Spacelike loxodromes on helicoidal surfaces in Lorentzian n-space

M. Babaarslan, B. B. Demirci and R. Genç

Abstract. In this paper, we study a special curve called a loxodrome on three types of helicoidal surfaces in Lorentzian n–space $\mathbb{E}_1^n$. First, we determine the parametrizations of spacelike loxodromes on spacelike helicoidal surfaces in $\mathbb{E}_1^n$ by solving related differential equations. Then, we get a classification for spacelike loxodromes on timelike helicoidal surfaces in $\mathbb{E}_1^n$. As a particular case, we obtain the parametrization of spacelike loxodromes on the right helicoidal surfaces in $\mathbb{E}_1^n$. Also, we find the length of the spacelike loxodromes on the right helicoidal surfaces in $\mathbb{E}_1^n$. Finally, we give an example in $\mathbb{E}_1^4$ to illustrate our main results by using Wolfram Mathematica 12.3.

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1 Introduction

A helicoidal surface is invariant under a motion which is a composition of a translation with a rotation. Thus, they are considered a kind of generalization of rotational surfaces. In Minkowski 4–space $\mathbb{E}_1^4$, there are three types of rotation such as elliptic, hyperbolic and screw rotation which leave spacelike, timelike and lightlike plane invariant, respectively. By using these rotations, M. Babaarslan and N. Sönmez [10] defined the helicoidal surfaces in $\mathbb{E}_1^4$.

In differential geometry, there are special curves defined on surfaces such as geodesics and helices. A helix is a curve whose tangent lines make a constant angle with the fixed direction in Euclidean 3–space $\mathbb{E}^3$. A loxodrome is also another special curve that makes a constant angle with all meridians on the Earth's surface. Even though the loxodromes are not useful to find the shortest route across the Earth's surface between two points as geodesics, they assure an efficient routing from one point to another by means of a constant course angle. Thus, the loxodromes are popular in navigation, see [1] and [13]. From mathematical point of view, geometers have studied loxodromes defined on different kind of surfaces in various ambient spaces, see [5, 6, 8, 9, 10, 11, 12, 15, 16].

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In [17], C. A. Noble obtained the equations of loxodromes on the rotational surfaces and he also found such curves on spheres and sphereoids. In addition, S. Kos et al. [14] calculated the arc–length of the loxodromes on a sphere and M. Petrović [18] also found the arc–length of the loxodromes on a sphereoid. Moreover, some results on the spacelike and timelike loxodromes on the rotational surfaces in Minkowski 3–space $\mathbb{E}^3_1$ were obtained in [2] and [4], respectively. In [20], D. W. Yoon studied loxodromes on rotational surfaces in the 3–dimensional simply isotropic space.

In [3], M. Babaarslan and Y. Yaylı studied the differential equation of loxodromes on helicoidal surfaces in $\mathbb{E}^3$. Recently, M. Babaarslan and M. Kayacık studied the spacelike loxodromes on helicoidal surfaces in $\mathbb{E}^3_1$ having spacelike meridians and timelike meridians in [7] and M. Babaarslan and N. Sönmez [10] also obtained the parametrization of spacelike and timelike loxodromes on the non–degenerate helicoidal surfaces in $\mathbb{E}^4_1$.

In this article, we consider three types of helicoidal surfaces in $\mathbb{E}^n_1$ such a generalization of helicoidal surfaces in $\mathbb{E}^4_1$. Then, we study a special curve called a loxodrome which makes a constant angle with a parameter curve, say meridian, of such helicoidal surfaces in $\mathbb{E}^n_1$. First, we obtain the related differential equations of spacelike loxodromes on the spacelike helicoidal surfaces in $\mathbb{E}^n_1$. Then, we give a result about the parametrizations of such loxodromes. After that, we make a similar computation for spacelike loxodromes on timelike helicoidal surfaces in $\mathbb{E}^n_1$. As a particular case, we study spacelike loxodromes on non–degenerate right helicoidal surfaces in $\mathbb{E}^n_1$. Moreover, we calculate the length of such spacelike loxodromes in $\mathbb{E}^n_1$. Finally, we give an example in $\mathbb{E}^4_1$ by using Wolfram Mathematica 12.3.

2 Preliminaries

For the vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in n–dimensional Euclidean space $\mathbb{R}^n$, the Lorentzian inner product $\langle \cdot, \cdot \rangle$ of x and y is defined by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_{n-1} y_{n-1} - x_n y_n. \tag{2.1}$$

$(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ with the Lorentzian inner product given above is said to be Lorentzian n–space. Then, it is denoted by $\mathbb{E}^n_1$.

The length of the vector x in $\mathbb{E}^n_1$ is defined by $||x|| = \sqrt{|\langle x, x \rangle|}$ and it is said to be a unit vector if $||x|| = 1$. A causal character of any arbitrary vector x in $\mathbb{E}^n_1$ is said to be spacelike (resp., timelike or lightlike) if $||x|| > 0$ or $x = 0$ (resp., $||x|| < 0$ or $||x|| = 0$ and $x \neq 0$).

Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^n_1$ be a smooth curve in $\mathbb{E}^n_1$. The curve α is spacelike (resp., timelike or lightlike) provided that α' is spacelike (resp., timelike or lightlike).

Suppose that $\mathbf{x} : M \longrightarrow \mathbb{E}^n_1$ is an isometric immersion from pseudo–Riemannian surface M to a Lorentzian n–space $\mathbb{E}^n_1$. Then, the coefficients of the first fundamental form of M are

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \;\; F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle \;\text{ and }\; G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle \tag{2.2}$$

for a coordinate system $\{u, v\}$ in M. Here, $\mathbf{x}_u$ and $\mathbf{x}_v$ denote the partial derivative of $\mathbf{x}$ with respect to u and v, respectively. Then, M is a spacelike surface if and only if $EG - F^2 > 0$; M is a timelike surface if and only if $EG - F^2 < 0$. For $EG - F^2 = 0$, the surface M is called degenerate. Throughout the article, we are not interested in degenerate case.

Moreover, the length of any non–lightlike curve α on the pseudo–Riemannian surface M in $\mathbb{E}_1^n$ defined by the isometric immersion $\mathbf{x}$ between two points u_0 and u_1 is given by

$$s = \int_{u_0}^{u_1} \sqrt{\left| E + 2F\frac{dv}{du} + G\left(\frac{dv}{du}\right)^2 \right|} du. \tag{2.3}$$

Assume that $\alpha(t) = \mathbf{x}(u(t), v(t))$ is a spacelike curve on the pseudo–Riemannian surface M in $\mathbb{E}_1^n$, that is,

$$E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2 > 0. \tag{2.4}$$

For later use, we also calculate the following equation

$$\langle \alpha'(t), \mathbf{x}_u \rangle = E\frac{du}{dt} + F\frac{dv}{dt}. \tag{2.5}$$

Definition 2.1. [19] Let x and y be vectors in a Lorentzian n–space $\mathbb{E}_1^n$. Then, we have the followings:

i. for spacelike vectors x and y that span a spacelike vector subspace, there is a unique Lorentzian spacelike angle θ between x and y such that

$$\langle x, y \rangle = ||x||||y|| \cos\theta, \quad \theta \in [0, \pi], \tag{2.6}$$

ii. for spacelike vectors x and y that span a timelike vector subspace, there is a unique Lorentzian timelike angle θ between x and y such that

$$\langle x, y \rangle = ||x||||y|| \cosh\theta, \tag{2.7}$$

iii. for a spacelike vector x and a positive timelike vector y that span a timelike vector subspace, there is a unique Lorentzian timelike angle θ between x and y such that

$$|\langle x, y \rangle| = ||x||||y|| \sinh\theta. \tag{2.8}$$

Now, we give the definition of the helicoidal surfaces in $\mathbb{E}_1^n$ as follows.

Assume that $\beta : I \subset \mathbb{R} \longrightarrow \Pi \subset \mathbb{E}_1^n$ is a smooth curve in a hyperplane $\Pi \subset \mathbb{E}_1^n$ and $\mathbf{P}$ is $(n-2)$–plane in the hyperplane $\Pi \subset \mathbb{E}_1^n$ and ℓ is a line which does not intersect the curve β and it is parallel to $\mathbf{P}$. A helicoidal surface in $\mathbb{E}_1^n$ is defined as a rotation of the curve β about $\mathbf{P}$ followed by a translation along a line ℓ. Also, the speed of such translation is proportional to the speed of this rotation. Hence, we obtain three types of helicoidal surfaces in $\mathbb{E}_1^n$ as follows (for the definition in $\mathbb{E}^n$, see [8]).

2.1 Helicoidal surface of type I

Let $\{e_1, e_2, \ldots, e_n\}$ be a standard orthonormal basis for $\mathbb{E}^n_1$, Π_I a hyperplane spanned by $\{e_1, e_3, \ldots, e_n\}$ and $\mathbf{P}_I$ $(n-2)$–plane spanned by $\{e_3, e_4, \ldots, e_n\}$. Assume that $\beta_I : I \longrightarrow \Pi_I \subset \mathbb{E}^n_1$, $\beta_I(u) = (x_1(u), 0, x_3(u), \ldots, x_n(u))$ is a smooth regular curve in Π_I and u is arc length parameter, that is, $x'^2_1(u) + x'^2_3(u) + \cdots - x'^2_n(u) = \varepsilon$ with $\varepsilon = \pm 1$.

Then, the parametrization of the helicoidal surface M_I obtained the rotation of the curve β_I which leaves the timelike plane $\mathbf{P}_I$ invariant followed by the translation along ℓ_I spanned by e_n can be given

$$\mathbf{x}_I(u,v) = \begin{bmatrix} \cos v & -\sin v & 0 & \cdots & 0 \\ \sin v & \cos v & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1(u) \\ 0 \\ x_3(u) \\ \vdots \\ x_n(u) \end{bmatrix} + cv \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{2.9}$$

i.e.,

$$M_I : \mathbf{x}_I(u,v) = (x_1(u)\cos v, x_1(u)\sin v, x_3(u), \ldots, x_n(u) + cv)$$

for $0 \leq v < 2\pi$ and a positive constant c. Formally, this helicoidal surface looks like the Euclidean case in [8]. By a direct computation, we have

$$\begin{aligned} (\mathbf{x}_I)_u &= (x'_1(u)\cos v, x'_1(u)\sin v, x'_3(u), \cdots, x'_n(u)), \\ (\mathbf{x}_I)_v &= (-x_1(u)\sin v, x_1(u)\cos v, 0, \cdots, c). \end{aligned} \tag{2.10}$$

Thus, the coefficients of the first fundamental form of M_I is given by

$$E = \varepsilon, \quad F = -cx'_n(u) \quad \text{and} \quad G = x_1^2(u) - c^2. \tag{2.11}$$

Due the fact that the surface M_I is non–degenerate in $\mathbb{E}^n_1$, $\varepsilon x_1^2(u) - c^2(\varepsilon + x'^2_n(u)) \neq 0$. If x_n is a constant function, then the surface M_I is called a right helicoidal surface of type I in $\mathbb{E}^n_1$.

2.2 Helicoidal surface of type II

Let $\{e_1, e_2, \ldots, e_n\}$ be a standard orthonormal basis for $\mathbb{E}^n_1$, Π_{II} a hyperplane spanned by $\{e_1, e_2, \ldots, e_{n-2}, e_n\}$ and $\mathbf{P}_{II}$ $(n-2)$–plane spanned by $\{e_1, e_2, \ldots, e_{n-2}\}$. Assume that $\beta_{II} : I \longrightarrow \Pi_{II} \subset \mathbb{E}^n_1$, $\beta_2(u) = (x_1(u), x_2(u), \ldots, x_{n-2}(u), 0, x_n(u))$ is a smooth regular curve in Π_{II} and u is an arc length parameter, that is, $x'^2_1(u) + x'^2_2(u) + \cdots - x'^2_n(u) = \varepsilon$ for $\varepsilon = \pm 1$.

Then, the parametrization of the helicoidal surface M_{II} obtained the rotation of the curve β_{II} which leaves the spacelike plane $\mathbf{P}_{II}$ invariant followed by the translation along ℓ_{II} spanned by e_1 can be given

$$\mathbf{x}_{II}(u,v) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cosh v & \sinh v \\ 0 & 0 & \cdots & \sinh v & \cosh v \end{bmatrix} \begin{bmatrix} x_1(u) \\ x_2(u) \\ \vdots \\ 0 \\ x_n(u) \end{bmatrix} + cv \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{2.12}$$

i.e.,

$$M_{II} : \mathbf{x}_{II}(u,v) = (x_1(u)+cv, x_2(u), \ldots, x_{n-2}(u), x_n(u)\sinh v, x_n(u)\cosh v)$$

for $v \in \mathbb{R}$ and a positive constant c. By a direct computation, we get

$$\begin{aligned} (\mathbf{x}_{II})_u &= (x_1'(u), x_2'(u), \cdots, x_n'(u)\sinh v, x_n'(u)\cosh v), \\ (\mathbf{x}_{II})_v &= (c, 0, \cdots, 0, x_n(u)\cosh v, x_n(u)\sinh v). \end{aligned} \tag{2.13}$$

Hence, the coefficients of the first fundamental form of M_{II} is given by

$$E = \varepsilon, \quad F = cx_1'(u) \text{ and } G = x_n^2(u) + c^2. \tag{2.14}$$

Since M_{II} is a non–degenerate helicoidal surface in $\mathbb{E}_1^n$, $\varepsilon x_n^2(u) + c^2(\varepsilon - x_1'^2(u)) \neq 0$. In particular, the surface M_{II} is called a right helicoidal surface of type II in $\mathbb{E}_1^n$ when x_1 is a constant function.

2.3 Helicoidal surface of type III

Let define a pseudo–orthonormal basis $\{e_1, e_2, \cdots, \xi_{n-1}, \xi_n\}$ for $\mathbb{E}_1^n$ using a standard orthonormal basis $\{e_1, e_2, \cdots, e_{n-1}, e_n\}$ for $\mathbb{E}_1^n$ such that

$$\xi_{n-1} = \frac{1}{\sqrt{2}}(e_n - e_{n-1}) \text{ and } \xi_n = \frac{1}{\sqrt{2}}(e_n + e_{n-1}) \tag{2.15}$$

where $\langle \xi_{n-1}, \xi_{n-1} \rangle = \langle \xi_n, \xi_n \rangle = 0$ and $\langle \xi_{n-1}, \xi_n \rangle = -1$. Assume that Π_{III} is a hyperplane spanned by $\{e_1, e_3, \cdots, e_{n-2}, \xi_{n-1}, \xi_n\}$ and $\mathbf{P}_{III}$ is $(n-2)$–plane spanned by $\{e_1, e_3, \cdots, \xi_{n-1}\}$. Suppose that $\beta_{III} : I \longrightarrow \Pi_{III} \subset \mathbb{E}_1^n$, $\beta_3(u) = x_1(u)e_1 + x_3(u)e_3 + \cdots + x_{n-1}(u)\xi_{n-1} + x_n(u)\xi_n$ is a smooth curve in Π_{III} parameterized by the arc–length parameter u, that is, $x_1'^2(u) + x_3'^2(u) + \cdots - 2x_{n-1}'(u)x_n'(u) = \varepsilon$ for $\varepsilon = \pm 1$.

Then, the position vector of the helicoidal surface M_{III} obtained a rotation of the curve β_{III} using the transformation T which leaves the degenerate plane $\mathbf{P}_{III}$ invariant followed by the translation along ℓ_{III} spanned by ξ_{n-1} can be given

$$\begin{aligned} \mathbf{x}_{III}(u,v) = & x_1(u)e_1 + \sqrt{2}vx_n(u)e_2 + x_3(u)e_3 + \cdots \\ & + x_{n-2}(u)e_{n-2} + (x_{n-1}(u) + v^2x_n(u) + cv)\xi_{n-1} + x_n(u)\xi_n. \end{aligned} \tag{2.16}$$

Here, T is an orthogonal transformation in $\mathbb{E}_1^n$ such that $T(e_1) = e_1$, $T(e_2) = e_2 + \sqrt{2}v\xi_{n-1}$, $T(e_3) = e_3$, $\ldots, T(\xi_{n-1}) = \xi_{n-1}$, $T(\xi_n) = \sqrt{2}ve_2 + v^2\xi_{n-1} + \xi_n$.

By a direct calculation, we find

$$(2.17)\quad \begin{aligned}(\mathbf{x}_{III})_u &= x_1'(u)e_1 + \sqrt{2}vx_n'(u)e_2 + x_3'(u)e_3 + \cdots + x_{n-2}'(u)e_{n-2}\\ &\quad + (x_{n-1}'(u) + v^2x_n'(u))\xi_{n-1} + x_n'(u)\xi_n,\\ (\mathbf{x}_{III})_v &= \sqrt{2}x_n(u)e_2 + (2vx_n(u) + c)\xi_{n-1},\end{aligned}$$

and then, the coefficients of the first fundamental form of M_{III} are given by

$$(2.18)\quad E = \varepsilon,\ \ F = -cx_n'(u),\ \text{ and }\ G = 2x_n^2(u).$$

Also, $2\varepsilon x_n^2(u) - c^2x_n'^2(u) \neq 0$ because M_{III} is a non–degenerate one. If x_n is a constant function, then the helicoidal surface M_{III} is called a right helicoidal surface of type III in $\mathbb{E}_1^n$.

Remark 2.2. It can be easily seen that the helicoidal surfaces M_I–M_{III} in $\mathbb{E}_1^n$ defined by (2.9), (2.12) and (2.16) reduce to the rotational surfaces in $\mathbb{E}_1^n$ for $c = 0$.

3 Spacelike loxodromes on spacelike helicoidal surfaces in $\mathbb{E}_1^n$

In this section, we obtain the parametrization of the spacelike loxodromes on the spacelike helicoidal surface of type I, type II and type III in a Lorentzian space $\mathbb{E}_1^n$ defined by (2.9), (2.12) and (2.16), respectively.

Let M_I be a spacelike helicoidal surface of type I in $\mathbb{E}_1^n$ defined by (2.9) having the spacelike meridians. That is, $\varepsilon = 1$. Then, using the equation (2.11) the induced metric g_I on M_I is obtained

$$(3.1)\quad g_I = du^2 - 2cx_n'(u)dudv + (x_1^2(u) - c^2)dv^2$$

with $x_1^2(u) - c^2(1 + x_n'^2(u)) > 0$. Assume that $\alpha_I(t) = \mathbf{x}_I(u(t), v(t))$ is a spacelike loxodrome on M_I, so that, the equation (2.4) becomes

$$(3.2)\quad \left(\frac{du}{dt}\right)^2 - 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} + (x_1^2(u) - c^2)\left(\frac{dv}{dt}\right)^2 > 0$$

and the equation (2.5) also gives

$$(3.3)\quad \langle \alpha_I'(t), (\mathbf{x}_I)_u\rangle = \frac{du}{dt} - cx_n'(u)\frac{dv}{dt}.$$

Since the loxodrome $\alpha_I(t)$ intersects the spacelike meridian of M_I with a constant Lorentzian spacelike angle denoting θ_0 at a point $p \in M_I$, from the equations (2.6), (3.2) and (3.3), we get

$$(3.4)\quad \cos\theta_0 = \frac{du - cx_n'(u)dv}{\sqrt{du^2 - 2cx_n'(u)dudv + (x_1^2(u) - c^2)dv^2}}.$$

After rearranging this equation, we obtain the following differential equation

$$(3.5)\quad (\cos^2\theta_0(x_1^2(u) - c^2) - c^2x_n'^2(u))\left(\frac{dv}{du}\right)^2 + 2c\sin^2\theta_0 x_n'(u)\frac{dv}{du} = \sin^2\theta_0.$$

Solving this one, we find the function $v = v(u)$ as

$$v(u) = \int_{u_0}^{u} \frac{-2c\sin^2\theta_0 x_n'(\xi) \pm \sqrt{(x_1^2(\xi) - c^2(1 + x_n'^2(\xi)))\sin^2 2\theta_0}}{2\cos^2\theta_0(x_1^2(\xi) - c^2) - 2c^2 x_n'^2(\xi)} d\xi \tag{3.6}$$

with $\cos^2\theta_0(x_1^2(\xi) - c^2) - c^2 x_n'^2(\xi) \neq 0$. For $\cos^2\theta_0(x_1^2(\xi) - c^2) - c^2 x_n'^2(\xi) = 0$, we determine the function $v = v(u)$ given by (3.18). Thus, we get such a parametrization of the spacelike loxodrome $\alpha_I(u) = \mathbf{x}_I(u, v(u))$ on the spacelike helicoidal surface M_I in $\mathbb{E}_1^n$ given as (i) of Theorem 3.1.

Similarly, for a spacelike helicoidal surface of type II M_{II} in $\mathbb{E}_1^n$ defined by (2.12), we have $\varepsilon x_n^2(u) + c^2(\varepsilon - x_1'^2(u)) > 0$ which implies $\varepsilon = 1$. Then, using the equation (2.14) the induced metric g_{II} on M_{II} is given by

$$g_{II} = du^2 + 2cx_1'(u)dudv + (x_n^2(u) + c^2)dv^2 \tag{3.7}$$

with $x_n^2(u) + c^2(1 - x_1'^2(u)) > 0$. Assume that $\alpha_{II}(t) = \mathbf{x}_{II}(u(t), v(t))$ is a spacelike loxodrome on M_{II}. Hence, the equations (2.4) and (2.5) give

$$\left(\frac{du}{dt}\right)^2 + 2cx_1'(u)\frac{du}{dt}\frac{dv}{dt} + (x_n^2(u) + c^2)\left(\frac{dv}{dt}\right)^2 > 0 \tag{3.8}$$

and

$$\langle \alpha_{II}'(t), (\mathbf{x}_{II})_u \rangle = \frac{du}{dt} + cx_1'(u)\frac{dv}{dt}, \tag{3.9}$$

respectively. We know that the spacelike curve $\alpha_{II}(t)$ also intersects the spacelike meridian of M_{II} with a constant Lorentzian spacelike angle at a point $p \in M_{II}$. Let say θ_0. Using the equations (3.8) and (3.9) in (2.6), we get

$$\cos\theta_0 = \frac{du + cx_1'(u)dv}{\sqrt{du^2 + 2cx_1'(u)dudv + (x_n^2(u) + c^2)dv^2}}. \tag{3.10}$$

After doing necessary calculation, we obtain the following differential equation

$$(\cos^2\theta_0(x_n^2(u) + c^2) - c^2 x_1'^2(u))\left(\frac{dv}{du}\right)^2 - 2c\sin^2\theta_0 x_1'(u)\frac{dv}{du} = \sin^2\theta_0. \tag{3.11}$$

For $\cos^2\theta_0(x_n^2(u) + c^2) - c^2 x_1'^2(u) = 0$, we determine the function $v = v(u)$ given by (3.21). Hence, we find the parametrization of the spacelike loxodrome $\alpha_{II}(u) = \mathbf{x}_{II}(u, v(u))$ on the spacelike helicoidal surface M_{II} in $\mathbb{E}_1^n$ given as (ii) of Theorem 3.1.

Finally, we consider a spacelike helicoidal surface of type III M_{III} in $\mathbb{E}_1^n$ defined by (2.16). Then, $2\varepsilon x_n^2(u) - c^2 x_n'^2(u) > 0$ which says that ε must be equal 1. Thus, the equation (2.18) gives the induced metric g_{III} on M_{III} as follows

$$g_{III} = du^2 - 2cx_n'(u)dudv + 2x_n^2(u)dv^2 \tag{3.12}$$

with $2x_n^2(u) - c^2 x_n'^2(u) > 0$. Assume that $\alpha_{III}(t) = \mathbf{x}_{III}(u(t), v(t))$ is a spacelike loxodrome on M_{III}. Again, the equation (2.4) implies

$$\left(\frac{du}{dt}\right)^2 - 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} + 2x_n^2(u)\left(\frac{dv}{dt}\right)^2 > 0 \tag{3.13}$$

and using the equation (2.5), we have

$$\langle \alpha'_{III}(t), (\mathbf{x}_{III})_u \rangle = \frac{du}{dt} - cx'_n(u)\frac{dv}{dt}. \tag{3.14}$$

Using the fact that the loxodrome $\alpha_{III}(t)$ meets the spacelike meridian of M_{III} with a constant Lorentzian spacelike angle at a point $p \in M_{III}$, denote θ_0, the equations (2.6), (3.13) and (3.14) give the following equation

$$\cos\theta_0 = \frac{du - cx'_n(u)dv}{\sqrt{du^2 - 2cx'_n(u)dudv + 2x_n^2(u)dv^2}}, \tag{3.15}$$

which is also expressed as

$$(2\cos^2\theta_0 x_n^2(u) - c^2 x_n'^2(u))\left(\frac{dv}{du}\right)^2 + 2c\sin^2\theta_0 x'_n(u)\frac{dv}{du} = \sin^2\theta_0. \tag{3.16}$$

For $2\cos^2\theta_0 x_n^2(u) - c^2 x_n'^2(u) = 0$, we find the function $v = v(u)$ given by (3.24). Therefore, we find the parametrization of the spacelike loxodrome $\alpha_{III}(u) = \mathbf{x}_{III}(u, v(u))$ on the spacelike helicoidal surface M_{III} in $\mathbb{E}_1^n$ given as (iii) of Theorem 3.1.

Thus, we give the following statement.

Theorem 3.1. *Let M be a spacelike helicoidal surface in Lorentzian n–space $\mathbb{E}_1^n$ defined by* (2.9), (2.12) *and* (2.16). *Then, the spacelike loxodrome on M has the following parametrization:*

i. for the helicoidal surface of type I,

$$\alpha_I(u) = (x_1(u)\cos v(u), x_1(u)\sin v(u), \ldots, x_n(u) + cv(u)) \tag{3.17}$$

where

$$v(u) = \pm\frac{1}{2\cos\theta_0}\int_{u_0}^{u} \frac{d\xi}{\sqrt{x_1^2(\xi) - c^2}} \tag{3.18}$$

or

$$v(u) = \int_{u_0}^{u} \frac{-2c\sin^2\theta_0 x'_n(\xi) \pm \sqrt{\sin^2 2\theta_0(x_1^2(\xi) - c^2(1 + x_n'^2(\xi)))}}{2\cos^2\theta_0(x_1^2(\xi) - c^2) - 2c^2 x_n'^2(\xi)}d\xi \tag{3.19}$$

provided that $\cos^2\theta_0(x_1^2(\xi) - c^2) - c^2 x_n'^2(\xi) \neq 0$,

ii. for the helicoidal surface of type II,

$$\alpha_{II}(u) = (x_1(u) + cv(u), x_2(u), \ldots, x_{n-2}(u), x_n(u)\sinh v(u), x_n(u)\cosh v(u)) \tag{3.20}$$

where

$$v(u) = \pm\frac{1}{2\cos\theta_0}\int_{u_0}^{u} \frac{d\xi}{\sqrt{x_n^2(\xi) + c^2}} \tag{3.21}$$

or

$$v(u) = \int_{u_0}^{u} \frac{2cx_1'^2(\xi)\sin^2\theta_0 \pm \sqrt{\sin^2 2\theta_0(x_n^2(\xi) + c^2(1 - x_1'^2(\xi)))}}{2\cos^2\theta_0(c^2 + x_n^2(\xi)) - 2c^2 x_1'^2(\xi)}d\xi \tag{3.22}$$

provided that $\cos^2\theta_0(c^2 + x_n^2(\xi)) - c^2 x_1'^2(\xi) \neq 0$.

iii. for the helicoidal surface of type III,

$$\begin{aligned}\alpha_{III}(u) =& x_1(u)e_1 + \sqrt{2}v(u)x_n(u)e_2 + x_3(u)e_3 + \cdots + x_{n-2}(u)e_{n-2} \\ &+ (x_{n-1}(u) + v^2(u)x_n(u) + cv(u))\xi_{n-1} + x_n(u)\xi_n\end{aligned} \tag{3.23}$$

where

$$v(u) = \pm\frac{c}{4A\cos^2\theta_0}e^{\pm\frac{\sqrt{2}\cos\theta_0}{c}(u-u_0)} \tag{3.24}$$

or

$$v(u) = \int_{u_0}^{u}\frac{-2cx_n'(\xi)\sin^2\theta_0 \pm \sqrt{\sin^2 2\theta_0(2x_n^2(\xi) - c^2x_n'^2(\xi))}}{4\cos^2\theta_0 x_n^2(\xi) - 2c^2x_n'^2(\xi)}d\xi \tag{3.25}$$

provided that $2\cos^2\theta_0 x_n^2(\xi) - c^2x_n'^2(\xi) \neq 0$ *for constants* $\theta_0 \neq \frac{\pi}{2}$ *and* A, $c > 0$.

As a particular case, we can give the parametrizations of spacelike loxodromes on spacelike right helicoidal surfaces in $\mathbb{E}_1^n$ as follows.

Corollary 3.2. *Let M be a spacelike right helicoidal surface in Lorentzian n–space $\mathbb{E}_1^n$ defined by (2.9), (2.12) and (2.16). Then, the spacelike loxodrome on M has the following parametrization:*

i. for the right helicoidal surface of type I,

$$\alpha_I(u) = (x_1(u)\cos v(u), x_1(u)\sin v(u), \ldots, x_n + cv(u)) \tag{3.26}$$

where x_n is a constant and

$$v(u) = \pm\tan\theta_0\int_{u_0}^{u}\frac{d\xi}{\sqrt{x_1^2(\xi) - c^2}}, \tag{3.27}$$

ii. for the right helicoidal surface of type II,

$$\alpha_{II}(u) = (x_1 + cv(u), x_2(u), \ldots, x_{n-2}(u), x_n(u)\sinh v(u), x_n(u)\cosh v(u)) \tag{3.28}$$

where x_1 is a constant and

$$v(u) = \pm\tan\theta_0\int_{u_0}^{u}\frac{d\xi}{\sqrt{x_n^2(\xi) + c^2}}, \tag{3.29}$$

iii. for the right helicoidal surface of type III,

$$\begin{aligned}\alpha_{III}(u) =& x_1(u)e_1 + \sqrt{2}v(u)x_ne_2 + x_3(u)e_3 + \cdots + x_{n-2}(u)e_{n-2} \\ &+ (x_{n-1}(u) + v^2(u)x_n + cv(u))\xi_{n-1} + x_n\xi_n\end{aligned} \tag{3.30}$$

where x_n is a non–zero constant and

$$v(u) = \pm\frac{\tan\theta_0}{\sqrt{2}x_n}(u - u_0) \tag{3.31}$$

for constants $\theta_0 \neq \frac{\pi}{2}$ *and* $c > 0$.

Also, using Corollary 3.2 and the equation (2.3), we get the following Corollary.

Corollary 3.3. *Let M be a spacelike right helicoidal surface in Lorentzian n–space $\mathbb{E}_1^n$ defined by* (2.9), (2.12) *and* (2.16)*. Then, the length of the spacelike loxodrome on M between two points u_0 and u_1 is given by*

$$s = \left| \frac{u_1 - u_0}{\cos\theta_0} \right| \tag{3.32}$$

where $\theta_0 \neq \frac{\pi}{2}$ is a constant.

4 Spacelike loxodromes on timelike helicoidal surfaces in $\mathbb{E}_1^n$

In this section, we obtain the parametrization of the spacelike loxodromes on the timelike helicoidal surface of type I, type II and type III defined by (2.9), (2.12) and (2.16), respectively.

For simplicity, define a constant Θ as

$$\Theta = \begin{cases} \cosh\theta_0 & \text{if } \varepsilon = 1, \\ \sinh\theta_0 & \text{if } \varepsilon = -1. \end{cases} \tag{4.1}$$

Consider a timelike helicoidal surface of type I M_I in $\mathbb{E}_1^n$ defined by (2.9) which means $\varepsilon x_1^2(u) - c^2(\varepsilon + x_n'^2(u)) < 0$. Thus, there are two cases occur according to the casual character of the meridian curve, i.e, $\varepsilon = 1$ or $\varepsilon = -1$.

Assume that $\alpha_I(t) = \mathbf{x}_I(u(t), v(t))$ is a spacelike loxodrome on M_I, so that, the equation (2.4) becomes

$$\varepsilon \left(\frac{du}{dt}\right)^2 - 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} + (x_1^2(u) - c^2)\left(\frac{dv}{dt}\right)^2 > 0 \tag{4.2}$$

and using the first one of (2.10), the equation (2.5) also gives

$$\langle \alpha_I'(t), (\mathbf{x}_I)_u \rangle = \varepsilon\frac{du}{dt} - cx_n'(u)\frac{dv}{dt}. \tag{4.3}$$

Since the loxodrome $\alpha_I(t)$ intersects the meridian of M_I with a constant Lorentzian timelike angle at a point $p \in M_I$, from the equations (2.7), (2.8), (4.2) and (4.3), we get

$$\Theta = \frac{\varepsilon du - cx_n'(u)dv}{\sqrt{\varepsilon du^2 - 2cx_n'(u)dudv + (x_1^2(u) - c^2)dv^2}}. \tag{4.4}$$

After rearranging this equation, we get the following differential equation

$$(\Theta^2(x_1^2(u) - c^2) - c^2x_n'^2(u))\left(\frac{dv}{du}\right)^2 + 2c(\varepsilon - \Theta^2)x_n'(u)\frac{dv}{du} = 1 - \varepsilon\Theta^2. \tag{4.5}$$

Solving this one, we obtain

$$v(u) = \int_{u_0}^{u} \frac{2cx_n'(\xi)(\Theta^2 - \varepsilon) \pm \sqrt{\sinh^2 2\theta_0(c^2(\varepsilon + x_n'^2(\xi)) - \varepsilon x_1^2(\xi))}}{2\Theta^2(x_1^2(\xi) - c^2) - 2c^2x_n'^2(\xi)} d\xi \tag{4.6}$$

with $\Theta^2(x_1^2(\xi)-c^2)-c^2x_n'^2(\xi) \neq 0$. For $\Theta^2(x_1^2(\xi)-c^2)-c^2x_n'^2(\xi) = 0$, we determine the function $v = v(u)$ given by (4.16). Thus, we find the parametrization of the spacelike loxodrome $\alpha_I(u) = \mathbf{x}_I(u, v(u))$ on the timelike helicoidal surface M_I in $\mathbb{E}_1^n$ given as (i) of Theorem 4.1.

Similarly, let M_{II} be a timelike helicoidal surface of type II in $\mathbb{E}_1^n$ defined by (2.12). That is $\varepsilon x_n^2(u)+c^2(\varepsilon - x_1'^2(u)) < 0$. Thus, there are two cases occur according to the casual character of the meridian curve, i.e, $\varepsilon = 1$ or $\varepsilon = -1$. Assume that $\alpha_{II}(t) = \mathbf{x}_{II}(u(t), v(t))$ is a spacelike loxodrome on M_{II}, so that the equation (2.4) and (2.5) imply

$$\varepsilon \left(\frac{du}{dt}\right)^2 + 2cx_1'(u)\frac{du}{dt}\frac{dv}{dt} + (x_n^2(u)+c^2)\left(\frac{dv}{dt}\right)^2 > 0 \tag{4.7}$$

and

$$\langle \alpha_{II}'(t), (\mathbf{x}_{II})_u \rangle = \varepsilon\frac{du}{dt} + cx_1'(u)\frac{dv}{dt}, \tag{4.8}$$

respectively. Due to the fact that the spacelike curve $\alpha_{II}(t)$ intersects the meridians of the timelike surface M_{II} with a constant Lorentzian timelike angle at a point $p \in M_{II}$, using the equations (4.7) and (4.8) in (2.7) and (2.8), we get

$$\Theta = \frac{\varepsilon du + cx_1'(u)dv}{\sqrt{\varepsilon du^2 + 2cx_1'(u)dudv + (x_n^2(u)+c^2)dv^2}}. \tag{4.9}$$

After rearranging this equation, we get the following differential equation

$$(\Theta^2(x_n^2(u)+c^2) - c^2x_1'^2(u))\left(\frac{dv}{du}\right)^2 + 2c(\Theta^2-\varepsilon)x_1'(u)\frac{dv}{du} = 1-\varepsilon\Theta^2. \tag{4.10}$$

For $\Theta^2(x_n^2(u)+c^2) - c^2x_1'^2(u) = 0$, we determine the function $v = v(u)$ given by (4.19). Solving this one, we find the parametrization of the spacelike loxodrome $\alpha_{II}(u) = \mathbf{x}_{II}(u, v(u))$ on the timelike helicoidal surface M_{II} in $\mathbb{E}_1^n$ given as (ii) of Theorem 4.1.

Let M_{III} be a timelike helicoidal surface of type III in $\mathbb{E}_1^n$ defined by (2.16). Thus, $2\varepsilon x_n^2(u) - c^2x_n'^2(u) < 0$ which implies $\varepsilon = 1$ or $\varepsilon = -1$. Suppose that $\alpha_{III}(t) = \mathbf{x}_{III}(u(t), v(t))$ is a spacelike loxodrome on M_{III}, so that the equation (2.4) implies

$$\varepsilon \left(\frac{du}{dt}\right)^2 - 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} + 2x_n^2(u)\left(\frac{dv}{dt}\right)^2 > 0 \tag{4.11}$$

and the equation (2.5) also gives

$$\langle \alpha_{III}'(t), (\mathbf{x}_{III})_u \rangle = \varepsilon\frac{du}{dt} - cx_n'(u)\frac{dv}{dt}. \tag{4.12}$$

Since the spacelike curve $\alpha_{III}(t)$ also meets the meridians of M_{III} with a constant Lorentzian timelike angle at a point $p \in M_{III}$, the equations (2.7), (2.8), (4.11) and (4.12) give

$$\Theta = \frac{\varepsilon du - cx_n'(u)dv}{\sqrt{\varepsilon du^2 - 2cx_n'(u)dudv + 2x_n^2(u)dv^2}} \tag{4.13}$$

which implies

$$(2\Theta^2 x_n^2(u) - c^2 x_n'^2(u))\left(\frac{dv}{du}\right)^2 + 2c(\varepsilon - \Theta^2)x_n'(u)\frac{dv}{du} = 1 - \varepsilon\Theta^2. \tag{4.14}$$

For $2\Theta^2 x_n^2(u) - c^2 x_n'^2(u) = 0$, we determine the function $v = v(u)$ given by (4.22). Hence, we find the parametrization of the spacelike loxodrome $\alpha_{III}(u) = \mathbf{x}_{III}(u, v(u))$ on the timelike helicoidal surface M_{III} in $\mathbb{E}_1^n$ given as (iii) of Theorem 4.1.

Thus, we have the following Theorem.

Theorem 4.1. *Let M be a timelike helicoidal surface in Lorentzian n–space $\mathbb{E}_1^n$ defined by* (2.9)*,* (2.12) *and* (2.16)*. Then, the spacelike loxodrome on M has the following parametrization*

i. for the helicoidal surface of type I,

$$\alpha_I(u) = (x_1(u)\cos v(u), x_1(u)\sin v(u), \ldots, x_n(u) + cv(u)) \tag{4.15}$$

where

$$v(u) = \frac{\varepsilon}{2\Theta}\int_{u_0}^{u}\frac{d\xi}{\sqrt{x_1^2(\xi) - c^2}} \tag{4.16}$$

or

$$v(u) = \int_{u_0}^{u}\frac{2cx_n'(\xi)(\Theta^2 - \varepsilon) \pm \sqrt{\sinh^2 2\theta_0(c^2(\varepsilon + x_n'^2(\xi)) - \varepsilon x_1^2(\xi))}}{2\Theta^2(x_1^2(\xi) - c^2) - 2c^2 x_n'^2(\xi)}d\xi \tag{4.17}$$

provided that $\Theta^2(x_1^2(\xi) - c^2) - c^2 x_n'^2(\xi) \neq 0$*,*

ii. for the helicoidal surface of type II,

$$\alpha_{II}(u) = (x_1(u) + cv(u), x_2(u), \ldots, x_{n-2}(u), x_n(u)\sinh v(u), x_n(u)\cosh v(u)) \tag{4.18}$$

where

$$v(u) = \frac{\varepsilon}{2\Theta}\int_{u_0}^{u}\frac{d\xi}{\sqrt{x_n^2(\xi) + c^2}} \tag{4.19}$$

or

$$v(u) = \int_{u_0}^{u}\frac{-2cx_1'(\xi)(\Theta^2 - \varepsilon) \pm \sqrt{\sinh^2 2\theta_0(c^2(x_1'^2(\xi) - \varepsilon) - \varepsilon x_n^2(\xi))}}{2\Theta^2(c^2 + x_n^2(\xi)) - 2c^2 x_1'^2(\xi)}dt \tag{4.20}$$

provided that $\Theta^2(c^2 + x_n^2(\xi)) - c^2 x_1'^2(\xi) \neq 0$*,*

iii. for the helicoidal surface of type III,

$$\begin{aligned}\alpha_{III}(u) =& x_1(u)e_1 + \sqrt{2}v(u)x_n(u)e_2 + x_3(u)e_3 + \cdots + x_{n-2}(u)e_{n-2} \\ &+ (x_{n-1}(u) + v^2(u)x_n(u) + cv(u))\xi_{n-1} + x_n(u)\xi_n\end{aligned} \tag{4.21}$$

where

$$v(u) = \pm\frac{\varepsilon c}{4A\Theta^2}e^{\pm\frac{\sqrt{2}\Theta}{c}(u-u_0)} \tag{4.22}$$

or

$$v(u) = \int_{u_0}^{u}\frac{2cx_n'(\xi)(\Theta^2-\varepsilon)\pm\sqrt{\sinh^2 2\theta_0(c^2x_n'^2(\xi)-2\varepsilon x_n^2(\xi))}}{4\Theta^2x_n^2(\xi)-2c^2x_n'^2(\xi)}d\xi \tag{4.23}$$

provided that $2\Theta^2x_n^2(\xi) - c^2x_n'^2(\xi) \neq 0$ *for constants* A, $c > 0$ *and* Θ *defined by* (4.1).

As a result, we can give the parametrizations of spacelike loxodromes on timelike right helicoidal surfaces in $\mathbb{E}_1^n$ as follows.

Corollary 4.2. *Let* M *be a timelike right helicoidal surface in Lorentzian n–space* $\mathbb{E}_1^n$ *defined by* (2.9) *and* (2.12). *Then, the spacelike loxodrome on* M *has the following parametrization*

i. for the right helicoidal surface of type I having spacelike meridian,

$$\alpha_I(u) = (x_1(u)\cos v(u), x_1(u)\sin v(u), \ldots, x_n + cv(u)) \tag{4.24}$$

where x_n *is a constant and*

$$v(u) = \pm\tanh\theta_0\int_{u_0}^{u}\frac{d\xi}{\sqrt{c^2-x_1^2(\xi)}}, \tag{4.25}$$

ii. for the right helicoidal surface of type II having timelike meridian,

$$\alpha_{II}(u) = (x_1 + cv(u), x_2(u), \ldots, x_{n-2}(u), x_n(u)\sinh v(u), x_n(u)\cosh v(u)) \tag{4.26}$$

where x_1 *is a constant*

$$v(u) = \pm\coth\theta_0\int_{u_0}^{u}\frac{d\xi}{\sqrt{c^2+x_n^2(\xi)}}, \tag{4.27}$$

for constants $\theta_0 \neq 0$ *and* $c > 0$.

Also, by using Corollary 4.2 and the equation (2.3), we give the following Corollary.

Corollary 4.3. *Let* M *be a timelike right helicoidal surface in Lorentzian n–space* $\mathbb{E}_1^n$. *Then, the length* s *of a spacelike loxodrome on* M *between two points* u_0 *and* u_1 *is given by the following:*

i. for a timelike right helicoidal surface of type I having spacelike meridian, $s = \left|\frac{u_1-u_0}{\cosh\theta_0}\right|$,

ii. for timelike right helicoidal surface of type II having timelike meridian, $s = \left|\frac{u_1-u_0}{\sinh\theta_0}\right|$

where θ_0 *is a non–zero constant.*

Example 4.1. We consider the following unit-speed spacelike profile curve in $\mathbb{E}^4_1$:

$$\beta_I(u) = \frac{\sqrt{2}}{2}\,(u, 0, u, 1)\,.$$

When take $c = \frac{\sqrt{2}}{2}$, we have the following parametrization of right helicoidal surface M_I:

$$x_I(u,v) = \frac{\sqrt{2}}{2}\,(u\cos v, u\sin v, u, 1+v)\,.$$

If we take $u \in (1,4)$, then M_I is a spacelike right helicoidal surface. Taking $u_0 = 1$ and $\theta_0 = \frac{\pi}{4}$, we get $v(u) = \sqrt{2}\ln(u + \sqrt{u^2-1})$. Thus, $v \in (0, 2.91814)$. Then, the parametrization of spacelike loxodrome is given by

$$\alpha_I(u) = \frac{\sqrt{2}}{2}\,(u\cos v(u), u\sin v(u), u, 1+v(u))$$

where $v(u) = \sqrt{2}\ln(u + \sqrt{u^2-1})$. Also, the arc-length of the spacelike loxodrome is equal to $3\sqrt{2}$.

The graphs of the projections of spacelike right helicoidal surface M_I, loxodrome and meridian ($v = 2$) in $\mathbb{E}^3_1$ can be drawn by using Mathematica plotting command

$$ParametricPlot3D[\{x_1(u,v) + x_2(u,v), x_3(u,v), x_4(u,v)\}, \{u, a, b\}, \{v, c, d\}]$$

as follows (Figure 1).

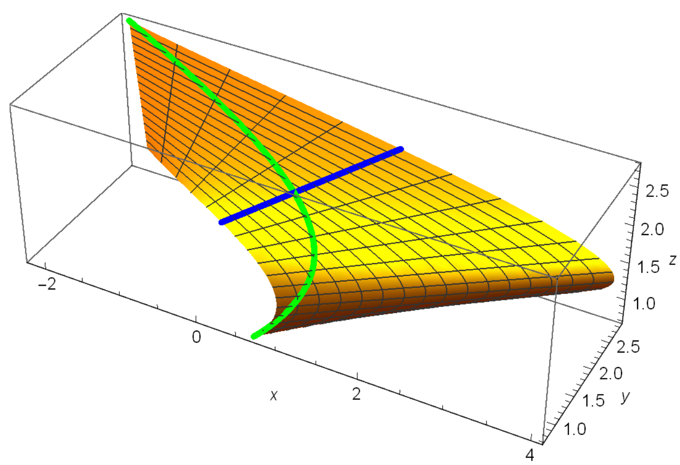

Figure 1: The spacelike right helicoidal surface, loxodrome (green), meridian (blue).

For the further examples in $\mathbb{E}^4_1$, we refer to [10].

5 Conclusion

Loxodromes in 3- and 4-dimensional different ambient spaces were studied by a lot of authors, see [1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 14, 15, 17, 18]. In [8], M. Babaarslan investigated the parametrizations of loxodromes on the helicoidal surfaces as well as the rotational surfaces in Euclidean n–space. In this paper, as a generalization of previous papers, we find the parametrizations of spacelike loxodromes on non-degenerate helicoidal surfaces in Lorentzian n–space $\mathbb{E}^n_1$. Also, as a particular case, we study spacelike loxodromes on non–degenerate right helicoidal surfaces in $\mathbb{E}^n_1$. Finally, we give an example in $\mathbb{E}^4_1$ to illustrate our main results via Wolfram Mathematica 12.3.

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Author's address:

Murat Babaarslan
Department of Mathematics, Faculty of Science and Letters,
Yozgat Bozok University, Yozgat, Turkey.
E-mail: murat.babaarslan@bozok.edu.tr

Burcu Bektaş Demirci
Department of Software Engineering, Faculty of Engineering,
Fatih Sultan Mehmet Vakıf University, Istanbul, Turkey.
E-mail: bbektas@fsm.edu.tr

Rukiye Genç
Department of Mathematics, Graduate School of Natural and Applied Sciences,
Yozgat Bozok University, Yozgat, Turkey.
E-mail: rkye.gnc@gmail.com