An investigation on the existence of totally contact umbilical screen-slant lightlike submanifolds of indefinite Sasakian manifolds

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Abstract. In this paper, we prove that every totally contact umbilical(TCU) proper screen-slant lightlike submanifold of an indefinite Sasakian manifold is totally contact geodesic. Further, we investigate existence of TCU proper screen-slant lightlike submanifolds of an indefinite Sasakian form $\bar{M}(c)$ with constant ϕ-sectional curvature $c \neq \epsilon$.

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1 Introduction

The geometry of degenerate (lightlike) submanifold was studied in detail by Duggal and Bejancu in [4]. A submanifold, with induced degenerate metric, of a semi-Riemannian manifold is known as degenerate or lightlike submanifold. Due to action of (1, 1)-tensor field ϕ on the distributions of any given lightlike submanifold, it is possible to define various classes of these submanifolds.
In 1960, Sasaki [14] introduced Sasakian structure on a differentiable manifold. The notion of semi-Riemannian metric on a Sasakian manifold was introduced by Takahasi [16]. Later, various submanifolds of indefinite sasakian manifold has been studied. Sahin et. al. [13] introduced screen-slant lightlike submanifolds of an indefinite Sasakian manifold. In paper [9] author study "totally contact umbilical proper slant lightlike submanifolds of an indefinite Sasakian manifold ". In this paper, author prove that every totally contact umbilical proper slant lightlike submanifold is totally contact geodesic. Apart from this some important results Theorem 4.4, 4.5. 4.6 are given. In [8, 10], author investigate similar work for slant lightlike submanifold of indefinite Kenmotsu manifold and Indefinite Cosymplectic submanifolds. By taking reference work of paper [8, 9] and the fact that there is no inclusion relation between slant and screen slant lightlike submanifolds of indefinite Sasakian manifold, we investigate this work for screen slant lightlike submanifold. The paper is organized in following setting:

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In section 2, we give a short overview of lightlike submanifolds and all required results for the current research work. In section 3, we explain the decomposition of vector fields in different distributions and state geometry of screen-slant lightlike submanifolds of an indefinite Sasakian manifold. In section 4, we prove that, "there does not exist any totally contact umbilical proper screen slant lightlike submanifold, other than totally contact geodesic, of indefinite Sasakian manifold". Further, we investigate existence of TCU proper screen-slant lightlike submanifolds of an indefinite Sasakian form $\bar{M}(c)$ with constant ϕ-sectional curvature $c \neq \epsilon$.

2 Preliminaries

A submanifold (M^m, g) of a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ with constant index q $(1 \leq q \leq m+n-1,\ m, n \geq 1)$ is known as degenerate(lightlike) submanifold, if the induced metric g from $\bar{g}$ is degenerate.
Due to degenerate induced metric on TM, there exist non zero intersection of T_xM ($m-$ dimensional) and $T_xM^{\perp}$ ($n-$ dimensional), which is denoted by $Rad(TM)$. A lightlike submanifold is known as r-lightlike, if there exist a smooth distribution $Rad(TM)$ of rank $r > 0$, such that every member of M goes to r-dimension subspace $Rad(T_xM)$ of T_xM. Then $S(TM)$(screen distribution) and $S(TM^{\perp})$(screen transversal distribution) are non-degenerate complementary subbundles of $Rad(TM)$ in TM and $TM^{\perp}$ respectively. Let $ltr(TM)$(lightlike transversal bundle) and $tr(TM)$(transversal bundle) be complementary but not orthogonal vector bundles to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and TM in $T\bar{M}|_M$ respectively.
Then, the orthogonal decomposition of $tr(TM)$ and $T\bar{M}|_M$ are given by(for detail see [4]) "

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}) \tag{2.1}$$

and

$$T\bar{M}|_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}) \tag{2.2}$$

respectively.
Let $\bar{\nabla}$ be a metric connection on $\bar{M}$. For $Y,\ Z \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_Y Z = \nabla_Y Z + h(Y, Z), \tag{2.3}$$

$$\bar{\nabla}_Y U = -A_U Y + \nabla^{\perp}_Y U, \tag{2.4}$$

where $\{\nabla_Y Z, A_U Z\}$ and $\{h(Y, Z), \nabla^{\perp}_Y U\}$ belong to $\Gamma(Rad(TM))$ and $\Gamma(tr(TM))$ respectively, ∇ is a torsion-free linear connection on M, h (known as second fundamental form) is a symmetric $\mathcal{F}(M)$ bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$, A_U (known as shape operator) is a linear endomorphism on $\Gamma(TM)$ and $\nabla^{\perp}$ is a linear connection on $tr(TM)$ which is known as the transversal linear connection.
Then (2.3) and (2.4) reduce to

$$\bar{\nabla}_Y Z = \nabla_Y Z + h^l(Y, Z) + h^s(Y, Z), \tag{2.5}$$

$$(2.6) \qquad \bar{\nabla}_Y N = -A_N Y + \nabla^l_Y(N) + D^s(Y, N),$$

$$(2.7) \qquad \bar{\nabla}_Y W = -A_W Y + \nabla^s_Y(W) + D^l(Y, W),$$

where $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$.
The equations (2.3), (2.5) are known as Gauss equations and (2.4), (2.6),(2.7) are known as Weingarten equations respectively, for the lightlike submanifold M of $\bar{M}$. Using metric connection $\bar{\nabla}$ and (2.3), (2.4)-(2.7), we get the following equations:

$$(2.8) \qquad \bar{g}(h^s(Y, Z), W') + \bar{g}(Z, D^l(Y, W')) = g(A_{W'} Y, Z),$$

$$(2.9) \qquad \bar{g}(h^l(Y, Z), \xi) + \bar{g}(Z, h^l(Y, \xi) = -g(Z, \nabla_Y \xi),$$

for $Y, Z \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $W' \in \Gamma(S(TM^\perp))$.
It is known that, in the study of non-degenerate submanifolds, the induced connection is always metric connection. But, in the case of lightlike submanifolds, it is not true. Hence, for $X_1, X_2, X_3 \in \Gamma(TM)$ and $U, U' \in \Gamma(tr(TM))$, we have following formulas

$$(2.10) \qquad (\nabla_{X_1} g)(X_2, X_3) = \bar{g}(h^l(X_1, X_2), X_3) + \bar{g}(h^l(X_1, X_3), X_2),$$

$$(2.11) \qquad (\nabla^t_{X_1} \bar{g})(U, U') = -\{\bar{g}(A_U X_1, U') + \bar{g}(A_{U'} X_1, U)\}.$$

For detail understanding of equations (2.3)-(2.11), see pg. 196-198 in[5].
Let $\bar{R}$ and R be the curvature tensors of the ambient space $\bar{M}$ and lightlike submanifolds M respectively, then [5]

$$\begin{aligned} \bar{R}(X, Y)Z = {} & R(X, Y)Z + A_{h^l(X,Z)} Y - A_{h^l(Y,Z)} X + A_{h^s(X,Z)} Y - A_{h^s(Y,Z)} X \\ & + (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z) + (\nabla_X h^s)(Y, Z) - (\nabla_Y h^s)(X, Z) + \\ (2.12) \qquad & D^l(X, h^s(Y, Z)) + D^l(Y, h^s(X, Z)) + D^s(X, h^l(Y, Z)) - D^s(Y, h^l(X, Z)), \end{aligned}$$

where

$$(2.13) \qquad (\nabla_X h^s)(Y, Z) = \nabla_X h^s(Y, Z) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z),$$

$$(2.14) \qquad (\nabla_X h^l)(Y, Z) = \nabla_X h^l(Y, Z) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z).$$

Definition 2.1. An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called (ϵ)-almost contact metric manifold [2, 3] if there exist $(1,1)$-tensor field ϕ, a vector field V(known as characteristic vector field), a 1-form η such that

$$(2.15) \qquad \bar{g}(\phi Y, \phi Z) = \bar{g}(Y, Z) - \epsilon \eta(Y)\eta(Z), \ \bar{g}(V, V) = \epsilon,$$

$$(2.16) \qquad \phi^2 Y = -Y + \eta(X)V, \ \eta(V) = \epsilon,$$

$$(2.17) \qquad d\eta(Y, Z) = \bar{g}(Y, \phi Z) = -\bar{g}(\phi Y, Z), \ \epsilon = 1 \ or \ -1.$$

for all $Y, Z \in \Gamma(T\bar{M})$.

It follows that

$$\eta \circ \phi = 0,\ \phi V = 0,\ \bar{g}(Y, V) = \epsilon \eta(Y).$$

Here $(\phi, V, \eta, \bar{g})$ is known as contact metric structure on $\bar{M}$. This structure is called normal contact structure on $\bar{M}$ if $N_\phi + d\eta \otimes V$, where N_ϕ is known as Nijenhuis tensor field of ϕ.

A normal contact metric manifold on $\bar{M}$ is called an indefinite Sasakian structure[16] if and only if, for any $Y, Z \in \Gamma(TM)$,

$$(\bar{\nabla}_Y \phi) Z = -\bar{g}(Y, Z)V + \epsilon \eta(Z) Y, \tag{2.18}$$

$$\bar{\nabla}_Y V = -\epsilon \phi Y. \tag{2.19}$$

Here, $\epsilon = 1$ represents spacelike contact metric manifold and $\epsilon = -1$ represents timelike contact metric manifold.
In this paper, we study spacelike indefinite Sasakian manifolds. The curvature tensor field $\bar{R}$ of an indefinite Sasakian manifold $\bar{M}(c)$ with constant holomorphic curvature c is given by

$$\begin{aligned}\bar{R}(X, Y, Z, W) = \frac{c}{4}\{&\bar{g}(X, W)\bar{g}(Y, Z) - \bar{g}(X, Z)\bar{g}(Y, W) + \bar{g}(X, \bar{J}W)\bar{g}(Y, \bar{J}Z) - \\ &\bar{g}(X, \bar{J}Z)\bar{g}(Y, \bar{J}W) - 2\bar{g}(X, \bar{J}Y)\bar{g}(Z, \bar{J}W)\} + \frac{1}{4}\{\bar{g}\big((\bar{\nabla}_X \bar{J})(W), (\bar{\nabla}_Y \bar{J})(Z)\big) \\ &-\bar{g}\big((\bar{\nabla}_X \bar{J})Z, (\bar{\nabla}_Y \bar{J})W\big) - 2\bar{g}\big((\bar{\nabla}_X \bar{J})Y, (\bar{\nabla}_Z \bar{J})W\big)\}.\end{aligned} \tag{2.20}$$

3 Screen-slant lightlike submanifolds

Definition 3.1. [13] Let M be a $2q-$lightlike submanifold of an indefinite Sasakian manifold $(\bar{M}, g, S(TM))$ of index $2q$ such that $2q < dim(M)$ with structure vector field V tangent to M. Then M is said to be a screen-slant lightlike submanifold of $\bar{M}$ if following conditions are satisfied:

- (i) $Rad(TM)$ is invariant with respect to ϕ.
- (ii) For any non-zero vector field Y tangent to $S(TM) = D \perp \{V\}$ at $y \in U \subset M$, the angle $\theta(Y)$ (known as slant angle) between ϕY and $S(TM)$ is constant, where D is the complementary distribution to V in screen distribution $S(TM)$ and Y and V are linearly independent.

From above definition, we have the following decomposition

$$TM = Rad(TM) \oplus_{orth.} D \oplus_{orth.} \{V\}.$$

Throughout the paper, $(M, g, S(TM))$ will be considered as a $2q$-lightlike submanifold of an indefinite Sasakian manifold with constant index $2q < dim(M)$ and structure vector field V is always tangent to M.

Let $(M, g, S(TM))$ be a screen-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$, P and Q are projection morphisms on the distribution $Rad(TM)$ and D, respectively. Then, for any $X \in \Gamma(TM)$, we have

$$X = PX + QX + \eta(X)V, \tag{3.1}$$

where $PX \in \Gamma(Rad(TM))$ and $QX \in \Gamma(D)$.
For any vector field $X \in \Gamma(TM)$, suppose T and w be the projection maps on TM and $tr(TM)$, then

$$\phi(X) = TX + wX, \tag{3.2}$$

For any vector field $U \in tr(TM)$, suppose B and C be the projection maps on $ltr(TM)$ and $S(TM)^{\perp}$, then

$$\phi(U) = BU + CU, \tag{3.3}$$

where $TX \in \Gamma(TM)$, $wX \in \Gamma(tr(TM))$, $BU \in \Gamma(S(TM))$ and $CU \in \Gamma(S(TM)^{\perp})$.
Applying ϕ on (3.1) and using (3.2), we obtain

$$\phi(X) = TPX + TQX + wQX. \tag{3.4}$$

The screen transversal bundle $S(TM^{\perp})$ can be decomposed as:

$$S(TM^{\perp}) = wQ(S(TM)) \perp \mu.$$

Then, for any $W \in S(TM^{\perp})$ and $N \in ltr(TM)$, we have

$$\phi(W) = BW + CW, \quad \phi(N) = CN, \tag{3.5}$$

where $BW \in S(TM)$, $CW \in S(TM^{\perp})$ and $CN \in ltr(TM)$.
Now, we state some important theorems and corollary (see detail in [6]), these results are used throughout the paper

Theorem 3.1. *[6] Let $(M, g, S(TM))$ be a $2q-$lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with constant index $2q < dim(M)$. Then M is said to be is a screen-slant lightlike submanifold if and only if there exists a constant $\lambda \in [-1, 0]$ such that*

$$(P \circ T)^2 = \lambda(-Y + \eta(Y)V), \tag{3.6}$$

for any $Y \in \Gamma(S(TM))$. Moreover, in this case $\lambda = -cos^2\theta|_{S(TM)}$.

Corollary 3.2. *[6] Let $(M, g, S(TM))$ be a screen-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, for any $Y, Z \in \Gamma(TM)$,*

$$g(TQY, TQZ) = cos^2\theta|_{S(TM)}[g(Y, Z) - \eta(Y)\eta(Z)], \tag{3.7}$$

$$g(wQY, wQZ) = sin^2\theta|_{S(TM)}[g(Y, Z) - \eta(Y)\eta(Z)]. \tag{3.8}$$

Theorem 3.3. *Let M be a screen-slant lightlike submanifold of an indefinite Sasakian manifold then, for any $Y, Z \in \Gamma(TM)$, the following equations hold:*

$$(\nabla_Y T)Z = A_{wQZ}Y + Bh^s(Y,Z) - g(Y,Z)V + \eta(Z)Y, \tag{3.9}$$

$$(\nabla_Y w)Z = Ch^l(Y,Z) - h^l(Y,TZ) - h^s(Y,TZ) - D^l(Y,wZ) + Ch^s(Y,Z) + Ch^s(Y,Z), \tag{3.10}$$

where $(\nabla_Y T)Z = \nabla_Y TZ - T\nabla_Y Z$ *and* $(\nabla_Y w)Z = \nabla^s_Y wQZ - wQ\nabla_Y Z$.

Proof. For any $Y, Z \in \Gamma(TM)$, using equation (2.18), we obtain

$$\bar{\nabla}_Y \phi Z = \phi\bar{\nabla}_Y Z - g(Y,Z)V + \eta(Z)Y.$$

Using (3.2) in above equation, we get

$$\bar{\nabla}_Y(TZ + wZ) = \phi\bar{\nabla}_Y Z - g(Y,Z)V + \eta(Z)Y,$$

$$\begin{aligned}\nabla_Y TZ + h^l(Y,TZ) + h^s(Y,TZ) - A_{wZ}Y + \nabla^s_Y wZ + D^l(Y,wZ) = T\nabla_Y Z + w\nabla_Y Z\\ +Ch^l(Y,Z) + Bh^s(Y,Z) + Ch^s(Y,Z) - g(Y,Z)V + \eta(Z)Y.\end{aligned} \tag{3.11}$$

Equating tangential and transversal components in (3.11), we obtain

$$(\nabla_Y T)Z = A_{wZ}Y + Bh^s(Y,Z) - g(Y,Z)V + \eta(Z)Y,$$

and

$$(\nabla_Y w)Z = Ch^l(Y,Z) - h^l(Y,TZ) - h^s(Y,TZ) - D^l(Y,wZ) + Ch^l(Y,Z) + Ch^s(Y,Z).$$

□

4 Totally contact umbilical screen-slant lightlike submanifolds

Definition 4.1. [17] Let M be a lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$, with structure vector field V tangent to M, is said to be totally contact umbilical manifold. If, for any vector fields Y, Z tangent to M and a vector field α normal to M, the second fundamental form is

$$h(Y,Z) = [g(Y,Z) - \eta(Y)\eta(Z)]\alpha + \eta(Y)h(Z,V) + \eta(Z)h(Y,V), \tag{4.1}$$

Equating $ltr(TM)$ and $S(TM^\perp)$ components in (4.1) , we get

$$h^l(Y,Z) = [g(Y,Z) - \eta(Y)\eta(Z)]\alpha_L + \eta(Y)h^l(Z,V) + \eta(Z)h^l(Y,V) \tag{4.2}$$

and

$$h^s(Y,Z) = [g(Y,Z) - \eta(Y)\eta(Z)]\alpha_S + \eta(Y)h^s(Z,V) + \eta(Z)h^s(Y,V), \tag{4.3}$$

where $\alpha_L \in \Gamma(ltr(TM))$ and $\alpha_S \in \Gamma(S(TM^{\perp}))$.
The curvature tensor $\bar{R}$ of an indefinite Sasakian space form $\bar{M}(c)$ is given by(for details see [7])

$$\begin{aligned}\bar{R}(X,Y)Z = \frac{c+3\epsilon}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \frac{c-\epsilon}{4}\{\eta(X)\eta(Z)Y \\ -\eta(Y)\eta(Z)X\} + \bar{g}(X,Z)\eta(Y)V - \bar{g}(Y,Z)\eta(X)V + \bar{g}(\phi Y,Z)\phi X \\ -\bar{g}(\phi X,Z)\phi Y - 2\bar{g}(\phi X,Y)\phi Z,\end{aligned} \tag{4.4}$$

where $X, Y, Z \in T\bar{M}$.

Theorem 4.1. *Let M be a totally contact umbilical screen-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then at least one of the following is true:*

(i) M is a screen real lightlike submanifold.

(ii) $D = \{0\}$.

(iii) If M is a proper screen-slant lightlike submanifold, then $\alpha_S \in \Gamma(\mu)$.

Proof. Since M is a TCU screen-slant lightlike submanifold, using (4.1), for any $Y = QY \in \Gamma(D)$, we have

$$h(TQY, TQY) = g(TQY, TQY)\alpha.$$

Using (2.3) and (3.7) in above equation, we obtain

$$\bar{\nabla}_{TQY}TQY - \nabla_{TQY}TQY = cos^2\theta(g(Y,Y))\alpha.$$

Now, using (2.18) in above equation, we get

$$\phi\bar{\nabla}_{TQY}QY - g(TQY,QY)V - \bar{\nabla}_{TQY}wQY - \nabla_{TQY}TQY = cos^2\theta(g(Y,Y))\alpha.$$

Using, for any $Y = QY \in \Gamma(D)$, $g(\phi(Y), QY) = g(TQY, QY) = 0$ and (2.5), (2.7) in above equation, we get

$$\begin{aligned}\phi(\nabla_{TQY}QY + h^l(TQY,QY) + h^s(TQY,QY)) + A_{wQY}TQY - \nabla^s_{TQY}wQY \\ -D^l(TQY,wQY) - \nabla_{TQY}TQY = cos^2\theta(g(Y,Y))\alpha,\end{aligned}$$

which reduces to

$$\begin{aligned}T\nabla_{TQY}QY + w\nabla_{TQY}QY + g(TQY,QY)\phi\alpha_L + g(TQY,QY)B\alpha_S \\ +g(TQY,QY)C\alpha_S + A_{wQY}TQY - \nabla^s_{TQY}wQY - D^l(TQY,wQY) \\ -\nabla_{TQY}TQY = cos^2\theta(g(Y,Y))\alpha.\end{aligned} \tag{4.5}$$

Equating transversal components in (4.5), we obtain

$$w\nabla_{TQY}QY - \nabla^s_{TQY}wQY - D^l(TQY,wQY) = cos^2\theta(g(Y,Y))\alpha. \tag{4.6}$$

On the other hand, taking covariant derivative of (3.8) with respect to TQY, we obtain

$$g(\nabla^s_{TQY}wQY, wQY) = sin^2\theta g(\nabla_{TQY}Y, Y). \tag{4.7}$$

Again, from (3.8), we get

$$g(w\nabla_{TQY}QY, wQY) = sin^2\theta g(\nabla_{TQY}Y, Y). \tag{4.8}$$

Now, taking the inner product of (4.6) with wQY, we obtain

$$g(w\nabla_{TQY}QY, wQY) - g(\nabla^s_{TQY}wQY, wQY) = cos^2\theta(g(Y, Y))g(\alpha_S, wQY). \tag{4.9}$$

Using (4.7) and (4.8) in (4.9), we obtain

$$cos^2\theta g(Y, Y)g(\alpha_S, wQY) = 0. \tag{4.10}$$

Thus from (4.9), it follows that either $\theta = \frac{\pi}{2}$ or $X = 0$ or $\alpha_S \in \Gamma(\mu)$.
This completes the proof. □

Theorem 4.2. *There does not exist any totally contact umbilical proper screen slant lightlike submanifold, other than totally contact geodesic, of indefinite Sasakian manifold.*

Proof. For any $Y \in \Gamma(D)$, from (2.18), we have

$$\bar{\nabla}_Y \phi Y = \phi \bar{\nabla}_Y Y - g(Y, Y)V + \eta(Y)Y,$$

which is equivalent to

$$\begin{aligned}\nabla_Y TQY + h(Y, TQY) - A_{wQY}Y + \nabla^s_Y wQY + D^l(Y, wQY) = T\nabla_Y Y \\ +w\nabla_Y Y + Ch^l(Y, Y) + Bh^s(Y, Y) + Ch^s(Y, Y) - g(Y, Y)V.\end{aligned}$$

Since $h(Y, TQY) = g(Y, TQY)\alpha = 0$, equating tangential components in above equation, we obtain

$$\nabla^s_Y wQY + D^l(Y, wQY) = w\nabla_Y Y + Ch^l(Y, Y) + Ch^s(Y, Y). \tag{4.11}$$

Taking scalar product of (4.11) with $\phi\xi$ and using (2.15), we obtain

$$\begin{aligned}\bar{g}(\phi h^l(Y, Y), \phi\xi) - \bar{g}(D^l(Y, wQY), \phi\xi) = \bar{g}(h^l(Y, Y), \xi) - \\ \bar{g}(D^l(Y, wQY), \phi\xi) = 0.\end{aligned} \tag{4.12}$$

From (2.8), we have

$$\bar{g}(h^s(Y, \phi(\xi)), wQY) + \bar{g}(D^l(Y, wQY), \phi\xi) = g(A_{wQY}Y, \phi(\xi)).$$

Using $h^s(Y, \phi(\xi)) = g(Y, \phi(\xi))\alpha_S = 0$ and $g(A_{wQY}Y, \phi(\xi)) = 0$ in above equation, we obtain

$$\bar{g}(D^l(Y, wQY), \phi\xi) = 0. \tag{4.13}$$

Hence, using (4.2)and (4.13) in (4.12), we obtain

$$g(Y, Y)g(\alpha_L, \xi) = 0.$$

Since M is a proper ($\theta \neq 0$, and $\theta \neq \pi/2$) and g is positive definite metric on D, from above equation, $\alpha_L = 0$.
On the other hand, for any $Y \in \Gamma(TM)$, using M is TCU proper screen-slant lightlike submanifold and g is positive definite metric on D in (4.10), we obtain

$$\bar{g}(\alpha_S, wQY) = 0.$$

This implies $\alpha_S \in \Gamma(\mu)$. Now, for any $Y, Z \in \Gamma(D)$, using the Sasakian property of $\bar{M}$, we have

$$\bar{\nabla}_Y \phi Z = \phi \bar{\nabla}_Y Z - g(Y,Z)V + \eta(Z)Y.$$

Using (3.4) and $\eta(Z) = 0$ in above equation, we get

$$\bar{\nabla}_Y (TQZ + wQZ) = \phi \bar{\nabla}_Y Z - g(Y,Z)V.$$

Using (2.5) and (2.7) in above equation, we get

$$\begin{aligned}\bar{\nabla}_Y (TQZ) + h(Y, TQZ) - A_{wQZ}Y + \nabla^s_Y wQZ +\\ D^l(Y, wQZ) = \phi(\nabla_Y Z + h(Y,Z)) - g(Y,Z)V.\end{aligned}$$

Using (4.1) in above equation, we obtain

$$\begin{aligned}\bar{\nabla}_Y TQZ + g(Y, TQZ)\alpha - A_{wQZ}Y + \nabla^s_Y wQZ + D^l(Y, wQZ) = T(\nabla_Y Z) +\\ w(\nabla_Y Z) + g(Y,Z)\phi\alpha - g(Y,Z)V.\end{aligned}$$

Now, taking scalar product of above equation with $\phi\alpha_S$ and using the fact that μ is invariant, we obtain

$$\bar{g}(\nabla^s_Y wQZ, \phi\alpha_S) = g(Y,Z)g(\alpha_S, \alpha_S). \tag{4.14}$$

Again, using (2.18), for any $Y \in \Gamma(D)$ and $\alpha_S \in \Gamma(\mu)$, we obtain

$$\bar{\nabla}_Y \phi\alpha_S = \phi \bar{\nabla}_Y \alpha_S.$$

Using (2.7) in above equation, we get

$$\begin{aligned}-A_{\phi\alpha_S Y} + \nabla^s_Y \phi\alpha_S + D^l(Y, \phi\alpha_S) = -TA_{\alpha_S Y} - wA_{\alpha_S Y} + B\nabla^s_Y \alpha_S +\\ C\nabla^s_Y \alpha_S + \phi D^l(Y, \alpha_S).\end{aligned} \tag{4.15}$$

Taking scalar product of (4.15) with wQZ and using $C\nabla^s_Y \phi\alpha_S \in \Gamma(\mu)$, we get

$$\bar{g}(\nabla^s_Y \phi\alpha_S, wQZ) = -g(wA_{\alpha_S Y}, wQZ) = -sin^2\theta(g(A_{\alpha_S}Y, Z)). \tag{4.16}$$

Since $\bar{\nabla}$ is a metric connection, $(\bar{\nabla}_X g)(wQZ, \phi\alpha_s) = 0$ this reduces to

$$\bar{g}(\nabla^s_Y wQZ, \phi\alpha_S) = \bar{g}(\nabla^s_Y \phi\alpha_S, wQZ). \tag{4.17}$$

Using (4.14) and (4.17) in (4.16), we get

$$g(Y,Z)g(\alpha_S, \alpha_S) = -sin^2\theta(g(A_{\alpha_S}Y, Z)), \tag{4.18}$$

and using (2.8) in (4.18), we obtain

$$g(A_{\alpha_S}Y, Z) = \bar{g}(h^s(Y, Z), \alpha_S). \tag{4.19}$$

Using (4.3) in (4.19), we get

$$g(A_{\alpha_S}Y, Z) = g(Y, Z)\bar{g}(\alpha_s, \alpha_S). \tag{4.20}$$

Using (4.20) in (4.18), we get

$$(1 + sin^2\theta)g(Y, Z)g(\alpha_S, \alpha_S) = 0. \tag{4.21}$$

Since M is a proper ($\theta \neq 0$, and $\theta \neq \pi/2$) and g is positive definite metric on D, from (4.21) α_S must be identically zero.
This completes the proof. □

Theorem 4.3. *There does not exist any TCU proper screen-slant lightlike submanifold of an indefinite Sasakian space form $\bar{M}(c)$ with constant ϕ-sectional curvature $c \neq \epsilon$.*

Proof. Let M be a TCU proper screen-slant lightlike submanifold of $\bar{M}(c)(c \neq \epsilon)$. By putting value, for any $Y \in \Gamma(D)$, $N \in \Gamma(ltr(TM))$ and $\xi \in \Gamma(Rad(TM))$, in (4.4) and taking inner product this with N, we obtain

$$\bar{g}(\bar{R}(Y, TQY)\xi, N) = -\frac{c-\epsilon}{2}\bar{g}(\phi Y, TQY)\bar{g}(\phi\xi, N). \tag{4.22}$$

Now, taking inner product of (2.12) with N, we get

$$\bar{g}(\bar{R}(Y, TQY)\xi, N) = \bar{g}((\nabla_Y h^l)(TQY, \xi), N) - \bar{g}((\nabla_{TQY} h^l)(Y, \xi), N) \tag{4.23}$$

Using (4.2), (4.3) in (2.13) (2.14) respectively, we obtain

$$\bar{g}((\nabla_Y h^l)(TQY, \xi), N) = -\bar{g}(TQY, \nabla_Y\xi)\alpha_L \tag{4.24}$$

and

$$\bar{g}((\nabla_{TQY} h^l)(Y, \xi), N) = -\bar{g}(Y, \nabla_{TQY}\xi)\alpha_L. \tag{4.25}$$

Since $\bar{g}(TQY, \nabla_Y\xi) = \bar{g}(\nabla_Y TQY, \xi) = 0$ and $\bar{g}(Y, \nabla_{TQY}\xi) = \bar{g}(\nabla_{TQY}Y, \xi) = 0$, therefore

$$(\nabla_Y h^l)(TQY, \xi) - (\nabla_{TQY} h^l)(Y, \xi) = 0. \tag{4.26}$$

From (4.23) and (4.26), we obtain

$$\bar{g}(\bar{R}(Y, TQY)\xi, N) = 0 \tag{4.27}$$

Using (4.27) in (4.22), we obtain

$$\frac{c-\epsilon}{2}\bar{g}(\phi Y, TQY)\bar{g}(\phi\xi, N) = 0. \tag{4.28}$$

Since g is positive definite metric on the distribution D and $\bar{g}(\phi Z, \xi)$ is not identically zero, we obtain, from (4.28) c must be equal to ϵ. This is a contradictory statement to our initial assumption $c = \epsilon$.

□

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