On almost Kenmotsu (κ, μ, ν)-spaces

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Abstract. The objective of the present research article is to investigate the characteristics of weakly symmetric and weakly concircular symmetric almost Kenmotsu (κ, μ, ν)-spaces admitting conformal Ricci solitons. In addition, we also discuss some results based on almost pseudo Ricci symmetric and weakly cyclic Z symmetric almost Kenmotsu (κ, μ, ν)-spaces.

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1 Introduction

In 1982, Hamilton [12] popularized the concept of Ricci flow principle and proved its existence. The Ricci flow is the evolution equation for the Riemannian manifold metrics given by

$$\frac{\partial}{\partial t} g = -2S, \tag{1.1}$$

where the Riemannian metric g and the Ricci tensor S. A self-similar approach to the Ricci flow [12], [34] is called Ricci soliton [11], if it only moves through a single family of diffeomorphism and scaling parameters. The equation, Ricci soliton is given by

$$\mathfrak{L}_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\mathfrak{L}$, V, and λ indicates a Lie derivative, a complete vector field, and a real scalar, respectively, on a Riemannian manifold. A Ricci soliton is also said to be expanding, shrinking, steady, and as λ is positive, negative and zero. A Ricci soliton reduced to Einstein equation with V=0. It has become much more necessary to solve the long-standing Poincar é conjecture posed in 1904 when Grigory Perelman implemented solitons to Ricci.
The notion of conformal Ricci flow was developed by Fischer [10], an alteration of the classical Ricci flow equation that rearranges the unit volume restriction of that

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equation to a scalar curvature constraint. In terms of equation, the conformal Ricci flow on a smooth connected n-manifold M [10] is defined as:

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg, \quad r(g) = -1, \tag{1.3}$$

where p is a non-dynamic field (time dependent scalar field), $r(g)$ is a n-dimensional multiple scalar curvature. It is similar to the Navier-Stokes fluid dynamics equations and because of this similarity the time-dependent scalar area p is called a conformal pressure and because of the actual physical pressure in fluid mechanics that serves to preserve the fluid's incompressibility, the conformal pressure acts as a Lagrange multiplier to deform the metric flow conformally so as to maintain the scalar curvature constraint. The equilibrium points of the conformal Ricci flow equations are Einstein metrics with Einstein constant -$\frac{1}{n}$. Thus the conformal pressure p at a point of equilibrium is zero and positive otherwise.
In 2015, Basu and Bhattacharyya [2] established the theory of conformal Ricci soliton and equation defined as

$$\mathfrak{L}_V g + 2S + \left[2\lambda - (p + \frac{2}{2n+1})\right] g = 0, \tag{1.4}$$

where λ is a constant . This equation is the generalized form of the Ricci soliton equation, and the conformal Ricci flow equation is also satisfied. Pigola et al. [25] initially introduced the concept of almost Ricci soliton. In addition, Sharma [28] also did an excellent job in nearly Ricci soliton.
A (M^n, g) Riemannian manifold is almost a Ricci soliton [4], if a complete vector field X exists and a smooth soliton function $\lambda : M^n \to \Re$ satisfying $R_{ij} + \frac{1}{2}(X_{ij} + X_{ji}) + \lambda g_{ij} = 0$, where $X_{ij} + X_{ji}$ and R_{ij} hold for the Lie derivative $(\mathfrak{L}_X g)$ and the Ricci tensor in local coordinates respectively. If $\lambda > 0, \lambda = 0$ or $\lambda < 0$; A conformal Ricci soliton is said to be almost conformal to Ricci soliton if it satisfies (1.4), it will be expanding, steady or shrinking, respectively.

In their articles [32] and [33], Tamassy and Binh respectively proposed the concept of weakly symmetrical manifolds and and weakly Ricci symmetric manifolds. There after many geometers studied these conditions on different manifolds [7], [22], [27], [39], [29]. The notion of weakly concircular symmetric manifold was introduced by Shaikh and Hui [26]. Recently, several authors investigated these condition on Kenmotsu manifolds [13], Trans-Sasakian manifolds [13],[24], Lorentzian concircular structure manifolds [21], generalized Sasakian space forms [31], (ϵ)-trans Sasakian manifolds [15], etc.
Recently, Mantica and Molinari [17] introduced weakly Z symmetric manifolds which generalize the term of weakly Ricci symmetric manifolds. Also De et al. [8] have proposed the idea that of weakly cyclic Z symmetric manifolds. Such a manifold is denoted by $(WCZS)_n$.
In 1972, Kenmotsu [14] introduced and studied well-known manifold called as Kenmotsu manifolds. The characteristics of Kenmotsu manifolds were examined by several writers such as [1], [36], [37], and others. Koufogiorgos el al. [16] introduced in the notion of (κ, μ, ν)-contact metric manifold defined as follow:

$$R(X,Y)\xi = \eta(Y)(kI + \mu h + \nu\varphi h)X - \eta(X)(kI + \mu h + \nu\varphi h)Y, \tag{1.5}$$

for some smooth functions k,μ and ν on M. Ozturk et al. [23] studied almost α-cosymplectic (κ,μ,ν)-space under different conditions (like η-parallelism) and gave an example in dimension three. These almost Kenmotsu manifolds whose almost Kenmotsu structures (φ,ξ,η,g) satisfy the condition

$$(1.6)\qquad \begin{aligned} R(\xi,X)Y &= k(g(Y,X)\xi-\eta(X)Y)+\mu(g(hY,X)\xi-\eta(Y)hX) \\ &\quad +\nu(g(\varphi hY,X)\xi-\eta(Y)\varphi hX), \end{aligned}$$

for $\kappa,\mu,\nu\in\Re_n(M^{2n+1})$, where $\Re_n(M^{2n+1})$ be the subring of the ring of smooth functions f on M^{2n+1} for which $df\wedge\eta=0$ [5], [16].

A non-flat differentiable manifold M^{2n+1} is called weakly symmetric if there exist a vector field P and 1-forms $\alpha,\beta,\gamma,\delta$ (not simultaneously zero) on M^{2n+1}, such that

$$(1.7)\qquad \begin{aligned} (\nabla_X R)(Y,Z)W &= \alpha(X)R(Y,Z)W+\beta(Y)R(X,Z)W \\ &\quad +\gamma(Z)R(Y,X)W+\delta(W)R(Y,Z)X \\ &\quad +g(R(Y,Z)W,X)P, \end{aligned}$$

holds for all vector fields $X,Y,Z,W\in\chi(M^{2n+1})$. A weakly symmetric manifold (M^{2n+1},g) is said to be pseudo-symmetric if $\beta=\gamma=\delta=\frac{1}{2}\alpha$ and $\alpha(X)=g(X,P)$, locally symmetric if $\alpha=\beta=\gamma=\delta=0$ and $P=0$. A weakly symmetric manifold is said to be proper if at least one of the 1-forms $\alpha,\beta,\gamma,\delta$ is not zero or $P\neq 0$.

A differentiable manifold M^{2n+1} is called weakly Ricci-symmetric if there exists 1-forms ε,σ,ρ on M^{2n+1} such that the condition

$$(1.8)\qquad (\nabla_X S)(Y,Z)=\varepsilon(X)S(Y,Z)+\sigma(Y)S(X,Z)+\rho(Z)S(X,Y),$$

holds for all vector fields $X,Y,Z,W\in\chi(M^{2n+1})$. If $\varepsilon=\sigma=\rho$, then M^{2n+1} is called pseudo Ricci-symmetric [6].
In view of (1.7), if M^{2n+1} is weakly symmetric, we have

$$(1.9)\qquad \begin{aligned} (\nabla_X S)(Z,W) &= \alpha(X)S(Z,W)+\beta(R(X,Z)W)+\gamma(Z)S(X,W) \\ &\quad +\delta(W)S(X,Z)+\rho(R(X,W)Z), \end{aligned}$$

where the 1-form ρ is defined by $\rho(X)=g(X,P)$ for all $X\in\chi(M^{2n+1})$.

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation. A concircular transformation is always a conformal transformation. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. A concircular curvature tensor of $(2n+1)$-dimensional almost Kenmotsu (κ,μ,ν)-space is given by[38]

$$(1.10)\qquad \begin{aligned} C(X,Y,Z,U) &= R(X,Y,Z,U) \\ &\quad -\frac{r}{2n(2n+1)}[g(Y,Z)g(X,U)-g(X,Z)g(Y,U)] \end{aligned}$$

If $\{e_i : i=1,2,3,\cdots,(2n+1)\}$ is an orthonormal basis of the tangent space at each point of the manifold and we define

$$(1.11)\qquad \hat{C}(X,U)=\sum_{i=1}^{2n+1}C(X,e_i,e_i,U).$$

In fact of (1.10) and (1.11), we get

$$\hat{C}(Y,Z)=S(Y,Z)-\frac{r}{2n+1}g(Y,Z). \tag{1.12}$$

In a Riemannian or a semi-Riemannian manifold (M^n,g), $(n>2)$, a $(0,2)$ symmetric tensor is a generalized $\hat{Z}$ tensor if

$$\hat{Z}(X,Y)=S(X,Y)+\pi g(X,Y), \tag{1.13}$$

where π is an arbitrary scalar function. The tensor $\hat{Z}$ was introduced in [18] and used in [19] and [20]. The classical Z-tensor is obtained with the choice If π=-$\frac{r}{n}$, where r is the scalar curvature. Hereafter we refer to the generalized Z-tensor simply as the Z-tensor. In particular, if the Z-tensor of a Riemannian manifold vanishes, then the manifold is Einstein. The scalar $\hat{z}$ is obtained by (1.13) as follows

$$\hat{z}=r+n\pi, \tag{1.14}$$

where the scalar curvature $r=\sum_{i=1}^{n}\epsilon_i S(e_i,e_i), g(e_i,e_i)=\epsilon_i, \epsilon_i=\pm 1$ and $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold. In a recent paper [17], the authors introduced weakly $\hat{Z}$ symmetric manifolds which is denoted by $(W\hat{Z}S)_n$. A Riemannian or a semi-Riemannian manifold is said to be weakly $\hat{Z}$ symmetric, denoted by $(W\hat{Z}S)_n$, if the generalized $\hat{Z}$ tensor satisfies the condition

$$(\nabla_X\hat{Z})(U,V)=A(X)\hat{Z}(U,V)+B(U)\hat{Z}(X,V)+D(V)\hat{Z}(U,X), \tag{1.15}$$

where A, B and D are 1-forms not simultaneously zero. If π=0, we recover from (1.15) a $(WRS)_n$, and as a particular case pseudo Ricci symmetric manifolds $(PRS)_n$ [6]. If π=-$\frac{r}{n}$ (classical Z tensor) and A is replaced by $2A$ and B and D are replaced by A, then $\hat{Z}(U,V)=\frac{n-1}{n}P(U,V)$, where $P(U,V)$ is the projective Ricci tensor considered by Chaki and Saha [6] and obtained by a contraction of the projective curvature tensor [9].

A non-flat Riemannian or a semi-Riemannian manifold (M^n,g),$(n>2)$ is called weakly cyclic $\hat{Z}$ symmetric if the generalized $\hat{Z}$ tensor is non-zero and satisfies the condition

$$\begin{aligned}&(\nabla_X\hat{Z})(U,V)+(\nabla_U\hat{Z})(V,X)+(\nabla_V\hat{Z})(X,U)\\ =\;&A(U)\hat{Z}(U,V)+B(U)\hat{Z}(V,X)+D(V)\hat{Z}(X,U),\end{aligned} \tag{1.16}$$

where $\hat{Z}$ is the generalized $\hat{Z}$ tensor and it is denoted by $(WC\hat{Z}S)_n$[8].

2 Almost Kenmotsu (κ,μ,ν)-space

Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a $(2n+1)$-dimensional almost contact Riemannian manifold, where φ is a $(1,1)$ tensor field, ξ is the structure vector field, η is a 1-form and g is Riemannian metric. Thus the almost contact structure (φ,ξ,η,g) satisfies

$$\eta(\xi)=1,\ \varphi\xi=0,\ \eta\circ\varphi=0, \tag{2.1}$$

$$\varphi^2X=-X+\eta(X)\xi,\qquad \eta(X)=g(X,\xi), \tag{2.2}$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X, Y on M^{2n+1}. The 2-form Ψ on M^{2n+1} defined by $\Psi(X, Y) = g(\varphi X, Y)$, is called the fundamental 2-form of the almost contact metric manifold M^{2n+1}. Almost contact metric manifolds such that $d\eta$=0 and $d\Psi$=$2\eta \wedge \Psi$ are almost Kenmotsu manifolds. Finally, a normal almost Kenmotsu manifold is called Kenmotsu manifold. An almost Kenmotsu manifold is a nice example of an almost contact manifold which is neither K-contact nor Sasakian manifolds. We recall some fundamental curvature properties of almost Kenmotsu manifolds which satisfy (1.5),(1.6) and the following properties

$$(\nabla_X \varphi)Y = g(\varphi X + hX, Y)\xi - \eta(Y)(\varphi X + hX), \tag{2.4}$$

$$\nabla_X \xi = -\varphi^2 X - \varphi hX, \tag{2.5}$$

$$S(X, \xi) = 2nk\eta(X), \tag{2.6}$$

$$Q\zeta = 2nk\zeta, \tag{2.7}$$

$$l = -k\varphi^2 + \mu h + \nu\varphi h, \tag{2.8}$$

$$l\varphi - \varphi l = 2\mu h\varphi + 2\nu h, \tag{2.9}$$

$$h^2 = (k+1)\varphi^2, k \leq -1, \tag{2.10}$$

$$\nabla_\xi h = -\mu\varphi h + (\nu - 2)h. \tag{2.11}$$

where Q is the Ricci operator, that is, $g(QX, Y)$=$S(X, Y)$, r is the scalar curvature of M^{2n+1} and l, h are the operators defined by $l(X)$=$R(X, \xi)\xi$ and h=$\frac{1}{2}\mathfrak{L}_\xi \varphi$, where $\mathfrak{L}$ is the Lie derivative operator.

3 Conformal Ricci soliton on almost Kenmotsu (κ, μ, ν)-space

Now, we recall the notion of conformal Ricci soliton on almost Kenmotsu (κ, μ, ν)-space. Then from (2.5), we have

$$\frac{1}{2}(\mathfrak{L}_\xi g)(X, Y) = g(X, Y) - \frac{1}{2}\{g(\varphi hX, Y) + g(\varphi hY, X)\} - \eta(X)\eta(Y), \tag{3.1}$$

In view of (1.4) and (3.1), we get

$$\begin{aligned} S(X, Y) \quad = \quad & \left[-1 + \left\{\lambda - (p + \frac{2}{2n+1})\right\}\right] g(X, Y) + \eta(X)\eta(Y) \\ & + \frac{1}{2}\{g(\varphi hX, Y) + g(\varphi hY, X)\}, \end{aligned} \tag{3.2}$$

which yields

$$S(X,\xi) = \left[\left\{\lambda - (p + \frac{2}{2n+1})\right\}\right]\eta(X) + \frac{1}{2}\eta(\varphi hX), \tag{3.3}$$

$$S(\xi,\xi) = \left\{\lambda - (p + \frac{2}{2n+1})\right\}, \tag{3.4}$$

$$QX = \left[-1 + \left\{\lambda - (p + \frac{2}{2n+1})\right\}\right]X + \eta(X)\xi + \varphi hX, \tag{3.5}$$

$$r = -2(n+1) + (2n+1)(\lambda - p), \tag{3.6}$$

At this glance, keeping in mind (3.6), we have

Proposition 3.1. *A conformal Ricci soliton (g,ξ,λ) on almost Kenmotsu (κ,μ,ν)-space is always expanding if $r \geq 0$.*

Corollary 3.2. *At an equilibrium stage if almost Kenmotsu (κ,μ,ν)-space admitting conformal Ricci soliton (g,ξ,λ) then soliton is expanding,shrinking or steady according as $r > 2(n+1)$,$r < 2(n+1)$,or r=$2(n+1)$ respectively.*

4 Weakly symmetric almost Kenmotsu (κ,μ,ν)-space admitting conformal Ricci soliton (g,ξ,λ)

In this section, we light the impact of conformal Ricci soliton (g,ξ,λ) on weakly symmetric almost Kenmotsu (κ,μ,ν)-space. So we have

Theorem 4.1. *If a weakly symmetric almost Kenmotsu (κ,μ,ν)-space admitting conformal Ricci soliton (g,ξ,λ) then, either the sum of 1-form is zero everywhere or the soliton is expanding.*

Proof. Let M^{2n+1} is a weakly symmetric almost Kenmotsu (κ,μ,ν)-space. Then substituting W=ξ in (1.9), we have

$$\begin{aligned}(\nabla_X S)(Z,\xi) &= \alpha(X)S(Z,\xi) + \beta(R(X,Z)\xi) \\ &\quad +\gamma(Z)S(X,\xi) + \delta(\xi)S(X,Z) + \rho(R(X,\xi)Z).\end{aligned} \tag{4.1}$$

In view of (1.5), (3.2) and (3.3), equation (4.1) reduces to

$$\begin{aligned}(\nabla_X S)(Z,\xi) &= \left\{\lambda - (p + \frac{2}{2n+1})\right\}\eta(Z)\alpha(X) + \frac{1}{2}\alpha(X)\eta(\varphi hZ) \\ &\quad +\kappa\eta(Z)\beta(X) + \mu\eta(Z)\beta(hX) + \nu\eta(Z)\beta(\varphi hX) \\ &\quad -\kappa\beta(Z)\eta(X) - \mu\eta(X)\beta(hZ) - \nu\eta(X)\beta(\varphi hZ) \\ &\quad +\left\{\lambda - (p + \frac{2}{2n+1})\right\}\eta(X)\gamma(Z) + \frac{1}{2}\gamma(Z)\eta(\varphi hX) \\ &\quad +\left\{\lambda - (p + \frac{2}{2n+1})\right\}g(X,Z)\delta(\xi) + \eta(X)\eta(Z)\delta(\xi) \\ &\quad +\frac{1}{2}\delta(\xi)\left\{g(\varphi hX,Z) + g(\varphi hZ,X)\right\} + \rho(R(X,\xi)Z).\end{aligned} \tag{4.2}$$

Taking covariant differentiation of the Ricci tensor S along the vector field X, we have

$$(\nabla_X S)(Z,\xi) = \nabla_X S(Z,\xi) - S(\nabla_X Z,\xi) - S(Z,\nabla_X \xi).$$

By the use of (2.5) and (3.3) above equation takes the form

$$\begin{aligned}(4.3)\quad (\nabla_X S)(Z,\xi) \;=\; & \left\{\lambda - (p + \frac{2}{2n+1})\right\}[g(\nabla_X Z,\xi) + g(Z,\nabla_X \xi)] \\ & + \frac{1}{2}\{g(\nabla_X \varphi)hZ,\xi) + g(\varphi(\nabla_X h)Z,\xi) + g(\varphi h(\nabla_X Z),\xi) \\ & + g(\varphi hZ,\nabla_X \xi)\} + \left\{\lambda - (p + \frac{2}{2n+1})\right\} g(\nabla_X Z,\xi) \\ & - \frac{1}{2} g(\varphi h \nabla_X Z,\xi) - S(Z,X) + \left\{\lambda - (p + \frac{2}{2n+1})\right\} \\ & \times \eta(Z)\eta(X) + \frac{1}{2} g(\varphi hZ,\xi)\eta(X) + S(Z,\varphi hX).\end{aligned}$$

Comparing the right hand sides of (4.2) and (4.3), we obtain

$$\begin{aligned}(4.4)\quad & \left\{\lambda - (p + \frac{2}{2n+1})\right\}\eta(Z)\alpha(X) + \frac{1}{2}\alpha(X)\eta(\varphi hZ) + \kappa\eta(Z)\beta(X) \\ & + \mu\eta(Z)\beta(hX) + \nu\eta(Z)\beta(\varphi hX) - \kappa\beta(Z)\eta(X) - \mu\eta(X)\beta(hZ) \\ & - \nu\eta(X)\beta(\varphi hZ) + \left\{\lambda - (p + \frac{2}{2n+1})\right\}\eta(X)\gamma(Z) \\ & + \frac{1}{2}\gamma(Z)\eta(\varphi hX)\left\{\lambda - (p + \frac{2}{2n+1})\right\} g(X,Z)\delta(\xi) + \eta(X)\eta(Z)\delta(\xi) \\ & + \frac{1}{2}\delta(\xi)\{g(\varphi hX,Z) + g(\varphi hZ,X)\} + \rho(R(X,\xi)Z) \\ & = \left\{\lambda - (p + \frac{2}{2n+1})\right\}[g(\nabla_X Z,\xi) + g(Z,\nabla_X \xi)] \\ & + \frac{1}{2}\{g(\nabla_X \varphi)hZ,\xi) + g(\varphi(\nabla_X h)Z,\xi) + g(\varphi h(\nabla_X Z),\xi) + g(\varphi hZ,\nabla_X \xi)\} \\ & + \left\{\lambda - (p + \frac{2}{2n+1})\right\} g(\nabla_X Z,\xi) - \frac{1}{2} g(\varphi h \nabla_X Z,\xi) - S(Z,X) \\ & + \left\{\lambda - (p + \frac{2}{2n+1})\right\}\eta(Z)\eta(X) + \frac{1}{2} g(\varphi hZ,\xi)\eta(X) + S(Z,\varphi hX).\end{aligned}$$

Setting $X{=}Z{=}\xi$ in (4.4) and on simplification, we yield

$$\left\{\lambda - (p + \frac{2}{2n+1})\right\}\{\alpha(\xi) + \gamma(\xi) + \delta(\xi)\} = 0.$$

This implies that either $\lambda = (p + \frac{2}{2n+1})$, or $\alpha(\xi){+}\gamma(\xi){+}\delta(\xi){=}0$. Since the vanishing the sum of 1-form $\alpha{+}\gamma{+}\delta$ over the vector field ξ necessary in order that M^{2n+1} be a conformal Ricci soliton on weakly symmetric almost Kenmotsu (κ, μ, ν)-space. Now we can easily show that, as similar to the previous calculation, $\left\{\lambda - (p + \frac{2}{2n+1})\right\}[\alpha(X) + \gamma(X) + \delta(X)] = 0$, holds for arbitrary vector field X on M^{2n+1}, which prove the Theorem 4.1. □

Ozturk et al. [23] proved that on an almost Kenmotsu (κ,μ,ν)-space of dimension $n \geq 5$, the function κ,μ,ν only vary in the direction of ξ, *i.e.*, $X(\kappa)$=$X(\mu)$=$X(\nu)$=0 for every vector field X orthogonal to ξ. Due to this fact and Theorem 4.1, we have the following corollaries.

Corollary 4.2. *Let M^{2n+1} be an almost Kenmotsu (κ,μ,ν)-space of dimension $n \geq 5$, the function κ,μ,ν only vary in the direction of ξ, i.e., $X(\kappa)$=$X(\mu)$=$X(\nu)$=0 for every vector field X orthogonal to ξ, then there does not exists weakly symmetric almost Kenmotsu (κ,μ,ν)-space $M^{2n+1}, (\kappa \leq -1)$, if $\alpha+\gamma+\delta$ is not everywhere zero.*

Corollary 4.3. *Let M^{2n+1} be an almost Kenmotsu $(\kappa,\mu,\nu)-$space of dimension $n \geq 5$, the function κ,μ,ν only vary in the direction of ξ, i. e., $X(\kappa)$=$X(\mu)$=$X(\nu)$=0 for every vector field X orthogonal to ξ, then there exist no weakly symmetric conformal Ricci soliton almost Kenmotsu (κ,μ,ν)-space $M^{2n+1}, (\kappa \leq -1)$, if the soliton is expanding in nature.*

Aktan et al. [1] proved that the Ricci tensor S on weakly symmetric almost Kenmotsu (κ,μ,ν)-spaces has the form

$$\begin{aligned}
(4.5)\quad S(X,Z) \;=\; & \frac{1}{\delta(\xi)}\{2nX(\kappa)\eta(Z)+2n\kappa g(Z,\nabla_X\xi)-S(Z,\nabla_X\xi)\\
& -2n\kappa\alpha(X)-\beta(\kappa)\eta(Z)X-\kappa\eta(Z)\beta(X)-\beta(\mu)\eta(Z)hX\\
& -\mu\eta(Z)\beta(hX)-\beta(\nu)\eta(Z)\varphi hX-\nu\eta(Z)\beta(\varphi hX)\\
& +\beta(\kappa)\eta(X)Z+\kappa\eta(X)\beta(Z)+\beta(\mu)\eta(X)hZ\\
& +\mu\eta(X)\beta(hZ)+\beta(\nu)\eta(X)\varphi hZ+\nu\eta(X)\beta(\varphi hZ)\\
& -2n\kappa\gamma(Z)\eta(X)+\rho(\kappa)(g(X,Z)\xi-\eta(Z)X)+\kappa(g(X,Z)\\
& \times\rho(\xi)-\eta(Z)\rho(X))+\rho(\mu)(g(hZ,X)\xi-\eta(Z)(hX))\\
& +\mu(g(hZ,X)\rho(\xi)-\eta(Z)\rho(hX))+\rho(\nu)(g(\varphi hZ,X)\xi\\
& -\eta(Z)(\varphi hX))+\nu(g(\varphi hZ,X)\rho(\xi)-\eta(Z)\rho(\varphi hX))\}
\end{aligned}$$

provided $\delta(\xi) \neq 0$. We suppose that $\hat{h}$ is a $(0,2)$ type symmetric parallel tensor field on an almost Kenmotsu (κ,μ,ν)-space $M^{2n+1}, (\kappa \leq -1)$, such that

$$(4.6)\qquad \hat{h}\,(\mathrm{X,Z}) = (\mathfrak{L}_\xi g)(X,Z)+2S(X,Z).$$

Setting X=Z=ξ in (4.6) and then using (2.5) and (4.5), we observe that

$$(4.7)\qquad \hat{h}\,(\xi{,}\xi) = \frac{4n}{\delta(\xi)}\{\xi(\kappa)-\kappa\{\alpha(\xi)+\gamma(\xi)\}\}.$$

If (g,ξ,λ) be a conformal Ricci soliton on almost Kenmotsu (κ,μ,ν)-space. Then from (3.2), we get

$$(4.8)\qquad \hat{h}(\xi,\xi) = -2\left\{\lambda-(p+\frac{2}{2n+1})\right\}.$$

In view of (4.7) and (4.8), we yields

$$(4.9)\qquad \lambda = \frac{1}{2}\left(p+\frac{2}{(2n+1)}\right)-\frac{2n}{\delta(\xi)}\{\kappa(\alpha(\xi)+\gamma(\xi))-\xi(\kappa)\}.$$

Thus, we can state the following:

Theorem 4.4. *If the tensor field* $\mathfrak{L}_\xi g + 2S$ *of type* $(0,2)$ *on a weakly symmetric almost Kenmotsu* (κ, μ, ν)*-space* $M^{2n+1}, (\kappa \leq -1)$, *with* $\delta(\xi) \neq 0$ *is parallel, then the conformal Ricci soliton* (g, ξ, λ) *is shrinking, steady or expanding according as* $\xi(k) \geq 0, \delta(\xi) > 0$, $\xi(k) = k\{\alpha(\xi) + \gamma(\xi)\}$ *or* $\xi(k) = 0, \delta(\xi) < 0$ *respectively.*

5 Weakly Ricci symmetric almost Kenmotsu (κ, μ, ν)-space admitting conformal Ricci soliton (g, ξ, λ)

This section deals with the conformal Ricci soliton (g, ξ, λ) on weakly Ricci symmetric almost Kenmotsu (κ, μ, ν)-space and we conclude the results.

Theorem 5.1. *Let* (g, ξ, λ) *be a conformal Ricci soliton on a weakly Ricci symmetric almost Kenmotsu* (κ, μ, ν)*-space, then the sum of* 1*-forms is zero, i.e.,* $\varepsilon + \sigma + \rho = 0$ *everywhere provided that the conformal Ricci soliton is to be either shrinking or expanding.*

Proof. Let M^{2n+1} is a weakly Ricci symmetric almost Kenmotsu (κ, μ, ν)-space. Putting $Z = \xi$ in (1.8) and by use of (3.3), we have

$$\begin{aligned}(\nabla_X S)(Y, \xi) &= [\varepsilon(X)\{\lambda - (p + \frac{2}{2n+1})\}\eta(Y) + \frac{1}{2}\eta(\varphi h Y)] \\ &\quad +[\sigma(Y)\{\lambda - (p + \frac{2}{2n+1})\}\eta(X)] \\ &\quad +\frac{1}{2}\eta(\varphi h X)\} + \rho(\xi) S(X, Y).\end{aligned} \tag{5.1}$$

Also replacing Z with Y in (4.3) and comparing the right hand sides of the equation (4.3) and (5.1), we obtain

$$\begin{aligned}&[\varepsilon(X)\{\lambda - (p + \frac{2}{2n+1})\}\eta(Y) + \frac{1}{2}\eta(\varphi h Y)] + [\sigma(Y)\{\lambda - (p + \frac{2}{2n+1})\}\eta(X)] \\ &+\frac{1}{2}\eta(\varphi h X)\} + \rho(\xi) S(X, Y) \\ &= \left\{\lambda - (p + \frac{2}{2n+1})\right\}[g(\nabla_X Y, \xi) + g(Y, \nabla_X \xi)] \\ &+\frac{1}{2}\{g(\nabla_X \varphi)hY, \xi) + g(\varphi(\nabla_X h)Y, \xi) + g(\varphi h(\nabla_X Y), \xi) + g(\varphi h Y, \nabla_X \xi)\} \\ &+\left\{\lambda - (p + \frac{2}{2n+1})\right\} g(\nabla_X Y, \xi) - \frac{1}{2} g(\varphi h \nabla_X Y, \xi) - S(Y, X) \\ &+\left\{\lambda - (p + \frac{2}{2n+1})\right\}\eta(Y)\eta(X) + \frac{1}{2} g(\varphi h Y, \xi)\eta(X) + S(Y, \varphi h X).\end{aligned} \tag{5.2}$$

Taking $X = Y = \xi$ in (5.2) and using (2.1) and (3.4), we get

$$\left\{\lambda - \left(p + \frac{2}{2n+1}\right)\right\}[\varepsilon(\xi) + \sigma(\xi) + \rho(\xi)] = 0. \tag{5.3}$$

Again putting $X{=}\xi$ in (5.1), we have

$$(5.4)\ \ \{\lambda-(p+\frac{2}{2n+1})\}\sigma(Y)\eta(Y)=-\sigma(\xi)\{-\{\lambda-(p+\frac{2}{2n+1})\}\eta(Y)+{}^1\!/\!{}_2\eta(\varphi hY)\}.$$

Replacing Y with X, we yield

$$(5.5)\ \ \{\lambda-(p+\frac{2}{2n+1})\}\sigma(X)\eta(X)=-\sigma(\xi)\{-\{\lambda-(p+\frac{2}{2n+1})\}\eta(X)+{}^1\!/\!{}_2\eta(\varphi hX)\}.$$

If we take $Y=\xi$ in (5.2), we obtain

$$(5.6)\ \ \{\lambda-(p+\frac{2}{2n+1})\}\varepsilon(X)\eta(X)=-\alpha(\xi)\{-\{\lambda-(p+\frac{2}{2n+1})\}\eta(X)+{}^1\!/\!{}_2\eta(\varphi hX)\}$$

and

$$(5.7)\ \ \{\lambda-(p+\frac{2}{2n+1})\}\rho(X)\eta(X)=-\rho(\xi)\{-\{\lambda-(p+\frac{2}{2n+1})\}\eta(X)+{}^1\!/\!{}_2\eta(\varphi hX)\}.$$

Taking the summation of (5.5), (5.6) and (5.7), using (5.3), gives

$$\{\lambda-(p+\frac{2}{2n+1})\}\eta(X)\{\sigma(X)+\varepsilon(X)+\rho(X)\}=0,$$

for all $X\in\chi(M^{2n+1})$.
So either $\lambda{=}(p+\frac{2}{2n+1})$ or $\sigma(X)+\varepsilon(X)+\rho(X)=0$. In general $\eta(X)\neq 0$ on almost Kenmotsu manifolds, thus the Theorem 5.1 is proved. □

In view of Theorem 5.1 and the results of Ozturk et al. [23], we state the corollary.

Corollary 5.2. *Let M be an almost Kenmotsu (κ,μ,ν)-space of dimension greater than or equal to 5, the function κ,μ,ν only vary in the direction of ξ, i.e., $X(\kappa){=}X(\mu)$ $=X(\nu){=}0$ for every vector field X orthogonal to ξ, then there does not exist conformal Ricci soliton on weakly Ricci symmetric almost Kenmotsu (κ,μ,ν)-space M^{2n+1}, $(\kappa\leq -1)$, if the sum of the 1−forms, i.e., $\varepsilon+\sigma+\rho$, is not everywhere zero.*

It is also observed that [1] the Ricci tensor S of a weakly Ricci symmetric almost Kenmotsu (κ,μ,ν)-space has the from

$$(5.8)\qquad \begin{aligned} S(X,Y) \;=\; &\frac{1}{\rho(\xi)}\{2nX(\kappa)\eta(Y)+2n\kappa g(Y,\nabla_X\xi)-S(Y,\nabla_X\xi)\\ &-2n\kappa\varepsilon(X)\eta(Y)-2n\kappa\sigma(Y)\eta(X)\},\end{aligned}$$

provided $\rho(\xi)\neq 0$. Again let $\hat{h}$ is a $(0,2)$-type symmetric parallel tensor field on an almost Kenmotsu (κ,μ,ν)-space M^{2n+1}, $(\kappa\leq -1)$, such that

$$(5.9)\qquad \hat{\mathrm{h}}\,(\mathrm{X},\mathrm{Y})=(\mathfrak{L}_\xi g)(X,Y)+2S(X,Y).$$

Taking $X{=}Y{=}\xi$ and using (3.1) and (5.8), equation(5.9) takes the form

$$(5.10)\qquad \hat{\mathrm{h}}\,(\xi,\xi)=\frac{4n}{\rho(\xi)}\left\{\xi(\kappa)-\kappa\{\varepsilon(\xi)+\sigma(\xi)\}\right\}.$$

In view of (4.8) and (5.10), we get

$$\lambda = \frac{1}{2}\left(p + \frac{2}{(2n+1)}\right) - \frac{2n}{\rho(\xi)}\left\{\kappa\{\varepsilon(\xi)+\sigma(\xi)\} - \xi(\kappa)\right\}. \tag{5.11}$$

Thus, we can state the following:

Theorem 5.3. *If the tensor field $\mathfrak{L}_\xi g + 2S$ of type $(0,2)$ on a weakly Ricci symmetric almost Kenmotsu (κ,μ,ν)-space $M^{2n+1}, (\kappa \leq -1)$, with $\rho(\xi) \neq 0$ is parallel, then conformal Ricci soliton (g,ξ,λ) is shrinking, steady and expanding according as $\xi(k) \geq 0, \rho(\xi) > 0$; $\xi(k) = k\{\alpha(\xi)+\gamma(\xi)\}$ and $\xi(k) = 0, \rho(\xi) < 0$ respectively.*

6 Weakly concircular symmetric almost Kenmotsu (κ,μ,ν)-space admitting conformal Ricci soliton (g,ξ,λ)

A Riemannian manifold (M^n, g), $(n > 2)$ is called weakly concircular symmetric manifold [23] if its concircular curvature tensor C of type $(0,4)$ is not identically zero and satisfies

$$\begin{aligned}(\nabla_X C)(Y,Z,U,V) \;=\; & A(X)C(Y,Z,U,V) + B(Y)C(X,Z,U,V) \\ & + H(Z)C(Y,X,U,V) + E(V)C(Y,Z,X,V) \\ & + D(V)C(Y,Z,U,X).\end{aligned} \tag{6.1}$$

In a weakly concircular symmetric manifold, it is also known that $B{=}H$ and $D{=}E$ [23]. Then

$$\begin{aligned}(\nabla_X C)(Y,Z,U,V) \;=\; & A(X)C(Y,Z,U,V) + B(Y)C(X,Z,U,V) \\ & + B(Z)C(Y,X,U,V) + D(V)C(Y,Z,X,V) \\ & + D(V)C(Y,Z,U,X),\end{aligned} \tag{6.2}$$

holds for all vector fields $X,Y,Z,U,V \in \chi(M^{2n+1})$.

Theorem 6.1. *In a weakly concircular symmetric almost Kenmotsu (κ,μ,ν)-space admitting conformal Ricci soliton (g,ξ,λ), then the relation (6.5) holds.*

Proof. We suppose that almost Kenmotsu (κ,μ,ν)-space admitting conformal Ricci soliton (g,ξ,λ) is weakly concircular symmetric then its satisfies (6.2) . For fix $Y{=}V{=}e_i$ in (6.2) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\begin{aligned}&(\nabla_X S)(Z,U) - \frac{dr(X)}{(2n+1)}g(Z,U) \\ &= A(X)[S(Z,U) - \frac{r}{(2n+1)}g(Z,U)] + B(Z)[S(X,U) - \frac{r}{(2n+1)}g(X,U)] \\ &+ D(U)[S(Z,X) - \frac{r}{(2n+1)}g(Z,X)] + B(R(X,Z)U) + D(R(X,U)Z) \\ &- \frac{r}{2n(2n+1)}[(B(X)+D(X))g(Z,U) - g(X,U)B(Z) - g(X,Z)D(U)].\end{aligned} \tag{6.3}$$

On substituting X=Z=U=ξ in (6.3) and using (3.4) and (4.3), we obtain

$$[\lambda-(p+\frac{2}{(2n+1)})][A(\xi)+B(\xi)+D(\xi)]=\frac{-dr(\xi)}{(2n+1)}, \tag{6.4}$$

which is equivalent to

$$A(\xi)+B(\xi)+D(\xi)=\frac{-dr(\xi)}{(2n+1)(\lambda-p)-2}. \tag{6.5}$$

This complete the prove. □

Also, we have the following result

Corollary 6.2. *If a weakly concircular symmetric almost Kenmotsu (κ,μ,ν)-space admitting conformal Ricci soliton (g,ξ,λ), then either the sum of the 1-forms A,B and D is zero everywhere over the filed ξ, or the soliton is always expanding provided the scalar curvature r of the manifold is constant.*

If we equated any two of the vector fields X,Z and U to ξ, using (3.3),(4.3), then from (6.3), one can easily obtain

$$D(U)=D(\xi)\eta(U), \tag{6.6}$$

$$B(Z)=B(\xi)\eta(Z), \tag{6.7}$$

$$A(X)=[\frac{dr(\xi)}{(2n+1)(\lambda-p)-2}-A(\xi)]\eta(X)+\frac{dr(X)}{(2n+1)(\lambda-p)-2}. \tag{6.8}$$

Thus, we state the following theorem:

Theorem 6.3. *In a weakly concircular symmetric almost Kenmotsu (κ,μ,ν)-space admitting conformal Ricci soliton (g,ξ,λ) the associated 1-forms are given by (6.6),(6.7) and(6.8) respectively.*

7 Weakly concircular Ricci almost Kenmotsu (κ,μ,ν)-space admitting conformal Ricci soliton (g,ξ,λ)

Right now, we light up weakly concircular Ricci almost Kenmotsu (κ,μ,ν)-space admitting conformal Ricci soliton (g,ξ,λ). Now, we recall the following.

Definition 7.1. A Riemannian manifold (M^n,g),$(n>2)$ is said to be weakly concircular Ricci symmetric manifold if its concircular Ricci curvature $\hat{C}$ of type $(0,2)$ is not identically zero and satisfies the condition [10]:

$$(\nabla_X\hat{C})(Y,Z)=A(X)\hat{C}(Y,Z)+B(Y)\hat{C}(X,Z)+D(Z)\hat{C}(X,Y), \tag{7.1}$$

holds for all vector fields $X,Y,Z\in\chi(M^{2n+1})$.

Theorem 7.1. *If a weakly concircular Ricci symmetric almost Kenmotsu (κ, μ, ν)-space admitting conformal Ricci soliton (g, ξ, λ),then the relation (7.1) holds.*

Proof. In view of (1.12) and (7.1), we obtain

$$\begin{aligned}(\nabla_X S)(Y,Z) - \frac{dr(X)}{(2n+1)}g(Y,Z) \\ = A(X)[S(Y,Z) - \frac{r}{(2n+1)}g(Y,Z)] + B(Y)[S(XZ) - \frac{r}{(2n+1)}g(X,Z)] \\ +D(Z)[S(X,Y) - \frac{r}{(2n+1)}g(X,Y)],\end{aligned} \tag{7.2}$$

On substituting $X{=}Y{=}Z{=}\xi$ in (7.2) and using (3.3) and (4.3), we get

$$[\lambda - (p + \frac{2}{(2n+1)})][A(\xi) + B(\xi) + D(\xi)] = \frac{-dr(\xi)}{(2n+1)}, \tag{7.3}$$

which implies that

$$A(\xi) + B(\xi) + D(\xi) = \frac{-dr(\xi)}{(2n+1)(\lambda - p) - 2}. \tag{7.4}$$

Again, we equated any two of the vector fields X,Y and Z to ξ, using (3.3),(4.3), then from equation (7.2), one can obtain

$$D(Z) = D(\xi)\eta(Z), \tag{7.5}$$

$$B(Y) = B(\xi)\eta(Y), \tag{7.6}$$

$$A(X) = [\frac{dr(\xi)}{(2n+1)(\lambda - p) - 2} + A(\xi)]\eta(X) + \frac{dr(X)}{(2n+1)(\lambda - p) - 2}. \tag{7.7}$$

Adding (7.5),(7.6),(7.7), and using (7.4), we have

$$A(X) + B(X) + D(X) = \frac{dr(X)}{(2n+1)(\lambda - p) - 2}.$$

□

This completes the proof of the Theorem 7.1. Also as per this sequel, we have the following corollary

Corollary 7.2. *If a weakly concircular Ricci symmetric almost Kenmotsu (κ, μ, ν)-space admitting conformal Ricci soliton (g, ξ, λ) the sum of the 1-forms A,B and D is zero everywhere if and only if the scalar curvature r of the manifold is constant.*

The notion of a special weakly Ricci symmetric manifold was introduced and studied by Singh and Quddus [30]. An n-dimensional Riemannian manifold (M, g) is called a special weakly concircular Ricci symmetric manifold $(SWRS)_n$ if

$$(\nabla_X \hat{C})(Y,Z) = 2\varepsilon(X)\hat{C}(Y,Z) + \varepsilon(Y)\hat{C}(X,Z) + \varepsilon(Z)\hat{C}(X,Y), \tag{7.8}$$

where ε is a 1-form and is defined by $\varepsilon(X){=}g(X, \rho)$, where ρ is the associated vector field. Keeping in mind we recall

Theorem 7.3. *If a special weakly concircular Ricci symmetric almost Kenmotsu (κ,μ,ν)-space with conformal Ricci soliton (g,ξ,λ) admits cyclic Ricci tensor then the associated 1-form A must vanishes, provided $\lambda\neq(p+\frac{2}{(2n+1)}+\frac{r}{(2n+1)})$.*

*Proof.*Let the manifold M^{2n+1} satisfies (7.8). Then taking cyclic sum of (7.8), we get

$$\begin{aligned}(\nabla_X\hat{C})(U,V)+(\nabla_U\hat{C})(X,V)+(\nabla_V\hat{C})(U,X)\\ =4[A(X)\hat{C}(U,V)+A(U)\hat{C}(X,V)+A(V)\hat{C}(U,X)]\end{aligned}\tag{7.9}$$

If M^{2n+1} admits a cyclic Ricci tensor. Then (7.9) reduces to

$$A(X)\hat{C}(U,V)+A(U)\hat{C}(X,V)+A(V)\hat{C}(U,X)=0.\tag{7.10}$$

Setting $U{=}V{=}\xi$ in (7.10), we have

$$[\lambda-(p+\frac{2}{(2n+1)})[A(X)+2A(\xi)\eta(X)]=0,\tag{7.11}$$

this implies that either $\lambda=(p+\frac{2}{(2n+1)}+\frac{r}{(2n+1)})$, or $A(X)+2A(\xi)\eta(X){=}0$.
Now, if $\lambda\neq(p+\frac{2}{(2n+1)}+\frac{r}{(2n+1)})$, then

$$A(X)+2A(\xi)\eta(X)=0\tag{7.12}$$

Again taking $X{=}\xi$ in (7.12), we obtain that $A(\xi)=0$. With reference to this and (7.12), we yield $A(X){=}0$, $\forall X$. The proof is completed.

Theorem 7.4. *A special weakly concircular Ricci symmetric almost Kenmotsu (κ,μ,ν)-space can not be an Einstein manifolds if the scalar curvature r of the manifold is constant.*

Proof. As we known that for Einstein manifold, we have $S(Y,Z){=}\tau g(Y,Z)$, and $(\nabla_X S)(Y,Z){=}0$. Thus for $(SWCRS)$ almost Kenmotsu (κ,μ,ν)-space, we get

$$\begin{aligned}-\frac{dr(X)}{(2n+1)}g(Y,Z) &= 2A(X)[\alpha-\frac{r}{(2n+1)}]g(Y,Z)\\ &\quad +A(Y)[\alpha-\frac{r}{(2n+1)}]g(X,Z)\\ &\quad +A(Z)[\alpha-\frac{r}{(2n+1)}]g(X,Y),\end{aligned}\tag{7.13}$$

On substituting $X{=}Y{=}Z{=}\xi$ in (7.13), we get

$$4A(\xi)[r-(2n+1)\alpha]=dr(\xi),\tag{7.14}$$

which implies that if r is constant then $\eta(\rho){=}0$, that is $A(Y){=}0$, $\forall Y$. This completes the proof of the theorem 7.4.

Corollary 7.5. *A special weakly concircular Ricci symmetric almost Kenmotsu (κ,μ,ν)-space admits cyclic Ricci tensor, can not be an Einstein manifolds if the scalar curvature r of the manifold is constant.*

Corollary 7.6. *In a special weakly concircular Ricci symmetric an Einstein almost Kenmotsu (κ,μ,ν)-space the 1-form A is given by $A(\xi){=}\frac{dr(\xi)}{4[(2n+1)\alpha-r]}$, provided $r\neq(2n+1)\alpha$.*

8 Almost pseudo Ricci symmetric almost Kenmotsu (κ, μ, ν)-space

Chaki and Kawaguchi [13] introduced the concept of almost pseudo Ricci symmetric manifolds as an extended class of pseudo symmetric manifolds. A Riemannian manifold (M, g) is called an almost pseudo Ricci symmetric manifold $(APRS)_n$, if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfying the following condition:

$$(\nabla_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + A(Y)S(X, Z) + A(Z)S(X, Y), \tag{8.1}$$

where A and B are two non-zero 1-forms defined by $A(X){=}g(X, \rho_1)$, $B(X){=}g(X, \rho_2)$.

Theorem 8.1. *There is no almost pseudo Ricci symmetric almost Kenmotsu (κ, μ, ν)-space admitting cyclic Ricci tensor, unless $3A{+}B$ vanishes everywhere on M^{2n+1}.*

*Proof.*Let M^{2n+1} is almost pseudo Ricci symmetric almost Kenmotsu (κ, μ, ν)-space. Then from (8.1), taking cyclic sum we have

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y)$$
$$= [3A(X) + B(X)]S(Y, Z) + [3A(Y) + B(Y)]S(X, Z) + [3A(Z) + B(Z)]S(X, Y)$$

Let M^{2n+1} admits a cyclic Ricci tensor, then

$$[3A(X) + B(X)]S(Y, Z) + [3A(Y) + B(Y)]S(X, Z) + [3A(Z) + B(Z)]S(X, Y) = 0$$

Replacing $X{=}Y{=}Z{=}\xi$ in above we conclude that

$$2nk[3\eta(\rho_1) + \eta(\rho_2)] = 0, \tag{8.2}$$

which implies that $3\eta(\rho_1) + \eta(\rho_2){=}0$, thus we get $3A(X) + B(X){=}0$. □ □

Corollary 8.2. *A conformal Ricci soliton on almost pseudo Ricci symmetric almost Kenmotsu (κ, μ, ν)-space admitting cyclic Ricci tensor, is always expanding, provided $3A + B \neq 0$ on M^{2n+1}.*

9 Weakly $\hat{Z}$ symmetric almost Kenmotsu (κ, μ, ν)-space admitting conformal Ricci soliton (g, ξ, λ)

In this section, we need to demonstrate some results for weakly $\hat{Z}$ symmetric almost Kenmotsu (κ, μ, ν)-space with conformal Ricci soliton (g, ξ, λ) . Thus we prove the following result

Theorem 9.1. *If a $(W\hat{Z}S)_{2n+1}$ almost Kenmotsu (κ, μ, ν)-space admitting conformal Ricci soliton (g, ξ, λ) then the relation (9.3) holds.*

Proof. Let M^{2n+1} is a $(W\hat{Z}S)_{2n+1}$ almost Kenmotsu (κ, μ, ν)-space. Then substituting $V{=}\xi$ in (1.15), and using (1.13), we have

$$\begin{aligned}(\nabla_X S)(U, \xi) + d\pi(X)g(U, \xi) \quad = \quad & A(X)[S(U, \xi) + \pi g(U, \xi)] \\ & +B(U)[S(X, \xi) + \pi g(X, \xi)] \\ & +D(\xi)[S(U, X) + \pi g(U, X)],\end{aligned} \tag{9.1}$$

Putting $X{=}U{=}\xi$ in (9.1) and keeping in mind (3.4) and (4.3), we obtain

$$[\lambda - (p + \frac{2}{(2n+1)}) + \pi][A(\xi) + B(\xi) + D(\xi)] = d\pi(\xi), \tag{9.2}$$

which implies that

$$A(\xi) + B(\xi) + D(\xi) = \frac{(2n+1)d\pi(\xi)}{(2n+1)(\lambda - p + \pi) - 2}. \tag{9.3}$$

This complete the desired result. □

We have the following corollary

Corollary 9.2. *If a $(W\hat{Z}S)_{2n+1}$ almost Kenmotsu (κ, μ, ν)-space admitting conformal Ricci soliton (g, ξ, λ) the sum of the 1-forms A,B and D is zero everywhere over the filed ξ if and only if the function π is constant.*

Corollary 9.3. *If a $(W\hat{Z}S)_{2n+1}$ almost Kenmotsu (κ, μ, ν)-space admitting conformal Ricci soliton (g, ξ, λ) then soliton is expanding, shrinking and steady according as*
i) $(p + \frac{2}{(2n+1)}) > \pi$,
ii) $(p + \frac{2}{(2n+1)}) < \pi$ *and*
iii) $(p + \frac{2}{(2n+1)}) = \pi$, *respectively.*

Theorem 9.4. *If a $(WC\hat{Z}S)_{n+1}$ almost Kenmotsu (κ, μ, ν)-space admitting conformal Ricci soliton (g, ξ, λ) then either the sum of 1-form is zero everywhere over the vector filed ξ or $\lambda = (p + \frac{2}{(2n+1)}) - \pi$.*

Proof. Let M^{2n+1} is a $(WC\hat{Z}S)_{2n+1}$ almost Kenmotsu (κ, μ, ν)-space admitting conformal Ricci soliton (g, ξ, λ) then from (1.13) and (1.15), we have

$$\begin{aligned} &A(U)[S(U,V) + \pi g(U,V)] + B(U)[S(V,X) \\ &+\pi g(V,X)] + D(V)[S(X,U) + \pi g(X,U)] = 0 \end{aligned} \tag{9.4}$$

Putting $X{=}U{=}V{=}\xi$ in (9.4) and keeping in mind (3.4), we yield

$$[\lambda - (p + \frac{2}{(2n+1)}) + \pi][A(\xi) + B(\xi) + D(\xi)] = 0, \tag{9.5}$$

which implies that either $\lambda = (p + \frac{2}{(2n+1)}) - \phi$, or $A(\xi) + B(\xi) + D(\xi){=}0$. This completes the proof. □

Corollary 9.5. *In a $(WC\hat{Z}S)_{2n+1}$ almost Kenmotsu (κ, μ, ν)-space admitting conformal Ricci soliton (g, ξ, λ) then soliton is expanding, shrinking or steady according as*
i). $(p + \frac{2}{(2n+1)}) > \pi$,
ii). $(p + \frac{2}{(2n+1)}) < \pi$ *or*
iii). $(p + \frac{2}{(2n+1)}) = \pi$,
provided the sum of 1-form is not zero everywhere over the vector filed ξ.

10 Example of almost Kenmotsu (κ, μ, ν)-space admitting conformal Ricci soliton (g, ξ, λ)

Let us assume a 3- dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0\}$, where (x, y, z) are the Cartesian coordinates in $\mathbb{R}^3$. We define three vector fields on M such as [16]

$$v_1 = \frac{\partial}{\partial x}, \quad v_2 = \frac{\partial}{\partial y}, \quad v_3 = -\frac{4}{z} e^f f_y \frac{\partial}{\partial x} + \psi \frac{\partial}{\partial y} + e^{\frac{f}{2}} \frac{\partial}{\partial z},$$

are linearly independent at each point of M, where $f = f(y, z) < 0$ for all (y, z) is a solution of the partial differential equation

$$2f_{yy} + f_y^2 = -ze^{-f}, \tag{10.1}$$

and the function $\psi = \psi(x, y, z)$ solves the system of partial differential equations

$$\psi_x = \frac{4}{zx^2} e^f, \quad \psi_y = \frac{1}{2z} e^{\frac{f}{2}} - \frac{f_z e^{f/z}}{2} - \frac{4e^f f_y}{xz}. \tag{10.2}$$

We define a Riemannian metric g on M such that $g(v_i, v_j) = \delta_{ij}$ for $i, j = 1, 2, 3$. We easily obtain

$$[v_1, v_2] = 0, \quad [v_1, v_3] = \frac{4e^f}{zx^2} v_2, \tag{10.3}$$

$$[v_2, v_3] = 2v_1 + \left(\frac{1}{2z} e^{f/z} - \frac{f_z}{2} e^{f/2} - \frac{4}{xz} e^f f_y - \frac{\psi f_y}{2} \right) + \frac{1}{2} f_y v_3.$$

Let η be the 1-from defined as $g(Z, v_1) = \eta(Z)$ for all $Z \in \chi(M)$ and φ be the tesnor field of type $(1, 1)$ defined by $\varphi v_1 = 0$, $\varphi v_2 = v_3$, and $\varphi v_3 = -v_2$. Using the linearity of φ, $d\eta$, and g, we easily find that $\eta(v_1) = 1$, $d\eta(U, Z) = g(\varphi U, Z)$, and $g(\varphi U, \varphi Z) = g(U, Z) - \eta(U)\eta(Z)$ for all vector fields U, Z on M. Hence $(M, \eta, \xi = v_1, \varphi, g)$ is contact metric manifolds. Let ∇ be the Levi-Civita connection to g and R be the Riemannian curvature tensor of g.
Setting $\xi = v_1$, $X = v_2$ and $\varphi X = v_3$ and using Koszul's formula

$$\begin{aligned} 2g(\nabla_E F, G) &= Eg(F, G) + Fg(G, E) - Gg(E, F) \\ &+ g([E, F], G) - g([F, G], E) + g([G, E], F) \end{aligned}$$

and also from equation (10.1) and (10.2), we compute the following values

$$\nabla_X \xi = \left(-\frac{2e^f}{zx^2} - 1 \right) v_3, \qquad \nabla_{\varphi X} \xi = \left(1 - \frac{2e^f}{zx^2} \right) v_2, \qquad \nabla_{\varphi X} \varphi X = \frac{1}{2} f_y v_2,$$

$$\nabla_\xi X = -\left(1 + \frac{2e^f}{zx^2} \right) \varphi X, \qquad \nabla_\xi \varphi X = \left(1 + \frac{2e^f}{zx^2} \right) X, \quad \nabla_\xi \xi = 0,$$

$$\nabla_{\varphi X} X = -\frac{1}{2} f_y \varphi X + \left(\frac{2e^f}{zx^2} - 1 \right) \xi, \ \nabla_X X = \left(-\frac{1}{2z} e^{f/2} + \frac{f_y}{2} e^{f/2} + \frac{4}{xz} e^f f_y + \frac{\psi f_y}{2} \right) \varphi X,$$

$$\nabla_X \varphi X = \left(\frac{1}{2z} e^{f/2} - \frac{f_y}{2} e^{f/2} - \frac{4}{xz} e^f f_y - \frac{\psi f_y}{2} \right) X + \left(\frac{2e^f}{zx^2} + 1 \right) \xi.$$

From the definition of the tensor field h and using relations from (10.3) we turn up $h\xi = 0$ and

$$hX = \frac{1}{2}(\mathfrak{L}_\xi \varphi)X = \frac{1}{2}([\xi, \varphi X] - \varphi[\xi, X]) = \frac{2e^f}{zx^2}X. \tag{10.4}$$

Similarly, we find that

$$h\varphi X = -\frac{2e^f}{zx^2}X.$$

Setting now $\kappa = 1 - (4e^{2f})/(z^2x^4)$, $\mu = 2(1 + 2e^f)/(zx^2)$, and $\nu = -\frac{2}{x}$. Now using the last two relation, we easily obtain the non-vanishing components of Riemannian curvature and Ricci tensor

$$R(X, \xi)\xi = -\frac{4}{x^3z}e^f\varphi X + \left(1 + \frac{2e^f}{zx^2}\right)^2 X$$

$$R(\varphi X, \xi)\xi = -\frac{4}{x^3z}e^f\varphi X - \left(1 + \frac{2e^f}{zx^2}\right)^2 \left(\frac{6e^f}{zx^2} - 1\right)$$

and

$$S(\xi, \xi) = \left(1 + \frac{2e^f}{zx^2}\right)$$

Adopting equation (3.4), we obtain

$$\lambda = \left(p + \frac{2}{2n+1}\right) + \left(1 + \frac{2e^f}{zx^2}\right)$$

Thus, any conformal Ricci soliton (g, ξ, λ) on almost Kenmotsu (κ, μ, ν)-space is expanding.

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