embedding of Y ([3, §4.3.5]) with $\mathbb{P}^N = \mathbb{P}(V_1 \otimes \cdots \otimes V_k)^\vee$. The elements of $\nu(Y)$ correspond to the rank 1 tensors with format $(n_1+1) \times \cdots \times (n_k+1)$ up to a non-zero multiplicative constant. More explicitly, choosing a system of coordinates $x_{i0}, \dots, x_{in_i}$ of the vector space V_i, if $A = v_1 \otimes \cdots v_k$ with $v_i = (a_{i0}, \dots, a_{in_i})$, then the point $[T] \in \nu(Y)$ corresponds to the point $(b_1, \dots, b_k) \in Y$ in which that i-th component b_i has $(a_{i0} : \cdots : a_{in_i})$ as its homogeneous coordinates. Thus if in (1.1) no two of the addenda T_i are proportional, they correspond to a set $S \subset Y$ such that $\#S = m$ and $[T] \in \langle \nu(S) \rangle$, where $\langle \ \rangle$ denote the linear span. The tensors T' such that $[T'] \in \langle \nu(S) \rangle$ are all tensors $T' = \sum_{i=1}^m c_i T_i$ with $c_i \in K$ and $T' \neq 0$. We cannot drop one of the points of S and get T' if and only if the tensors $T_1, \dots, T_m$ are linearly independent (i.e. the set $\nu(S) \subset \mathbb{P}^N$ is linearly independent) and $c_i \neq 0$ for all i. In the following we assume that we have a representation (1.1) with $T_1, \dots, T_m$ linearly independent. The test (if it satisfied) says that every $T' = \sum_{i=1}^m c_i T_i$ with $c_i \neq 0$ for all i is concise. Note that (assuming $\nu(S)$ linearly independent) $T' = \sum_{i=1}^m c_i T_i$ with $c_i \neq 0$ for all i if and only if $[T'] \in \langle \nu(S) \rangle$ and $[T'] \notin \langle \nu(S') \rangle$ for any $S' \subsetneq S$. In this case we say that $\nu(S)$ irredundantly spans $[T']$.

To state our result we introduce the following notation. For any $i = 1, \dots, k$ set $Y_i := \prod_{h \neq i} \mathbb{P}^{n_h}$ and let $\eta_i : Y \to Y_i$ denote the surjection which to each $p = (p_1, \dots, p_k) \in Y$ delete the i-th component of p. Since $k \geq 2$, Y_i is a multiprojective space with $k-1$ factors. Let ν_i denote the Segre embedding of Y_i.

Theorem 1.1. *Let $T \in V_1 \otimes \cdots \otimes V_k$ be a tensor with a rank 1 decomposition corresponding to a finite set $S \subset Y$. Assume that Y is the minimal multiprojective space containing S and that $\nu(S)$ irredundantly spans $[T]$, i.e. there is no $S' \subsetneq S$ such that $[T] \in \langle S \rangle$. Assume that for all $i = 1, \dots, k$ the following conditions hold:*

1. *$\eta_{i|S}$ is injective;*
2. *$\nu_i(\eta_i(S_i))$ is linearly independent.*

Then T is concise.

The proof of the sufficient condition is just a few lines, due to previous work, but if we have the finite set S it is quick to check the criterion for S and, if it does not fail, we get the conciseness for all tensors $U \neq 0$, with $[U]$ irredundantly spanned by S, i.e. such that $[U] \in \langle \nu(S) \rangle$ and $[U] \notin \langle S' \rangle$ for any $S' \subsetneq S$.

Remark 1.1. The assumption that Y is the minimal multiprojective space containing S is obviously a necessary condition for the conciseness of T. It is easy to test this condition, because $\prod_{i=1}^k \langle \pi_i(S) \rangle$ is the minimal multiprojective subspace of Y containing S. Thus Y is the minimal multiprojective space containing S if and only if $\langle \pi_i(S) \rangle = \mathbb{P}^{n_i}$ for all i.

At the end of this short note we give 3 observations on the information we get about the tensor T if the test fails (Remarks 2.1, 2.2 and 2.3). In particular we observe that if condition (1) of Theorem 1.1 fails for some i, say for x indices i, then T has tensor rank at most $m - x < m$ (Remark 2.2).

2 The proof and further remarks

For any $i \in \{1, \dots, k\}$ let ϵ_i denote the multiindex $(a_1, \dots, a_k) \in \mathbb{N}^k$ such that $a_j = 0$ for all $j \neq i$ and $a_i = 1$. Thus $\mathcal{O}_Y(\epsilon_i) \cong \pi_i^*(\mathcal{O}_{\mathbb{P}^{n_i}}(1))$.

Proof of Theorem 1.1: Assume that T is not concise, i.e. assume the existence of a multiprojective space $Y' \subsetneq Y$ such that $q := [T] \in \nu(Y')$. By concision there is $A \subset Y'$ such that $\#A$ is the tensor rank of T and $q \in \langle \nu(A) \rangle$. Set $B := A \cup S$. Since Y is the minimal multiprojective space containing S, $A \neq S$. Since Y' is a multiprojective subspace of Y, each $\pi_i(Y')$ is a linear subspace of $\mathbb{P}^{n_i}$. Since $Y' \subsetneq Y$, there is $i \in \{1, \dots, k\}$ such that $\pi_i(Y')$ is a proper linear subspace of $\mathbb{P}^{n_i}$. Let $M \subset \mathbb{P}^{n_i}$ be a hyperplane containing $\pi_i(Y')$. Set $H := \pi_i^{-1}(M)$. Note that $H \in |\mathcal{O}_Y(\epsilon_i)|$ and that $A \subset H$. Since Y is the minimal multiprojective space containing S, $S \nsubseteq H$. Thus $B \neq B \cap H$. Note that $B \setminus B \cap H = S \setminus S \cap H$. The contradiction comes from [1, Lemma 5.1] or [2, Lemmas 2.4 or 2.5]. □

Remark 2.1. Take a tensor T irredundantly spanned by the set $\nu(S)$ for some finite set $S \subset Y$. Assume that the test of Theorem 1.1 fails and call $E \subseteq \{1, \dots, k\}$ the set of all $i \in \{1, \dots, k\}$ such that one of the two conditions in Theorem 1.1 is not satisfied. Assume $E \neq \emptyset$, but also assume $E \neq \{1, \dots, k\}$. Let $Y' \subseteq Y$ be the minimal multiprojective space such that $T \in \langle \nu(Y') \rangle$, i.e. let $\nu(Y')$ be the concise Segre of T. Fix $i \in \{1, \dots, k\} \setminus E$. The proof of Theorem 1.1 gives $\pi_i(Y') = \mathbb{P}^{n_i}$.

Remark 2.2. Take a tensor T irredundantly spanned by the set $\nu(S)$ for some finite set $S \subset Y$. Assume that condition (1) the test of Theorem 1.1 fails and call $E \subseteq \{1, \dots, k\}$ the set of all $i \in \{1, \dots, k\}$ such that (1) fails. Set $x := \#E$. By assumption $1 \leq x \leq k$.

Claim: T has tensor rank at most $m - x$.

Proof of the claim: To prove the claim it is sufficient to find a set $A \subset Y$ such that $\#A = m - x$ and $T \in \langle \nu(A) \rangle$. Fix $i \in E$ and take $u, v \in S$ such that $u \neq v$ and $\eta_i(u) = \eta_i(v)$. Set $B := S \setminus \{u, v\}$. Write $u = (u_1, \dots, u_k)$ and $v = (v_1, \dots, v_k)$. Since $u \neq v$ and $\eta_i(u) = \eta_i(v)$, $u_j = v_j$ for all $j \neq i$ and $u_i \neq v_i$. Let $L \subseteq \mathbb{P}^{n_i}$ denote the line spanned by $\{u_i, v_i\}$. Let $Y' \subseteq Y$ be the multiprojective space such that $\pi_j(Y') = \{u_j\}$ for all $j \neq i$ and $\pi_i(Y') = L$. Note that $\nu(Y')$ is the line spanned by $\nu(u)$ and $\nu(v)$. Since $\{\nu(u), \nu(v)\} \subset \nu(Y')$ T is contained in the linear span of $\nu(B)$ and $\nu(Y')$. Since $\nu(Y')$ is the line, there is $o \in Y'$ such that $T \in \langle \nu(B) \cup \{o\} \rangle$. Since $\#(B \cup \{o\}) \leq \#B + 1 = m - 1$, the tensor rank of T is at most $m - 1$. If $x = 1$, then the claim is proved. Now assume $x \geq 2$ and take $j \in E \setminus \{i\}$. Note that condition (1) of Theorem 1.1 fails for B and hence it fails for $B \cup \{o\}$. We use j as we used i and conclude the proof of the claim by induction on x.

Remark 2.3. Assume that the test fails for S and fix $p \in S$. Set $S' := S \setminus \{p\}$. Since S irredundantly spans $[T]$, there is a unique tensor U such that $\nu(S')$ irredundantly spans $[U]$ and $[T] \in \langle \{[U], \nu(p)\} \rangle$. Let $Y' \subseteq Y$ be the multiprojective subspace of Y with $\nu(Y')$ the concise Segre of U. If the test works for S', then Y' is the minimal multiprojective subspace of Y containing S' and hence it is easily determined (Remark 1.1). Write $Y' = \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_k}$ in which we allow the case $m_h = 0$ for some h. Thus $0 \leq m_h \leq n_h$ for all h. Let $Y'' = \mathbb{P}^{s_1} \times \cdots \times \mathbb{P}^{s_k}$ with $0 \leq s_h \leq n_h$ for all h be the multiprojective space with $\nu(Y'')$ the concise Segre of T. Take $A \subset Y'$

evincing the tensor rank of U. Let $W \subseteq Y$ be the minimal multiprojective subspace containing $A \cup \{p\}$. Take $B \subset Y''$ evincing the tensor rank of T. By concision ([3, Proposition 3.2.1.1]) Y' is the minimal multiprojective space containing A and Y'' is the minimal multiprojective space containing B. Write $W = \mathbb{P}^{w_1} \times \cdots \times \mathbb{P}^{w_k}$ for some $0 \le w_h \le n_k$. Since $[T] \in \langle \nu(A \cup \{p\}) \rangle$, it is easy to see that $s_h \le \min\{n_h, m_h + 1\}$ for all h. Since $[U] \in \langle \{\nu(B \cup \{p\}\}) \rangle$, we get $w_h \ge m_h - 1$ and $s_h \ge m_h - 1$ for all h. Instead of deleting one point of S we may delete several points, but the informations we would obtain quickly decrease.

References

[1] E. Ballico, A. Bernardi, *Stratification of the fourth secant variety of Veronese varieties via the symmetric rank*, Advances in Pure and Applied Mathematics, 4 (2013), 215-250.

[2] E. Ballico, A. Bernardi, M. Christandl and F. Gesmundo, *On the partially symmetric rank of tensor products of W-states and other symmetric tensors*, Rend. Lincei Mat. Appl. 30, 1 (2019), 93-124.

[3] J.M. Landsberg,*Tensors: Geometry and Applications*, Graduate Studies in Mathematics, Vol. 128, Amer. Math. Soc., Providence, 2012.

Author's address:

Edoardo Ballico
Department of Mathematics, University of Trento,
via Sommarive 14, 38123, Trento, Italy.
E-mail: edoardo.ballico@unitn.it