

# Conformal Einstein soliton within the framework of para-Kähler manifolds

Soumendu Roy, Santu Dey and Arindam Bhattacharyya

**Abstract.** The object of the present paper is to study some properties of para-Kähler manifold whose metric is conformal Einstein soliton. We have studied some certain curvature properties of para-Kähler manifold admitting conformal Einstein soliton. Also, we have enriched the importance of the Laplace equation in physics and gravity, satisfies conformal Einstein soliton.

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**Key words:** Einstein soliton; conformal Einstein soliton; Einstein manifold; para-Kähler manifold.

## 1 Introduction

The notion of Einstein soliton was introduced by G. Catino and L. Mazzieri [3] in 2016, which generates self-similar solutions to Einstein flow,

$$(1.1) \quad \frac{\partial g}{\partial t} = -2(S - \frac{r}{2}g),$$

where  $S$  is Ricci tensor,  $g$  is Riemannian metric and  $r$  is the scalar curvature. The equation of the Einstein soliton [2] is given by,

$$(1.2) \quad \mathcal{L}_V g + 2S + (2\lambda - r)g = 0,$$

where  $\mathcal{L}_V$  is the Lie derivative along the vector field  $V$ ,  $S$  is the Ricci tensor,  $r$  is the scalar curvature of the Riemannian metric  $g$ , and  $\lambda$  is a real constant.

In 2015, N. Basu and A. Bhattacharyya [1] introduced the notion of conformal Ricci soliton equation [7], [8], given by

$$(1.3) \quad \mathcal{L}_V g + 2S = [2\lambda - (p + \frac{2}{n})]g,$$

where  $\mathcal{L}_V$  is the Lie derivative along the vector field  $V$ ,  $S$  is the Ricci tensor,  $\lambda$  is constant,  $p$  is a scalar non-dynamical field(time dependent scalar field) and  $n$  is the dimension of the manifold.

So we introduce the notion of conformal Einstein soliton as:

**Definition 1.1.** A Riemannian or pseudo-Riemannian manifold  $(M, g)$  of dimension  $n$  is said to admit conformal Einstein soliton if

$$(1.4) \quad \mathcal{L}_V g + 2S + [2\lambda - r + (p + \frac{2}{n})]g = 0,$$

where  $\mathcal{L}_V$  is the Lie derivative along the vector field  $V$ ,  $S$  is the Ricci tensor,  $r$  is the scalar curvature of the Riemannian metric  $g$ ,  $\lambda$  is real constant,  $p$  is a scalar non-dynamical field (time dependent scalar field) and  $n$  is the dimension of manifold. Also it is said to be shrinking, steady or expanding according as  $\lambda < 0$ ,  $\lambda = 0$  and  $\lambda > 0$  respectively.

In the present paper we study conformal Einstein soliton on para-Kähler manifold. The paper is organized as follows:

After introduction, section 2 is devoted for preliminaries on  $n$ - dimensional para-Kähler manifold, where  $n$  is even. In section 3, we have studied conformal Einstein soliton on para-Kähler manifold. Here we proved if a  $n$ - dimensional para-Kähler manifold admits conformal Einstein soliton then the vector field associated with the soliton is solenoidal depends on the scalar curvature. We have also characterized the nature of the manifold if the manifold is quasi conformally flat, pseudo-projectively flat and  $W_2$ - flat. Section 4 deals with the application of Laplace equation in physics and gravity of conformal Einstein soliton.

## 2 Preliminaries

Let  $M$  be a connected differentiable manifold of dimension  $n = 2m$ ,  $m \geq 2$ ,  $F$  be a (1,1)-tensor field and  $g$  be a pseudo-Riemannian metric on  $M$ . Then  $(M, F, g)$  is said to be a para-Kähler manifold if the following conditions hold:

$$(2.1) \quad F^2 = I, \quad g(FX, FY) = -g(X, Y), \quad \nabla F = 0.$$

for any  $X, Y \in \chi(M)$ , being the Lie algebra of vector fields on  $M$ ,  $\nabla$  is the Levi-Civita connection of  $g$  and  $I$  is the identity operator.

In a para-Kähler manifold  $(M, F, g)$ , the Riemannian curvature tensor  $R$ , the Ricci tensor  $S$  and the scalar curvature  $r$  are defined by:

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= g(R(X, Y)Z, W), \\ S(X, Y) &= \text{trace}\{Z \rightarrow R(Z, X)Y\}, \\ r &= \text{trace}S. \end{aligned}$$

Also the following properties are satisfied in a para-Kähler manifold:

$$(2.2) \quad R(FX, FY)Z = -R(X, Y)Z,$$

$$(2.3) \quad R(FX, Y)Z = -R(X, FY)Z,$$

$$(2.4) \quad S(FX, Y) = -S(FY, X),$$

$$(2.5) \quad S(FX, FY) = -S(X, Y).$$

### 3 Some results on conformal Einstein soliton within the framework of para-Kähler manifold

In this section we prove the following:

**Theorem 3.1.** *If the metric of an  $n$ -dimensional para-Kähler manifold satisfies a conformal Einstein soliton then the vector field associated with the soliton is solenoidal iff the scalar curvature is  $\frac{2\lambda n}{n-2} + \frac{n(p+\frac{2}{n})}{n-2}$ , provided  $n > 2$ .*

*Proof.* From equation (1.4), we can write,

$$(3.1) \quad (\mathcal{L}_V g)(X, Y) + 2S(X, Y) + [2\lambda - r + (p + \frac{2}{n})]g(X, Y) = 0,$$

for any  $X, Y \in \chi(M)$ , being the Lie algebra of vector fields on  $M$ .

Taking  $X = e_i, Y = e_i$  in the above equation and summing over  $i = 1, 2, \dots, n$ , we get,

$$(3.2) \quad \text{div}V + r + [\lambda - \frac{r}{2} + \frac{1}{2}(p + \frac{2}{n})]n = 0.$$

Now if  $V$  is solenoidal then  $\text{div}V = 0$  and so from the above equation we have  $r = \frac{2\lambda n}{n-2} + \frac{n(p+\frac{2}{n})}{n-2}$ . Again if  $r = \frac{2\lambda n}{n-2} + \frac{n(p+\frac{2}{n})}{n-2}$  then (3.2) gives  $\text{div}V = 0$ .

Hence the proof. □

From the above theorem we can state:

**Theorem 3.2.** *If the metric of an  $n$ -dimensional para-Kähler manifold satisfies a conformal Einstein soliton, whose potential vector field  $V$  is the gradient of a smooth function  $f$ , then the Laplacian equation satisfied by  $f$  is,*

$$\Delta(f) = -r - [\lambda - \frac{r}{2} + \frac{1}{2}(p + \frac{2}{n})]n,$$

provided  $n > 2$ .

*Proof.* As the vector field  $V$  is of gradient type i.e  $V = \text{grad}(f)$ , for  $f$  is a smooth function on  $M$ , then (3.2) gives,

$$(3.3) \quad \Delta(f) = -r - [\lambda - \frac{r}{2} + \frac{1}{2}(p + \frac{2}{n})]n,$$

where  $\Delta(f)$  is the Laplacian equation satisfied by  $f$ .

This completes the proof. □

**Definition 3.1.** The notion of quasi-conformal curvature tensor was introduced by Yano and Sawaki [9] and it is defined by:

$$(3.4) \quad \begin{aligned} C(X, Y)Z &= \alpha R(X, Y)Z + \beta[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &- g(X, Z)QY] - \frac{r}{n}(\frac{\alpha}{n-1} + 2\beta)[g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

where  $\alpha, \beta$  are constants,  $Q$  is the Ricci operator, defined by  $g(QX, Y) = S(X, Y)$  and  $n$  is the dimension of the manifold.

Moreover if  $\alpha = 1$  and  $\beta = -\frac{1}{n-2}$ , the above equation reduces to conformal curvature tensor [4].

Again a manifold  $(M^n, g)$  where  $n > 3$ , is said to be quasi conformally flat if  $C = 0$ .

Using the above definition we have,

**Theorem 3.3.** *If the metric of an  $n$ -dimensional quasi-conformally flat para-Kähler manifold satisfies a conformal Einstein soliton then the vector field associated with the soliton is solenoidal iff  $\lambda + \frac{1}{2}(p + \frac{2}{n}) = 0$ .*

*Proof.* In an  $n$ - dimensional para-Kähler manifold, we can define the Ricci tensor  $S$  as:

$$(3.5) \quad S(X, Y) = \frac{1}{2} \sum_{i=1}^n \epsilon_i \tilde{R}(e_i, Fe_i, X, FY),$$

where  $\{e_1, e_2, \dots, e_n\}$  is an orthonormal frame and  $\epsilon_i$  is the indicator of  $e_i$ ,  $\epsilon_i = g(e_i, e_i) = 1$ .

Taking inner product in (3.4) by  $W$ , we get,

$$(3.6) \quad \begin{aligned} g(C(X, Y)Z, W) &= \alpha \tilde{R}(X, Y, Z, W) + \beta[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) \\ &+ g(Y, Z)S(X, W) - g(X, Z)S(Y, W)] \\ &- \frac{r}{n} \left( \frac{\alpha}{n-1} + 2\beta \right) [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned}$$

Now as the manifold is quasi-conformally flat then the above equation reduces to,

$$(3.7) \quad \begin{aligned} \alpha \tilde{R}(X, Y, Z, W) + \beta[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) \\ + g(Y, Z)S(X, W) - g(X, Z)S(Y, W)] - \frac{r}{n} \left( \frac{\alpha}{n-1} + 2\beta \right) [g(Y, Z)g(X, W) \\ - g(X, Z)g(Y, W)] = 0. \end{aligned}$$

Putting  $X = e_i, Y = Fe_i, W = FW$  in the above equation and summing over  $i = 1, 2, \dots, n$  and also using (3.5), (2.4), we get,

$$(3.8) \quad \begin{aligned} 2\alpha S(Z, W) + 2\beta[S(Z, W) - S(FZ, FW)] \\ - \frac{r}{n} \left( \frac{\alpha}{n-1} + 2\beta \right) [g(Z, W) - g(FZ, FW)] = 0. \end{aligned}$$

Again using (2.5) and (2.1) in the above equation, we obtain,

$$(3.9) \quad 2\alpha S(Z, W) + 4\beta S(Z, W) - \frac{2r}{n} \left( \frac{\alpha}{n-1} + 2\beta \right) g(Z, W) = 0,$$

which reduces to,

$$(3.10) \quad (\alpha + 2\beta)S(Z, W) = \frac{r}{n} \left( \frac{\alpha}{n-1} + 2\beta \right) g(Z, W).$$

Taking  $Z = e_i, W = e_i$  in the above equation and summing over  $i = 1, 2, \dots, n$ , we have,

$$(3.11) \quad r = 0,$$

since  $\alpha \neq 0$ .

Now if  $r = 0$ , then from (3.10), we get  $S = 0$ , provided  $\alpha + 2\beta \neq 0$  i.e the manifold is locally flat if  $\alpha + 2\beta \neq 0$ .

Then (1.4) becomes,

$$(3.12) \quad (\mathcal{L}_V g)(X, Y) + [2\lambda + (p + \frac{2}{n})]g(X, Y) = 0$$

for any  $X, Y \in \chi(M)$ , being the Lie algebra of vector fields on  $M$ .

Putting  $X = Y = e_i$  and summing over  $i = 1, 2, \dots, n$ , we get,

$$(3.13) \quad \sum_{i=1}^n \epsilon_i (\mathcal{L}_V g)(e_i, e_i) + \sum_{i=1}^n \epsilon_i [2\lambda + (p + \frac{2}{n})]g(e_i, e_i) = 0,$$

which reduces to,

$$(3.14) \quad \text{div}V + [\lambda + \frac{1}{2}(p + \frac{2}{n})]n = 0.$$

Now if  $V$  is solenoidal then  $\text{div}V = 0$  and so from (3.14), we have  $\lambda + \frac{1}{2}(p + \frac{2}{n}) = 0$ . Also if  $\lambda + \frac{1}{2}(p + \frac{2}{n}) = 0$ , (3.14) reduces to  $\text{div}V = 0$ .

Hence the proof. □

**Definition 3.2.** The pseudo-projective curvature tensor  $\bar{P}$  [6] is given by:

$$(3.15) \quad \begin{aligned} \bar{P}(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] \\ &- \frac{r}{n}(\frac{a}{n-1} + b)[g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

where  $a, b \neq 0$  are constants.

Also a manifold  $(M^n, g)$ , is said to be pseudo-projectively flat if  $\bar{P} = 0$ .

**Theorem 3.4.** *If the metric of an  $n$ -dimensional pseudo-projectively flat para-Kähler manifold satisfies a conformal Einstein soliton then the vector field associated with the soliton is solenoidal iff  $\lambda + \frac{1}{2}(p + \frac{2}{n}) = 0$ .*

*Proof.* Taking inner product in (3.15) by  $W$ , we get,

$$(3.16) \quad \begin{aligned} g(\bar{P}(X, Y)Z, W) &= a\tilde{R}(X, Y, Z, W) + b[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)] \\ &- \frac{r}{n}(\frac{a}{n-1} + b)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned}$$

Now as the manifold is pseudo-projectively flat then the above equation reduces to,

$$(3.17) \quad \begin{aligned} a\tilde{R}(X, Y, Z, W) + b[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)] \\ - \frac{r}{n}(\frac{a}{n-1} + b)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] = 0. \end{aligned}$$

Putting  $X = e_i, Y = Fe_i, W = FW$  in the above equation and summing over  $i = 1, 2, \dots, n$  and also using (3.5), (2.4), we get,

$$(3.18) \quad 2aS(Z, W) + b[S(Z, W) - S(FZ, FW)] - \frac{r}{n}(\frac{a}{n-1} + b)[g(Z, W) - g(FZ, FW)] = 0.$$

Again using (2.5) and (2.1) in the above equation, we obtain,

$$(3.19) \quad (a+b)S(Z, W) - \frac{r}{n} \left( \frac{a}{n-1} + b \right) g(Z, W) = 0.$$

Taking  $Z = e_i, W = e_i$  in the above equation and summing over  $i = 1, 2, \dots, n$ , we have,

$$(3.20) \quad r = 0,$$

since  $a \neq 0$ .

Now if  $r = 0$ , then from (3.19), we get  $S = 0$ , provided  $a + b \neq 0$  i.e the manifold is locally flat if  $a + b \neq 0$ .

Then (1.4) becomes,

$$(3.21) \quad (\mathcal{L}_V g)(X, Y) + \left[ 2\lambda + \left( p + \frac{2}{n} \right) \right] g(X, Y) = 0$$

for any  $X, Y \in \chi(M)$ , being the Lie algebra of vector fields on  $M$ .

Putting  $X = Y = e_i$  and summing over  $i = 1, 2, \dots, n$ , we get,

$$(3.22) \quad \sum_{i=1}^n \epsilon_i (\mathcal{L}_V g)(e_i, e_i) + \sum_{i=1}^n \epsilon_i \left[ 2\lambda + \left( p + \frac{2}{n} \right) \right] g(e_i, e_i) = 0,$$

which reduces to,

$$(3.23) \quad \text{div}V + \left[ \lambda + \frac{1}{2} \left( p + \frac{2}{n} \right) \right] n = 0.$$

Now if  $V$  is solenoidal then  $\text{div}V = 0$  and so from (3.23) we have  $\lambda + \frac{1}{2} \left( p + \frac{2}{n} \right) = 0$ . Also if  $\lambda + \frac{1}{2} \left( p + \frac{2}{n} \right) = 0$ , (3.23) reduces to  $\text{div}V = 0$ .

This completes the proof.  $\square$

**Definition 3.3.** The  $W_2$ -curvature tensor ( $n > 2$ ) [5] is given by:

$$(3.24) \quad W_2(X, Y)Z = R(X, Y)Z + \frac{1}{n-1} [g(X, Z)QY - g(Y, Z)QX].$$

Moreover a manifold is  $W_2$ - flat if  $\widetilde{W}_2(X, Y, Z, U) = g(W_2(X, Y)Z, U) = 0$ .

**Theorem 3.5.** *If the metric of an  $n$ -dimensional  $W_2$ - flat para-Kähler manifold satisfies a conformal Einstein soliton then the vector field associated with the soliton is solenoidal iff  $\lambda + \frac{1}{2} \left( p + \frac{2}{n} \right) = 0$ .*

*Proof.* Taking inner product in (3.24) by  $U$ , we get,

$$(3.25) \quad g(W_2(X, Y)Z, U) = \widetilde{R}(X, Y, Z, U) + \frac{1}{n-1} [g(X, Z)S(Y, U) - g(Y, Z)S(X, U)].$$

Now if the manifold is  $W_2$ - flat then the above equation reduces to,

$$(3.26) \quad \widetilde{R}(X, Y, Z, U) + \frac{1}{n-1} [g(X, Z)S(Y, U) - g(Y, Z)S(X, U)] = 0.$$

Putting  $X = e_i, Y = Fe_i, U = FU$  in the above equation and summing over  $i = 1, 2, \dots, n$  and also using (3.5), (2.4), we get,

$$(3.27) \quad S(Z, U) + \frac{1}{n-1}[S(FZ, FU) - S(Z, U)] = 0.$$

Again using (2.5) in the above equation, we obtain,

$$(3.28) \quad (n-3)S(Z, U) = 0.$$

Then we have  $S(Z, U) = 0$ , for any  $Z, U \in \chi(M)$ , being the Lie algebra of vector fields on  $M$ , since  $n$  is even.

Hence from the above equation we can get,  $r = 0$ .

Then (1.4) becomes,

$$(3.29) \quad (\mathcal{L}_V g)(X, Y) + [2\lambda + (p + \frac{2}{n})]g(X, Y) = 0$$

for any  $X, Y \in \chi(M)$ , being the Lie algebra of vector fields on  $M$ .

Putting  $X = Y = e_i$  and summing over  $i = 1, 2, \dots, n$ , we get,

$$(3.30) \quad \sum_{i=1}^n \epsilon_i (\mathcal{L}_V g)(e_i, e_i) + \sum_{i=1}^n \epsilon_i [2\lambda + (p + \frac{2}{n})]g(e_i, e_i) = 0,$$

which reduces to,

$$(3.31) \quad \text{div}V + [\lambda + \frac{1}{2}(p + \frac{2}{n})]n = 0.$$

Now if  $V$  is solenoidal then  $\text{div}V = 0$  and so from (3.31) we have  $\lambda + \frac{1}{2}(p + \frac{2}{n}) = 0$ . Also if  $\lambda + \frac{1}{2}(p + \frac{2}{n}) = 0$ , (3.31) reduces to  $\text{div}V = 0$ .

Hence the proof. □

## 4 Application of Laplace equation in physics and gravity

Laplace equation, a second order P.D.E widely useful in physics as its solution, which is known as harmonic functions occur in problems of electrical, magnetic and gravitational potentials of steady state temperatures and of hydrodynamics.

- The real and imaginary parts of complex analytic function both satisfy Laplace equation. That is if  $z = x + iy$  and  $f(x, y) = u(x, y) + iv(x, y)$ , then the necessary condition of  $f(z)$  to be analytic is that  $u$  and  $v$  and that be C.R equation be satisfied,  $u_x = v_y, u_y = -v_x$ , where  $u_x, u_y$  is the first partial derivatives of  $u$  with respect to  $x, y$  respectively and  $v_x, v_y$  is the first partial derivatives of  $v$  with respect to  $x, y$  respectively. It follows that  $u_{yy} = (-v_x)_y = -(v_y)_x = -(u_{xx})$ . Therefore,  $u$  satisfies Laplace equation.

- If we have a region where the charge density is zero (there may be non-zero charge

densities at the boundaries), the electric potential  $V$  satisfies Laplace equation inside the region. Solving Laplace equation, we get electric potential, which is very important quantity as we can use it to compute the electric field very easily,  $E = \nabla V$  and therefore the force  $\vec{F} = qE$ . There are many interesting cases in physics, where we are concerned with the potential in regions with zero charged density. Classic examples include the region inside and outside a hollow charged sphere, or the region outside charged metal plates. Each of the cases come with different set of boundary conditions on what makes Laplace equation interesting.

In general, for a given charged density,  $L(x, y, z)$ , electric (and gravitational) potentials satisfy poisson's equation,  $\nabla^2 V = L(x, y, z)$ . Laplace equation or poisson's equation are the simplest examples of a class of P.D.Es called elliptical P.D.Es. A lot of interesting mathematical techniques used to solve electrical P.D.Es are first introduced by Laplace equation.

- In electrostatics, according to Maxwell's equation, a electric fluid  $(u, v)$  in two space dimensions, that is independent of time satisfies,

$$\nabla \times (u, v, 0) = (v_x - u_y)\hat{k} = 0,$$

and

$$\nabla \cdot (u, v) = L,$$

where  $L$  is the charge density.

The Laplace equation can be used in three dimension problems in electrostatics and fluid flow just as in two dimensions.

- It has applications in gravity also. Let  $\tilde{g}, \tilde{\rho}, G$  be the gravitational field, mass density and gravitational constant. Then Gauss's law for gravitation in differential form is:

$$\nabla \cdot \tilde{g} = -4\pi G\tilde{\rho}.$$

Also we have,  $\nabla^2 V = 4\pi G\tilde{\rho}$ , which is Poisson's equation for gravitational fields.

This physical significance is directly equivalent to Theorem (3.2) and equation (3.3), which is a Laplace equation with potential vector field of gradient type.

In empty space  $\tilde{\rho} = 0$ , we have  $\nabla^2 V = 0$ , which is Laplace equation for gravitational fields.

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*Author's address:*

Soumendu Roy and Arindam Bhattacharyya  
Department of Mathematics, Jadavpur University, Kolkata-700032, India.  
E-mail: soumendu1103mtma@gmail.com, bhattachar1968@yahoo.co.in

Santu Dey  
Department of Mathematics, Bidhan Chandra College,  
Asansol - 4, West Bengal-713304, India.  
E-mail: santu.mathju@gmail.com