On the structure of lightlike hypersurfaces of an indefinite Kaehler statistical manifold

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Abstract. This paper introduces the geometry of lightlike hypersurfaces of an indefinite Kaehler statistical manifold and investigates their structural properties with respect to the dual connections. Some conditions for the integrability and parallelism of the almost complex distribution have been derived. Further, the induced local lightlike parallel vector fields in the lightlike hypersurfaces have been characterized. It is also shown that a Lie-recurrent structure tensor field in the lightlike hypersurface of an indefinite Kaehler statistical manifold is Lie-parallel.

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1 Introduction

The existence of lightlike geometry in the basic structure of the theory of relativity has led to the development of quite an interesting branch of study in geometry known as lightlike hypersurfaces. Initiated by [3] and further studied elaborately by many geometers, various results of considerable importance in this theory have been worked upon so far.[6] studied the theory of real lightlike hypersurfaces of an indefinite Kaehler manifold and derived interesting results using the induced structure. He further introduced Hopf, recurrent and Lie-recurrent hypersurfaces of the indefinite Kaehler manifold and developed their structural properties in [7].

For the statistical manifolds, which have developed from the investigation of geometric structures on sets of certain probability distributions, the theory of hypersurfaces was initiated and studied extensively by [4]. Later on [5] et al introduced Sasakian statistical structure and developed results open for further research in this field. In the direction of developing its lightlike theory, [1] initiated the concept of lightlike hypersurfaces of statistical manifolds and further introduced the same for an indefinite Sasakian statistical manifold in [2]. This motivated us to develop the lightlike hypersurfaces for its even dimensional counterpart - the indefinite Kaehler statistical manifold, whose structure was introduced in [8] and [9].

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In this paper, we introduce the concept of lightlike hypersurface of an indefinite Kaehler statistical manifold using the dual connections and derive the structural properties for the same. We develop conditions relating the integrability and the parallelism of distributions with geodesicity of these hypersurfaces. Further, we characterize the Lie-recurrent structure tensor field of a lightlike hypersurface of the indefinite Kaehler statistical manifold.

2 Lightlike hypersurface

Some notations regarding the structure of a lightlike hypersurface following [3] are given as:

Consider an $(m+2)$ dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ of constant index $q \geq 1$. Let (M, g) be a lightlike hypersurface of $(\bar{M}, \bar{g})$ with $g = \bar{g} \mid M$. If the induced metric g on M is degenerate, then M is called a lightlike or degenerate hypersurface of $\bar{M}$ .There exists a vector field $\xi \neq 0$ on M such that $g(\xi, X) = 0 \quad \forall X \in \Gamma(TM)$.

The null space or radical space of $T_x(M)$ at each point $x \in M$ is a subspace $RadT_x(M)$ defined as

$RadT_xM = \{\xi \in T_x(M) : g_x(\xi, X) = 0 \quad \forall\ X \in \Gamma(TM)\}$ whose dimension is called the nullity degree of g.

Since g is degenerate and any null vector is perpendicular to itself, therefore $T_xM^{\perp}$ is also null and

$$RadT_xM = T_xM \cap T_xM^{\perp}.$$

For a hypersurface M, dimension of $T_xM^{\perp}$ equals 1 which implies that the dimension of $RadT_xM$ is also 1 and $RadT_xM = T_xM^{\perp}$. Here $RadTM$ is called a radical distribution of M.

Now consider $S(TM)$, called screen distribution, as a complementary vector bundle of $Rad(TM)$ in TM , such that

$$TM = RadTM \perp S(TM) \tag{2.1}$$

It follows that $S(TM)$ is a non-degenerate distribution. Thus, we have

$$TM_{|M} = S(TM) \perp S(TM)^{\perp}$$

where $S(TM)^{\perp}$, known as screen transversal vector bundle, is the orthogonal complement to $S(TM)$ in $TM_{|M}$.

Theorem 2.1. *[3] Let (M, g) be a lightlike hypersurface of $(\bar{M}, \bar{g})$. Then there exists a unique vector bundle $tr(TM)$ known as lightlike transversal vector bundle of rank 1 over M, such that for any non-zero local normal section ξ of $Rad(TM)$, there exist a unique section N of $tr(TM)$ satisfying*

$$\bar{g}(N, \xi) = 1 \tag{2.2}$$

$$\bar{g}(N, N) = 0, \quad \bar{g}(N, W) = 0 \ \forall\, W \in \Gamma(S(TM)).$$

Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follows:

$$T\bar{M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM)$$

3 Lightlike hypersurface of an indefinite statistical manifold

This section deals with the basic structure of a lightlike hypersurface of an indefinite statistical manifold.

Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold, $\bar{D}$ be an affine connection on $\bar{M}$ and $\bar{g}$ be a semi-Riemannian metric on $\bar{M}$. A semi-Riemannian manifold $(\bar{M}, \bar{g})$ which admits an affine connection $\bar{D}$ such that for all $X, Y, Z \in \Gamma(T\bar{M})$
i) $\bar{D}_X Y - \bar{D}_Y X = [X, Y]$;
ii) $(\bar{D}_X \bar{g})(Y, Z) = (\bar{D}_Y \bar{g})(X, Z)$ hold,
is said to be an indefinite statistical manifold and is denoted by $(\bar{M}, \bar{g}, \bar{D})$.
Moreover, there exists $\bar{D}^*$ which is a dual connection of $\bar{D}$ with respect to $\bar{g}$, satisfying

$$X\bar{g}(Y, Z) = \bar{g}(\bar{D}_X Y, Z) + \bar{g}(Y, \bar{D}^*_X Z) \quad X, Y, Z \in \Gamma(T\bar{M}). \tag{3.1}$$

If $(\bar{M}, \bar{g}, \bar{D})$ is an indefinite statistical manifold, then so is $(\bar{M}, \bar{g}, \bar{D}^*)$. Hence the indefinite statistical manifold is denoted by $(\bar{M}, \bar{g}, \bar{D}, \bar{D}^*)$.

Let (M, g) be a lightlike hypersurface of a statistical manifold $(\bar{M}, \bar{g})$. Then the Gauss and Weingarten formulae with respect to dual connections as given by [10],[4],[1] are as follows:

$$\bar{D}_X Y = D_X Y + h(X, Y), \quad \bar{D}_X N = -A^*_N X + D^{\perp}_X N, \tag{3.2}$$

$$\bar{D}^*_X Y = D^*_X Y + h^*(X, Y), \quad \bar{D}^*_X N = -A_N X + D^{\perp *}_X N, \tag{3.3}$$

for $X,\ Y \in \Gamma(TM)$, $N \in \Gamma(T^{\perp}M)$, where $D_X Y, D^*_X Y, A_N X, A^*_N X \in \Gamma(TM)$ and $h(X, Y), h^*(X, Y), D^{\perp}_X N, D^{\perp *}_X N \in \Gamma(tr(TM))$.
Here, D, D^* are called induced connections on M and A_N, A^*_N are called shape operators with respect to $\bar{D}$ and $\bar{D}^*$ respectively. Also, we denote by B and B^*, the second fundamental forms with respect to $\bar{D}$ and $\bar{D}^*$.
Then we have

$$B(X, Y) = \bar{g}(\bar{D}_X Y, \xi), \quad \tau^*(X) = \bar{g}(\bar{D}_X N, \xi)$$

$$B^*(X, Y) = \bar{g}(\bar{D}^*_X Y, \xi), \quad \tau(X) = \bar{g}(\bar{D}^*_X N, \xi)$$

It follows that

$$h(X, Y) = B(X, Y)N, \quad h^*(X, Y) = B^*(X, Y)N$$

$$D^{\perp}_X N = \tau^*(X)N, \quad D^{\perp *}_X N = \tau(X)N$$

Hence,

$$\bar{D}_X Y = D_X Y + B(X, Y)N, \quad \bar{D}_X N = -A^*_N X + \tau^*(X)N \tag{3.4}$$

$$\bar{D}^*_X Y = D^*_X Y + B^*(X, Y)N, \quad \bar{D}^*_X N = -A_N X + \tau(X)N \tag{3.5}$$

Using Gauss formula and the relation between dual connections, we have from [1]

$$Xg(Y,Z) = g(\bar{D}_X Y, Z) + g(Y, \bar{D}^*_X Z)$$

$$= g(D_X Y, Z) + g(Y, D^*_X Z) + B(X,Y)\eta(Z) + B^*(X,Z)\eta(Y)$$

From the above equation, we obtain that induced connections D and D^* are not dual connections and a lightlike hypersurface of a statistical manifold need not a statistical manifold with respect to the dual connections. Also, the induced connections D and D^* and the second fundamental forms B and B^* are symmetric.

Using Gauss and Weingarten formulae, we have

$$(D_X g)(Y,Z) + (D^*_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y) + B^*(X,Y)\eta(Z) + B^*(X,Z)\eta(Y)$$

Let P denote the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.1). Then we have

$$D_X PY = \nabla_X PY + h'(X,PY), \quad D^*_X PY = \nabla^*_X PY + h^{*\prime}(X,PY)$$

$$D_X \xi = -A'_\xi X + \nabla'^t_X \xi, \quad D^*_X \xi = -A^{*\prime}_\xi X + \nabla^{*\prime t}_X \xi$$

for all $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\nabla_X PY$, $\nabla^*_X PY$, $A'_\xi X$ and $A^{*\prime}_\xi X \in \Gamma(S(TM))$, ∇, ∇^* and $\nabla'^t, \nabla^{*\prime t}$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively. Here $h', h^{*\prime}$ and $A', A^{*\prime}$ are respectively called screen second fundamental forms and screen shape operators of $S(TM)$.
We define the local second fundamental forms of $S(TM)$ as

$$C(X,PY) = \bar{g}(h'(X,PY),N), \quad C^*(X,PY) = \bar{g}(h^{*\prime}(X,PY),N)$$

$$\epsilon(X) = g(\nabla'^t_X \xi, N), \quad \epsilon^*(X) = g(\nabla^{*\prime t}_X \xi, N) \quad \forall\, X,Y \in \Gamma(TM)$$

Therefore, we have

$$h'(X,PY) = C(X,PY)\xi, \quad h^{*\prime}(X,PY) = C^*(X,PY)\xi$$

$$\nabla'^t_X \xi = -\tau(X)\xi, \quad \nabla^{*\prime t}_X \xi = -\tau^*(X)\xi$$

$$D_X PY = \nabla_X PY + C(X,PY)\xi \quad D^*_X PY = \nabla^*_X PY + C^*(X,PY)\xi$$

$$D_X \xi = -A'_\xi X - \tau(X)\xi, \quad D^*_X \xi = -A^{*\prime}_\xi X - \tau^*(X)\xi \quad \forall X,Y \in \Gamma(TM) \tag{3.6}$$

where $\epsilon(X) = -\tau(X)$

Using above equation, the induced objects are related as:

$$B(X,\xi) + B^*(X,\xi) = 0, \quad g(A_N X + A^*_N X, N) = 0, \tag{3.7}$$

$$C(X,PY) = g(A_N X, PY), \quad C^*(X,PY) = g(A^*_N X, PY)$$

From the equations (2.2), (3.1),(3.4),(3.5) and (3.6), the following propositions hold:

Proposition 3.1. *Let (M, g) be a lightlike hypersurface of a statistical manifold $(\bar{M}, \bar{g}, \bar{D}, \bar{D}^*)$. Then the second fundamental forms B and B^* are related to the shape operators $A'_\xi X$ and $A^{*\prime}_\xi X$ of $S(TM)$ as follows:*

$$g(A'_\xi X, PY) = B^*(X, PY), \quad g(A^{*\prime}_\xi X, PY) = B(X, PY) \tag{3.8}$$

Therefore, from equation (3.8), we obtain

$$B(A^{*\prime}_\xi X, Y) = B(X, A^{*\prime}_\xi Y), \quad B^*(A'_\xi X, Y) = B^*(X, A'_\xi Y)$$

Proposition 3.2. *Let (M, g) be a lightlike hypersurface of a statistical manifold $(\bar{M}, \bar{g}, \bar{D}, \bar{D}^*)$. Then the shape operator of any screen distribution of a lightlike hypersurface is symmetric with respect to the second fundamental form of M.*

Also for the dual connections, [1] gave

$$B(X, Y) = g(A^{*\prime}_\xi X, Y) + B^*(X, \xi)$$

$$B^*(X, Y) = g(A'_\xi X, Y) + B(X, \xi)$$

Using above equations, we have

$$A^{*\prime}_\xi \xi + A'_\xi \xi = 0$$

From (3.4), (3.5),(3.6) and (3.7), we obtain

$$\bar{D}_X \xi + \bar{D}^*_X \xi = D_X \xi + B(X, \xi) + D^*_X \xi + B^*(X, \xi)$$

$$= -\bar{A}_\xi X - \tau(X)\xi + B(X, \xi) - \bar{A}^*_\xi X - \tau^*(X)\xi + B^*(X, \xi)$$

which implies

$$\bar{D}_X \xi + \bar{D}^*_X \xi = -\bar{A}_\xi X - \tau(X)\xi - \bar{A}^*_\xi X - \tau^*(X)\xi$$

4 Indefinite Kaehler statistical manifold

Consider a Levi-Civita connection $\bar{D}^\circ$ w.r.t $\bar{g}$ such that $\bar{D}^\circ = \frac{1}{2}(\bar{D} + \bar{D}^*)$.
For a statistical manifold $(\bar{M}, \bar{g}, \bar{D}, \bar{D}^*)$, the difference $(1, 2)$ tensor K of a torsion free affine connection $\bar{D}$ and Levi-Civita connection $\bar{D}^\circ$ is defined as

$$K(X, Y) = K_X Y = \bar{D}_X Y - \bar{D}^\circ_X Y$$

Since $\bar{D}$ and $\bar{D}^\circ$ are torsion free, we have

$$K(X, Y) = K(Y, X), \quad \bar{g}(K_X Y, Z) = \bar{g}(Y, K_X Z)$$

for any $X, Y, Z \in \Gamma(TM)$.
Also

$$K(X, Y) = \bar{D}^\circ_X Y - \bar{D}^*_X Y.$$

From the above equations, we get

$$K(X, Y) = \frac{1}{2}(\bar{D}_X Y - \bar{D}^*_X Y).$$

Let $(\bar{J}, \bar{g})$ be an indefinite almost Hermitian structure with an almost complex structure $\bar{J}$ and Hermitian metric $\bar{g}$ such that for all $X, Y \in \Gamma(T\bar{M})$,

$$\bar{J}^2 = -I, \qquad \bar{g}(\bar{J}X, \bar{J}Y) = \bar{g}(X, Y). \tag{4.1}$$

An indefinite almost Hermitian structure on $\bar{M}$ is called an indefinite Kaehler structure if $\bar{J}$ is parallel with respect to $\bar{D}^{\circ}$, i.e.,

$$(\bar{D}^{\circ}_X \bar{J})Y = 0$$

Definition 4.1. [8] A triplet $(\bar{D} = \bar{D}^{\circ} + K, \bar{g}, \bar{J})$ is called an indefinite Kaehler statistical structure on $\bar{M}$ if
(i) $(\bar{g}, \bar{J})$ is an indefinite Kaehler structure on $\bar{M}$
(ii)$(\bar{D}, \bar{g})$ is a statistical structure on $\bar{M}$
and the condition

$$K(X, \bar{J}Y) = -\bar{J}K(X, Y)$$

holds for any $X, Y \in \Gamma(T\bar{M})$.
Then $(\bar{M}, \bar{D}, \bar{g}, \bar{J})$ is called an indefinite Kaehler statistical manifold. If $(\bar{M}, \bar{D}, \bar{g}, \bar{J})$ is an indefinite Kaehler statistical manifold, then so is $(\bar{M}, \bar{D}^*, \bar{g}, \bar{J})$.

Equivalently

$$(\bar{D}^{\circ}_X \bar{J})Y = 0$$

for all $X, Y \in \Gamma(T\bar{M})$ which implies that

$$\bar{D}^{\circ}_X \bar{J}Y - \bar{J}\bar{D}^{\circ}_X Y = 0$$

$$\bar{D}_X \bar{J}Y - K(X, \bar{J}Y) - \bar{J}(\bar{D}^*_X Y + K(X, Y)) = 0$$

$$\bar{D}_X \bar{J}Y - \bar{J}\bar{D}^*_X Y = K(X, \bar{J}Y) + \bar{J}K(X, Y)$$

5 Structure of a hypersurface in an indefinite Kaehler statistical manifold

Let (M, g) be a real lightlike hypersurface of an indefinite Kaehler statistical manifold $\bar{M}$, where g is the degenerate metric induced on M. Then, as given by [3], $S(TM)$ splits in the following way:
Let (ξ, N) be a pair of local sections of $TM^{\perp} \oplus tr(TM)$. Then we have

$$\bar{g}(\bar{J}\xi, \xi) = \bar{g}(\bar{J}\xi, N) = \bar{g}(\bar{J}N, \xi) = \bar{g}(\bar{J}N, N) = 0, .$$

This shows that $\bar{J}\xi$ and $\bar{J}N$ are vector fields tangent to M. Thus $\bar{J}(TM^{\perp})$ and $\bar{J}(tr(TM))$ are distributions on M such that $TM^{\perp} \cap \bar{J}(TM^{\perp}) = 0$ and $TM^{\perp} \cap \bar{J}(tr(TM)) = 0$. As ξ and N are null vector fields satisfying $\bar{g}(\xi, N) = 1$, from (4.1), it follows that $\bar{J}\xi$ and $\bar{J}N$ are null vector field satisfying $\bar{g}(\bar{J}\xi, \bar{J}N) = 1$. Then one can choose a screen distribution $S(TM)$ such that it contains $\bar{J}(TM^{\perp})$ and $\bar{J}(tr(TM))$ as vector subbundles. Thus $\bar{J}(TM^{\perp}) \oplus \bar{J}(tr(TM))$ is a vector subbundle of $S(TM)$ of rank 2. There exists a non-degenerate almost complex distribution $E_{\circ}$ on M with respect to the almost complex structure tensor $\bar{J}$ of $\bar{M}$, i.e., $J(E_{\circ}) = E_{\circ}$, such that

$$S(TM) = \{\bar{J}(TM^{\perp}) \oplus \bar{J}(tr(TM))\} \perp E_{\circ}$$

and

$$TM = \{\bar{J}(TM^{\perp}) \oplus \bar{J}(tr(TM))\} \perp E_{\circ} \perp TM^{\perp}$$

Now consider the almost complex distribution E such that

$$E = \{TM^{\perp} \perp \bar{J}(TM^{\perp})\} \perp E_{\circ}$$

and the local lightlike vector fields U and V such that

$$U = -\bar{J}N, \quad V = -\bar{J}\xi. \tag{5.1}$$

Denote by S, the projection morphism of TM on E. Then

$$X = SX + u(X)U, \tag{5.2}$$

where u is a 1-form locally defined on M by

$$u(X) = g(X, V). \tag{5.3}$$

Apply $\bar{J}$ to (5.2) and obtain

$$\bar{J}X = FX + u(X)N, \tag{5.4}$$

where F is a tensor field of type (1,1) globally defined on M by

$$FX = \bar{J}SX, \quad \forall X \in \Gamma(TM). \tag{5.5}$$

Using equations (5.2), (5.3) and (5.5) and taking into account that $(E, \bar{J})$ is an almost complex distribution, we have

$$F^2X = -X + u(X)U; \quad u(U) = 1. \tag{5.6}$$

Also,

$$g(FX, FY) = g(X, Y) - u(X)v(Y) - u(Y)v(X)$$

for any $X, Y \in \Gamma(TM)$, where v is a 1-form locally defined on M by

$$v(X) = g(X, U)$$

Lemma 5.1. *Let (M, g) be a lightlike real hypersurface of an indefinite Kaehler statistical manifold $(\bar{M}, \bar{J}, \bar{g})$. Then, for any $X, Y \in \Gamma(TM)$, we have the following identities :*

$$D_X FY - FD^*_X Y = u(Y)A^*_N X - B^*(X, Y)U \tag{5.7}$$

and

$$Xu(Y) - uD^*_X Y = -u(Y)\tau^*(X) - B(X, FY) \tag{5.8}$$

Also

$$D^*_X FY - FD_X Y = u(Y)A_N X - B(X, Y)U \tag{5.9}$$

and

$$Xu(Y) - uD_X Y = -u(Y)\tau(X) - B^*(X, FY) \tag{5.10}$$

Proof. $\bar{M}$ being an indefinite Kaehler Statistical manifold implies that

$$\bar{D}_X \bar{J} Y = \bar{J} \bar{D}^*_X Y$$

for any $X, Y \in \Gamma(TM)$. Using equations (5.4), (3.2),(3.3) and (5.1), we get

$$D_X FY - FD^*_X Y + Xu(Y)N - u(D^*_X Y)N = u(Y)A^*_N X - u(Y)\tau^*(X)N$$

$$-B(X, FY)N - B^*(X, Y)U$$

Corresponding to the tangential and normal parts, we get the required identities (5.7) and (5.8). Also the fact that

$$\bar{D}^*_X \bar{J} Y = \bar{J} \bar{D}_X Y$$

for any $X, Y \in \Gamma(TM)$ results in the identities (5.9) and (5.10). □

This lemma leads to the following theorem.

Theorem 5.2. *Let (M, g) be a lightlike real hypersurface of an indefinite Kaehler statistical manifold $(\bar{M}, \bar{J}, \bar{g})$. Then M is totally geodesic*

1. *with respect to $\bar{D}^*$, if and only if, we have*

$$D_X FY = FD^*_X Y, \qquad \forall\, X \in \Gamma(TM), Y \in \Gamma(E),$$

and

$$A^*_N X = -F(D^*_X U), \qquad \forall\, X \in \Gamma(TM).$$

2. *with respect to $\bar{D}$, if and only if, we have*

$$D^*_X FY = FD_X Y, \qquad \forall\, X \in \Gamma(TM), Y \in \Gamma(E),$$

and

$$A_N X = -F(D_X U), \qquad \forall\, X \in \Gamma(TM).$$

Proof. From equations (5.4) and (5.7), we obtain

$$D_X FY - FD^*_X Y = g(Y, V)A^*_N X - B^*(X, Y)U, \quad \forall\, X \in \Gamma(TM), Y \in \Gamma(E).$$

Since $Y \in \Gamma(E)$ and V is a lightlike vector field, therefore

(5.11) $$D_X FY - FD^*_X Y = -B^*(X, Y)U, \quad \forall\, X \in \Gamma(TM).$$

Also, by replacing Y by U in (5.7) and using (5.4) and (5.6), we obtain

$$D_X FU - FD^*_X U = u(U)A^*_N X - B^*(X, U)U$$

(5.12) $$-FD^*_X U = A^*_N X - B^*(X, U)U$$

Using the fact that M is totally geodesic and from the equations (5.11), (5.12), we get the required identities. Similarly, the coresponding case for the dual connection also holds.
□

Lemma 5.3. *Let (M, g) be a lightlike real hypersurface of an indefinite Kaehler statistical manifold $(\bar{M}, \bar{J}, \bar{g})$. Then, for the local lightlike vector fields U and V, we have following identities:*

$$\tau^*(X) = u(D^*_X U), \tag{5.13}$$

$$D_X V = F(A^{*\prime}_\xi X) - \tau^*(X)V + B^*(X, \xi)U, \tag{5.14}$$

and

$$\tau(X) = u(D_X U), \tag{5.15}$$

$$D^*_X V = F(A'_\xi X) - \tau(X)V + B(X, \xi)U, \quad \forall\, X \in \Gamma(TM). \tag{5.16}$$

Proof: Replacing Y by U in (5.10), we derive

$$Xu(U) - u(D_X U) = -u(U)\tau(X) - B^*(X, FU)$$

From (5.4) and (5.6), we have

$$\tau(X) = u(D_X U), \quad \forall\, X \in \Gamma(TM).$$

Taking $Y = \xi$ in (5.9) and using (3.6),(5.3) and (5.4), we derive

$$D^*_X F\xi - FD_X\xi = u(\xi)A_N X - B(X, \xi)U$$

$$D^*_X V = F(A'_\xi X) - \tau(X)V + B(X, \xi)U, \qquad \forall\, X \in \Gamma(TM).$$

Hence the result. Similarly, we obtain (5.15) and (5.16).

Theorem 5.4. *Let (M, g) be a lightlike real hypersurface of an indefinite Kaehler statistical manifold $(\bar{M}, \bar{J}, \bar{g})$. Then*
The vector field U is parallel

1. *with respect to $\bar{D}^*$, if and only if*

$$A^*_N X = u(A^*_N X)U, \quad \tau^*(X) = 0, \quad \forall\, X \in \Gamma(TM). \tag{5.17}$$

2. *with respect to $\bar{D}$, if and only if*

$$A_N X = u(A_N X)U, \quad \tau(X) = 0 \quad \forall\, X \in \Gamma(TM). \tag{5.18}$$

Proof. Applying F to (5.12) and using equations (5.4), (5.6) and (5.13) , we have $\forall\, X \in \Gamma(TM)$,

$$F(A^*_N X) + F^2(D^*_X U) = F(B^*(X, U)U),$$
$$F(A^*_N X) - D^*_X U + u(D^*_X U)U = B^*(X, U)FU,$$
$$F(A^*_N X) - D^*_X U + \tau^*(X)U = 0,$$
$$D^*_X U = \bar{J}(A^*_N X) - u(A^*_N X)N + \tau^*(X)U,$$

From the hypothesis of parallelism of U with respect to the connection $\bar{D^*}$, the required result follows. The similar logic holds for the dual connection $\bar{D}$. □

Theorem 5.5. *Let (M, g) be a lightlike real hypersurface of an indefinite Kaehler statistical manifold $(\bar{M}, \bar{J}, \bar{g})$. Then*
The vector field V is parallel

1. *with respect to D^*, if and only if*

$$A_\xi^{*\prime} X = u(A_\xi^{*\prime} X)U; \quad \tau^*(X) = 0, \qquad \forall\, X \in \Gamma(TM). \tag{5.19}$$

2. *with respect to D, if and only if*

$$A'_\xi X = u(A'_\xi X)U; \quad \tau(X) = 0, \qquad \forall\, X \in \Gamma(TM). \tag{5.20}$$

Proof. Applying F to (5.14), we derive $\forall\, X \in \Gamma(TM)$,

$$F(D_X V) = F^2(A_\xi^{*\prime} X) - F(\tau^*(X)V) + F(B^*(X, \xi)U), \quad \forall X \in \Gamma(TM).$$

Further, equations (5.1), (5.4) and (5.6) imply that

$$F(D_X V) = -A_\xi^{*\prime} X + u(A_\xi^{*\prime} X)U + \tau^*(X)\xi$$

Hence the required identity (5.19) follows from the hypothesis using the above equation.
We obtain (5.20) for the corresponding dual connection proceeding in the same way.
□

Theorem 5.6. *Let M be a real lightlike hypersurface of an indefinite Kaehler statistical manifold $\bar{M}$.The distribution E is integrable with respect to $\bar{D}$ (resp. $\bar{D}^*$) if and only if*

$$B^*(X, FY) = B^*(FX, Y) \;\; (resp. B(X, FY) = B(FX, Y)), \quad \forall\, X, Y \in \Gamma(E)$$

Also, if M is totally geodesic with respect to $\bar{D}^*$ (resp. $\bar{D}$), then E is a parallel distribution with respect to $\bar{D}$ (resp. $\bar{D}^*$) on E.

Proof. Consider $Y \in \Gamma(E)$. Then $FY = JY \in \Gamma(E)$. Applying $\bar{J}$ to (3.4) with $Y \in \Gamma(E)$ and using the concept of indefinite Kaehler statistical manifold alongwith equations (5.1),(5.4), we derive

$$\bar{J}(\bar{D}_X Y) = \bar{J}(D_X Y) + \bar{J}(B(X, Y)N)$$

$$\bar{D}^*_X \bar{J} Y = F(D_X Y) + u(D_X Y)N + B(X, Y)\bar{J}N$$

$$D^*_X \bar{J} Y + B^*(X, \bar{J}Y)N = F(D_X Y) + u(D_X Y)N - B(X, Y)U$$

$$D^*_X FY + B^*(X, FY)N = F(D_X Y) + u(D_X Y)N - B(X, Y)U$$

Comparing tangential and normal parts and using (5.3), we get

$$D^*_X FY - F(D_X Y) = -B(X, Y)U, \quad B^*(X, FY) = g(D_X Y, V)$$

Now

$$B^*(X, FY) - B^*(FX, Y) = g(D_X Y, V) - g(D_Y X, V) = g([X, Y], V)$$

If the distribution E is integrable with respect to $\bar{D}$, then $g([X,Y],V)=0$. Therefore, $B^*(X,FY)=B^*(FX,Y)$

Conversely, if $B^*(X,FY)=B^*(FX,Y)$ for all $X,Y\in\Gamma(E)$, we have $g([X,Y],V)=0$ which implies that the distribution E is integrable with respect to $\bar{D}$. The corresponding result for the dual connection $\bar{D^*}$ also holds by the similar procedure.

Moreover, if M is totally geodesic with respect to $\bar{D^*}$, then

$$g(D_XY,V)=0,\quad \forall\ X\in\Gamma(TM)\ ,Y\in\Gamma(E).$$

This implies that $D_XY\in\Gamma(E)$ for all $X,Y\in\Gamma(E)$.
Hence E is a parallel distribution with respect to D on E. Therefore, if M is totally geodesic with respect to $\bar{D}$, then E is a parallel distribution with respect to D^* on E. □

Lemma 5.7. *Let M be a real lightlike hypersurface of an indefinite Kaehler statistical manifold $\bar{M}$.Then, for any $X,Y\in\Gamma(TM)$, we have*

$$D^*_XU=F(A^*_NX)+\tau^*(X)U,\quad B^*(X,U)=C^*(X,V) \tag{5.21}$$

$$D_XU=F(A_NX)+\tau(X)U,\quad B(X,U)=C(X,V) \tag{5.22}$$

Proof. From equations (5.1), (5.4), (3.5) after applying $\bar{J}$ to (3.4), we derive for an indefinite Kaehler statistical structure ,

$$\bar{J}(\bar{D}_XN)=-\bar{J}(A^*_NX)+\bar{J}(\tau^*(X)N)$$

$$\bar{D}^*_X(\bar{J}N)=-F(A^*_NX)-u(A^*_NX)N+\tau^*(X)\bar{J}N$$

$$-D^*_XU-B^*(X,U)N=-F(A^*_NX)-u(A^*_NX)N-\tau^*(X)U$$

Hence the tangential and transversal parts on both sides of the equation lead to the desired results for the connections dual to each other. □

Lemma 5.8. *Let M be a real lightlike hypersurface of an indefinite Kaehler statistical manifold $\bar{M}$.Then, for any $X,Y\in\Gamma(TM)$, we have*

$$D^*_XV=F(A'_\xi X)-\tau(X)V+B(X,\xi)U,\quad B^*(X,V)=u(A'_\xi X) \tag{5.23}$$

$$D_XV=F(A^{*\prime}_\xi X)-\tau^*(X)V+B^*(X,\xi)U,\quad B(X,V)=u(A^{*\prime}_\xi X) \tag{5.24}$$

Proof. After applying $\bar{J}$ to (3.4), we obtain from equations (3.6), (5.1), (5.4) using the concept of indefinite Kaehler statistical manifold,

$$\bar{J}(\bar{D}_X\xi)=-\bar{J}(A'_\xi X)-\bar{J}(\tau(X)\xi)+\bar{J}(B(X,\xi)N)$$

$$\bar{D}^*_X(\bar{J}\xi)=-F(A'_\xi X)-u(A'_\xi X)N-\tau(X)\bar{J}\xi+B(X,\xi)\bar{J}N$$

$$-D^*_XV-B^*(X,V)N=-F(A'_\xi X)-u(A'_\xi X)N+\tau(X)V-B(X,\xi)U$$

Hence the required results follow using the tangential and transversal parts. □

6 Recurrent and Lie recurrent structure tensor field

[7] defined recurrent and Lie-recurrent lightlike hypersurfaces of an indefinite Kaehler manifold and developed results related to their structure. We characterize the Lie-recurrent structure tensor field of the lightlike hypersurfaces in the statistical manifold corresponding to the dual connections.

Definition 6.1. The structure tensor field F of a lightlike hypersurface M with respect to $\bar{D}$ (resp. $\bar{D^*}$) is said to be recurrent if there exist a 1-form ω on M such that

$$(D_X F)Y = \omega(X)FY, \qquad (resp.(D^*_X F)Y = \omega^*(X)FY).$$

We define

$$\sigma(X) = B(X,U) = C(X,V) \ \ and \ \ \sigma^*(X) = B^*(X,U) = C^*(X,V),$$

where σ and σ^* are 1-forms on M.

If the lightlike hypersurface M of an indefinite Kaehler statistical manifold $\bar{M}$ admits a recurrent structure tensor field , it is called a recurrent lightlike hypersurface.

Theorem 6.1. *For a lightlike hypersurface M of an indefinite Kaehler statistical manifold $\bar{M}$, if F is recurrent, then*

1. $(D_X F)Y + (D^*_X F)Y = 0$
2. $A_N X + A^*_N X = (\sigma^*(X) + \sigma(X))U$
3. $A'_\xi X + A^{*\prime}_\xi X = (\sigma(X) + \sigma^*(X))V$

Proof. From lemma 5.1 we get,

$$(D_X F)Y + (D^*_X F)Y = u(Y)(A_N X + A^*_N X) - (B^*(X,Y) + B(X,Y))U$$

Since M is a recurrent lightlike hypersurface, therefore

$$\omega(X)FY + \omega^*(X)FY = u(Y)(A_N X + A^*_N X) - (B^*(X,Y) + B(X,Y))U \tag{6.1}$$

Replacing Y by ξ in the above equation and using the fact that $F\xi = -V$, we derive

$$(\omega(X) + \omega^*(X))V = 0 \tag{6.2}$$

Taking scalar product of U with (6.2), we get
$(D_X F)Y + (D^*_X F)Y = 0$
Replacing Y by U and using equation (5.6), we obtain

$$A_N X + A^*_N X = (\sigma^*(X) + \sigma(X))U$$

Now, by taking scalar product of V with (6.1) such that $\omega(X) + \omega^*(X) = 0$, we get

$$g(Y,V)g((\sigma^*(X) + \sigma(X))U, V) = g(U,V)(g(A'_\xi X) + g(A^{*\prime}_\xi X))$$
$$g((\sigma^*(X) + \sigma(X))V, Y) = g(A'_\xi X + A^{*\prime}_\xi X, Y)$$

which proves the assertion. □

Definition 6.2. The structure tensor field F of a lightlike hypersurface M is said to be Lie recurrent if there exist a $1-form\ \omega$ on M such that

$$(L_X F)Y = \theta(X)FY,$$

where L_X stands for the Lie derivative on M with respect to X. In particular, if $(L_X F)Y = 0$, then F is called Lie parallel.

$$(L_X F)Y = [X, FY] - F[X, Y]$$

As defined by [7], the lightlike hypersurface admitting a Lie-recurrent structure tensor field is a Lie-recurrent lightlike hypersurface.

Theorem 6.2. *A structure tensor field in a Lie-recurrent lightlike hypersurface M of an indefinite Kaehler statistical manifold $\bar{M}$ is Lie-parallel.*

Proof. Since θ is a Lie-recurrent structure tensor field, we derive

$$\theta(X)FY = u(Y)A^*_N X - B^*(X,Y)U - FD_X Y + FD^*_X Y - D_{FY}X + FD_Y X \tag{6.3}$$

$$\theta(X)FY = u(Y)A_N X - B(X,Y)U - FD^*_X Y + FD_X Y - D^*_{FY}X + FD^*_Y X \tag{6.4}$$

Replacing Y by ξ in above equations and using equations (3.7) and (5.3), we obtain

$$2\theta(X)F\xi = -D_{F\xi}X + FD_\xi X - D^*_{F\xi}X + FD^*_\xi X$$

Putting $F\xi = -V$ in the above equation and then taking its scalar product with V , we have

$$u(D_V X) + u(D^*_V X) = 0$$

On the other hand, replacing Y by V in equations (6.3) and (6.4) and then using equation (5.3) , we get

$$2\theta(X)\xi = -(B^*(X,V) + B(X,V))U - D_\xi X + FD_V X - D^*_\xi X + FD^*_V X$$

Finally, applying F to above equation , we obtain

$$2\theta(X)V = D_V X + FD_\xi X + D^*_V X + FD^*_\xi X$$

This implies that $\theta = 0$. Hence the result. □

7 Conclusion and scope

In this paper, the concept of lightlike hypersurfaces of the indefinite Kaehler statistical manifold has been initiated and results related to the geodesicity and paralleism of vector fields therein have been obtained. Also, some properties of the recurrent lightlike hypersurfaces have been worked upon. This study can be considered as a tool to explore more properties of lightlike hypersurfaces in the indefinite Kaehler manifold as well as in its odd dimensional counterpart.The applications of the statistical manifold endowed with the lightlike geometry in various brances of mathematics and physics can motivate the geometers to work further on the stucture of the indefinite Kaehler statistical manifold.

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