On the Ricci curvature of a homogeneous Finsler space with Randers change of square metric

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Abstract. The study of curvature properties of homogeneous Finsler spaces with (α, β)-metrics is one of the central problems in Riemann-Finsler geometry. In this paper, we consider a homogeneous Finsler space with Randers change of square metric. First, we establish an explicit formula for Ricci curvature of a homogeneous Finsler space with the metric under consideration. Next, we find a necessary and sufficient condition under which a homogeneous Finsler space with Randers change of square metric metric is of vanishing S-curvature. A formula for Ricci curvature of a homogeneous Finsler space with aforesaid metric having vanishing S-curvature is established. Finally, we prove that the afore said space having vanishing S-curvature and negative Ricci curvature must be Riemannian.

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1 Introduction

Finsler geometry is an interesting and active area of research for both pure and applied reasons (for detail, see [1, 2, 15, 19]). One special class of Finsler spaces is homogeneous and symmetric Finsler spaces, which is an active area of research nowadays. Many authors (for example, see [10, 14, 17, 26, 29, 35]) have worked in this area. Ricci curvature, denoted by $Ric(x, y)$ is an important entity in Riemann-Finsler geometry. It is the mathematical object that controls growth rate of volume of a metric ball in a manifold. The Ricci curvature of a Finsler space (M, F) can always be expressed in the form $Ric(x, y) = (n-1)\lambda(x, y)F^2$, where $\lambda(x, y)$ is a scalar function on TM, called Einstein scalar. If the Einstein scalar $\lambda(x, y)$ depends only on x, then the Finsler metric F is called an Einstein metric. Einstein manifolds play an important role in Riemann-Finsler geometry. Motivated by the open problem

"Does every smooth manifold admits an Einstein Finsler metric?"

posed by Chern, so many geometers (for example, see [5, 7, 8, 13, 24, 31, 38]) have established many results on Einstein Finsler metrics.

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In [27], Rezaei et al. studied some Einstein (α, β)-metrics. Further, in [33], Wang and Shen have studied Einstein-Randers metrics on homogeneous Riemannian spaces. After that, Deng and Hu [12] extended the work done in [33] to homogeneous Finsler spaces.

In [4], Bao and Robles proved that a compact Einstein-Randers space without boundary and with negative Ricci curvature must be Riemannian. The problem of negative Ricci curvature was further studied by Mo and Yu in [22].

In [39], Zhou studied Finsler spaces with (α, β)-metrics and introduced formulae for Riemann curvature and Ricci curvature for these metrics. Further, Cheng et al. [8] studied Einstein (α, β)-metrics. During their study, they found that the formulae given in [39] are incorrect and they provided correct version of these formulae. Based on these formulae, a formula for Ricci curvature of homogeneous (α, β)-metrics was constructed by Yan and Deng [36]. Recently, Kaur and Shanker [18] have worked on Ricci curvature of a homogeneous Finsler space with exponential metric.

In 1972, Matsumoto [20] introduced the concept of $(\alpha, \beta)-$metrics which are the generalizations of Randers metric introduced by Randers [25]. The main aim of this paper is to establish an explicit formula for Ricci-curvature of a homogeneous Finsler space with Randers change of square metric, and the same for that metric having vanishing S-curvature.

The simplest non-Riemannian metrics are the Randers metrics given by $F = \alpha + \beta$ with $\|\beta\|_\alpha < 1$, where α is a Riemannian metric and β is a 1-form. Besides Randers metrics, other interesting kind of non-Riemannian metrics are square metrics. Berwald's metric, constructed by Berwald [6] in 1929 as

$$F = \frac{\left(\sqrt{(1-|x|^2)\,|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle\right)^2}{\left(1-|x|^2\right)^2 \sqrt{(1-|x|^2)\,|y|^2 + \langle x, y\rangle^2}}$$

is a classical example of square metric. Berwald's metric can be rewritten as follows:

$$F = \frac{(\alpha + \beta)^2}{\alpha}, \tag{1.1}$$

where

$$\alpha = \frac{\sqrt{(1-|x|^2)\,|y|^2 + \langle x, y\rangle^2}}{\left(1-|x|^2\right)^2},$$

and

$$\beta = \frac{\langle x, y\rangle}{\left(1-|x|^2\right)^2}.$$

Equation (1.1) with the condition $\|\beta\|_\alpha < 1$ satisfies Lemma 2.1 which is a necessary and sufficient condition for F to be a Finsler metric. Therefore, the metric given by equation (1.1) is an (α, β)-metric. An (α, β)-metric expressed in the form (1.1) is called square metric [32]. Square metrics play an important role in Finsler geometry. The importance of square metric can be seen in papers [32, 34, 37]. Square metrics can also be expressed in the form

$$F = \frac{\left(\sqrt{(1-b^2)\,\alpha^2 + \beta^2} + \beta\right)^2}{\left(1-b^2\right)^2 \sqrt{(1-b^2)\,\alpha^2 + \beta^2}},$$

where $b := \|\beta_x\|_\alpha$ is the length of β [37].
In this case, $F = \alpha\phi\left(b^2, \dfrac{\beta}{\alpha}\right)$, where $\phi = \phi(b^2, s)$ is a smooth function, is called general (α,β)-metric. If $\phi = \phi(s)$ is independent of b^2, then F is called an (α,β)-metric.
If (M, F) is an n-dimensional Finsler space and $\beta = b_i(x)y^i$ is a 1-form on M. Then $F \longrightarrow \bar{F}$ is called Randers change if

$$\bar{F} = F + \beta \tag{1.2}$$

Above change has been introduced by Matsumoto [21], and it was named as "Randers change" by Hashiguchi and Ichijyō [16]. In the current paper, we deal with Randers change of square metric

$$F = \frac{(\alpha+\beta)^2}{\alpha} + \beta = \alpha\phi(s), \text{where} \ \ \phi(s) = 1 + s^2 + 3s.$$

The paper is organized as follows:
In section 2, we discuss some basic definitions and results to be used in subsequent sections. An explicit formula for Ricci curvature of a homogeneous Finsler space with Randers change of square metric has been established in section 3. In section 4, we find a necessary and sufficient condition under which a homogeneous Finsler space with Randers change of square metric is of vanishing S-curvature. After that, a formula for Ricci curvature of a homogeneous Finsler space with that metric having vanishing S-curvature is established. Finally, we prove that the afore said space having vanishing S-curvature and negative Ricci curvature must be Riemannian.

2 Preliminaries

In this section, we discuss some basic definitions and results required for subsequent sections. We refer [3, 9, 11] for notations and further details.

Definition 2.1. An n-dimensional real vector space V is said to be a **Minkowski space** if there exists a real valued function $F : V \longrightarrow [0, \infty)$, called Minkowski norm, satisfying the following conditions:

- F is smooth on $V\backslash\{0\}$,
- F is positively homogeneous, i.e., $F(\lambda v) = \lambda F(v) \ \ \forall\ \lambda > 0$,
- For any basis $\{u_1,\ u_2,\ ...,\ u_n\}$ of V and $y = y^i u_i \in V$, the Hessian matrix $\left(g_{ij}\right) = \left(\dfrac{1}{2}F^2_{y^i y^j}\right)$ is positive-definite at every point of $V\backslash\{0\}$.

Definition 2.2. Let M be a connected smooth manifold. If there exists a function $F : TM \longrightarrow [0, \infty)$ which is smooth on slit tangent bundle $TM\backslash\{0\}$ and the restriction of F to any $T_xM,\ x \in M$, is a Minkowski norm, then M is called a Finsler space and F is called a Finsler metric.

An (α, β)-metric on a connected smooth manifold M is a Finsler metric F constructed from a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and a one-form $\beta = b_i(x)y^i$ on M and is of the form $F = \alpha\phi\left(\dfrac{\beta}{\alpha}\right)$, where ϕ is a smooth function on M. Basically, (α, β)-metrics are the generalization of Randers metrics. Many authors (for example, see [14, 17, 28, 30, 35]) have worked on (α, β)-metrics. Let us recall following (Shen's) lemma which gives necessary and sufficient condition for a function of α and β to be a Finsler metric:

Lemma 2.1. *[9] Let $F = \alpha\phi(s),\ s = \beta/\alpha$, where ϕ is a smooth function on an open interval $(-b_0, b_0)$, α is a Riemannian metric and β is a 1-form with $\|\beta\|_\alpha < b_0$. Then F is a Finsler metric if and only if the following conditions are satisfied:*

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + \left(b^2 - s^2\right)\phi''(s) > 0 \ \ \forall \ \ |s| \le b < b_0.$$

2.1 Ricci curvature of an (α, β)-metric

The notion of Riemannian curvature for Riemannian spaces can be extended to Finsler spaces. For an n-dimensional Finsler space (M, F), $x \in M$, $y(\neq 0) \in T_xM$, the Riemannian curvature is a linear map $R_y : T_xM \longrightarrow T_xM$ defined as

$$R_y(v) = R^i_k(y)v^k \frac{\partial}{\partial x^i}, \quad v = v^i \frac{\partial}{\partial x^i},$$

where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k},$$

and G^i are geodesic coefficients given by

$$G^i \ := \frac{1}{4} g^{im} \left\{ \left(F^2\right)_{x^k y^m} y^k - \left(F^2\right)_{x^m} \right\}, \quad i = 1, 2, ..., n.$$

Definition 2.3. Let (M, F) be an n-dimensional Finsler space. For $x \in M$ and non-zero $y \in T_xM$, the map defined on tangent bundle TM of M as

$$Ric(y) = tr(R_y) \ \forall \ y \in T_xM$$

is called **Ricci-curvature** of (M, F).

For an (α, β)-metric $F = \alpha\phi(s),\ s = \dfrac{\beta}{\alpha}$ on a Finsler manifold M, let us take

$$r_{ij} = \frac{b_{i;j} + b_{j;i}}{2}, \qquad s_{ij} = \frac{b_{i;j} - b_{j;i}}{2},$$
$$r^i_j = a^{i\ell} r_{\ell j}, \qquad s^i_j = a^{i\ell} s_{\ell j},$$
$$r_i = b^\ell r_{\ell i} = b_j r^j_i, \qquad s_i = b^\ell s_{\ell i} = b_j s^j_i,$$
$$r = r_{ij} b^i b^j = b^i r_i, \qquad r_{00} = r_{ij} y^i y^j,$$
$$r_{i0} = r_{ij} y^j, \qquad s_{i0} = s_{ij} y^j,$$
$$r_0 = r_i y^i, \qquad s_0 = s_i y^i,$$

where ; denotes the covariant derivative w.r.t. Levi-Civita connection on Riemannian metric α and $a^{ij} = \left(a_{ij}\right)^{-1}, \ \ b^i = a^{ij} b_j$.

Theorem 2.2. *[8] Let F be an (α,β)-metric on a Finsler space M and ${}^{\alpha}Ric$ be Ricci curvature of α. Then Ricci curvature of F is given by $Ric ={}^{\alpha} Ric + RT^l_l$, with*

$$
\begin{aligned}
RT^l_l =&\frac{1}{\alpha^2}\Big\{ (n-1)\,\zeta_1+\zeta_2\Big\}r_{00}^2\\
&+\frac{1}{\alpha}\Big[\Big((n-1)\,\zeta_3+\zeta_4\Big)r_{00}s_0+\Big((n-1)\,\zeta_5+\zeta_6\Big)r_{00}r_0+\Big((n-1)\,\zeta_7+\zeta_8\Big)r_{00;0}\Big]\\
&+\Big((n-1)\,\zeta_9+\zeta_{10}\Big)s_0^2+\big(rr_{00}-r_0^2\big)\,\zeta_{11}+\Big((n-1)\,\zeta_{12}+\zeta_{13}\Big)r_0s_0\\
&+\Big(r_{00}r^l_l-r_{0l}r^l_0+r_{00;l}b^l-r_{0l;0}b^l\Big)\zeta_{14}+\Big((n-1)\,\zeta_{15}+\zeta_{16}\Big)r_{0l}s^l_0\\
&+\Big((n-1)\,\zeta_{17}+\zeta_{18}\Big)s_{0;0}+s_{0l}s^l_0\zeta_{19}+\alpha\Big[rs_0\ \zeta_{20}+\Big((n-1)\,\zeta_{21}+\zeta_{22}\Big)s_ls^l_0\Big]\\
&+\alpha\Big((3s_lr^l_0-2s_0r^l_l+2r_ls^l_0-2s_{0;l}b^l+s_{l;0}b^l)\ \zeta_{23}+s^l_{0;l}\zeta_{24}\Big)\\
&+\alpha^2\left(s_ls^l\ \zeta_{25}+s^i_ls^l_i\ \zeta_{26}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta_1 =&2\,\psi\,\Theta_s\left(B-s^2\right)-2\,s\psi\,\Theta+\Theta^2-\Theta_s,\\
\zeta_2 =&2\,\psi\,\psi_{ss}\left(B-s^2\right)^2-(6\,s\psi\,\psi_s+\psi_{ss})\left(B-s^2\right)+2\,s\psi_s,\\
\zeta_3 =&-4\,\left(2\,Q\Theta_s+Q_s\Theta\right)\psi\,\left(B-s^2\right)+4\,Q\Theta_s+2\,Q_s\Theta+4\,Q\Theta\,\left(s\psi-\Theta\right)-2\,\Theta_B,\\
\zeta_4 =&-4\,\psi\,\left(2\,Q\psi_{ss}+Q_s\psi_s+Q_{ss}{\psi_s}^2\right)\left(B-s^2\right)^2\\
&+\left(-4\,\psi^2\left(Q-sQ_s\right)+4\,Q_{ss}\psi+2\,Q_s\psi_s+4\,Q\psi_{ss}-2\,\psi_{sB}+20\,sQ\psi\,\psi_s\right)\left(B-s^2\right)\\
&+2\,\psi\,\left(Q-sQ_s\right)-4\,\psi_s-Q_{ss}-10\,sQ\psi_s,\\
\zeta_5 =&4\,\psi\,\Theta-2\,\Theta_B,\\
\zeta_6 =&2\,\left(2\,\psi\,\psi_s-\psi_{sB}\right)\left(B-s^2\right)-2\,\psi_s,\\
\zeta_7 =&-\Theta,\\
\zeta_8 =&-\psi_s\,\left(B-s^2\right),\\
\zeta_9 =&8\,Q\psi\,\left(Q\Theta_s+Q_s\Theta\right)\left(B-s^2\right)+4\,Q^2\left(\Theta^2-\Theta_s\right)+4\,Q\left(\Theta_B-Q_s\right),\\
\zeta_{10} =&\left(4\,\psi^2\left(2\,QQ_{ss}-{Q_s}^2\right)+8\,Q\psi\,\left(Q\psi_{ss}+Q_s\psi_s\right)-4\,Q^2{\psi_s}^2\right)\left(B-s^2\right)^2\\
&+\Big(-16\,sQ\psi\,\left(Q\psi_s+Q_s\psi\right)-4\,\psi\,\left(2\,QQ_{ss}-{Q_s}^2\right)-4\,Q\left(Q\psi_{ss}+Q_s\psi_s\right)\\
&+4\,Q\psi_{sB}+4\,Q_s\psi_B\Big)\left(B-s^2\right)-4\,s^2Q^2\psi^2+4\,\left(2+3\,sQ\right)\left(Q\psi_s+Q_s\psi\right)\\
&-8\,Q^2\psi+2\,QQ_{ss}-{Q_s}^2+4\,sQ\psi_B,\\
\zeta_{11} =&4\,\psi^2+4\,\psi_B,\\
\zeta_{12} =&4\,Q\left(-2\,\psi\Theta+\Theta_B\right),\\
\zeta_{13} =&\Big(8\,\psi\,\left(Q_s\psi-Q\psi_s\right)+4\,Q\psi_{sB}+4\,Q_s\psi_B\Big)\left(B-s^2\right)+8\,sQ\psi^2+4\,Q\psi_s-4\,\left(1-sQ\right)\psi_B,\\
\zeta_{14} =&2\,\psi,\\
\zeta_{15} =&4\,Q\,\Theta,
\end{aligned}
$$

$$\begin{aligned}
\zeta_{16} =& 4\,(Q\psi_s - Q_s\psi)\left(B-s^2\right) + 2\,Q_s - 2\,(1+2\,sQ)\,\psi,\\
\zeta_{17} =& 2\,Q\,\Theta,\\
\zeta_{18} =& 2\,(Q\psi_s + Q_s\psi)\left(B-s^2\right) + 2\,sQ\psi - Q_s,\\
\zeta_{19} =& 2\,(1+sQ)\,Q_s - 2\,Q^2,\\
\zeta_{20} =& -8\,\left(\psi^2+\psi_B\right)Q,\\
\zeta_{21} =& -4\,Q^2\Theta,\\
\zeta_{22} =& 2\,Q\psi - 4\,Q^2\psi_s\left(B-s^2\right),\\
\zeta_{23} =& 2\,Q\,\psi,\\
\zeta_{24} =& 2\,Q,\\
\zeta_{25} =& -4\,Q^2\psi,\\
\zeta_{26} =& -Q^2,\\
Q =& \frac{\phi'}{\phi - s\phi'},\\
\Theta =& \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\,(\phi - s\phi' + (B-s^2)\phi'')},\\
\psi =& \frac{\phi''}{2\phi\,(\phi - s\phi' + (B-s^2)\phi'')},\\
B =& b^2.
\end{aligned}$$

2.2 Homogeneous Finsler spaces

Let G be a Lie group and M, a smooth manifold. Then a smooth map $f : G\times M \longrightarrow M$ satisfying

$$f(g_2, f(g_1, x)) = f(g_2 g_1, x), \quad \text{for all} \quad g_1, g_2 \in G, \quad \text{and} \quad x \in M$$

is called a smooth action of G on M.

Definition 2.4. Let M be a smooth manifold and G, a Lie group. If G acts smoothly on M, then G is called a **Lie transformation group** of M.

The following theorem gives us a differentiable structure on the coset space of a Lie group.

Theorem 2.3. *Let G be a Lie group and H, its closed subgroup. Then there exists a unique differentiable structure on the left coset space G/H with the induced topology that turns G/H into a smooth manifold such that G is a Lie transformation group of G/H.*

Definition 2.5. Let (M, F) be a connected Finsler space and $I(M, F)$ the group of isometries of (M, F). If the action of $I(M, F)$ is transitive on M, then (M, F) is said to be a **homogeneous Finsler space**.

Let G be a Lie group acting transitively on a smooth manifold M. Then for $a \in M$, the isotropy subgroup G_a of G is a closed subgroup and by Theorem 2.3, G is a Lie transformation group of G/G_a. Further, G/G_a is diffeomorphic to M.

Theorem 2.4. *[11] Let (M, F) be a Finsler space. Then $G = I(M, F)$, the group of isometries of M is a Lie transformation group of M. Let $a \in M$ and $I_a(M, F)$ be the isotropy subgroup of $I(M, F)$ at a. Then $I_a(M, F)$ is compact.*

Let (M, F) be a homogeneous Finsler space, i.e., $G = I(M, F)$ acts transitively on M. For $a \in M$, let $H = I_a(M, F)$ be a closed isotropy subgroup of G which is compact. Then H is a Lie group itself being a closed subgroup of G. Write M as the quotient space G/H.

Definition 2.6. [23] Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H respectively. Then the direct sum decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$, where $\mathfrak{k}$ is a subspace of $\mathfrak{g}$ such that $\mathrm{Ad}(h)(\mathfrak{k}) \subset \mathfrak{k} \ \forall\, h \in H$, is called a reductive decomposition of $\mathfrak{g}$, and if such decomposition exists, then $(G/H, F)$ is called reductive homogeneous space.

Therefore, we can write, any homogeneous Finsler space as a coset space of a connected Lie group with an invariant Finsler metric. Here, the Finsler metric F is viewed as G invariant Finsler metric on M.

We can identify $\mathfrak{k}$ with the tangent space $T_H(G/H)$ of G/H at the origin H. Let $\left\{w_1, w_2, w_3, ..., w_n = \dfrac{w}{c}\right\}$ be the orthonormal basis of $\mathfrak{k}$, where w is the invariant vector field of length c corresponding to the 1-form β, i.e., $\langle w, Y\rangle = \beta(Y) \ \forall\, Y \in \mathfrak{k}$, and $\langle\ ,\ \rangle$ is the restriction of Riemannian metric α to $\mathfrak{k}$.

3 Ricci curvature of a homogeneous Finsler space with Randers change of square metric

Γ^m_{ij} are the Christoffel symbols given by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^m_{ij} \frac{\partial}{\partial x^m}.$$

To compute Γ^k_{ij}, we need following notations:
For $1 \le i,\ j,\ m \le n$, define the the structure constants of $\mathfrak{k}$ as follows:

$$C^m_{ij} = \left\langle [w_i, w_j]_{\mathfrak{k}}, w_m \right\rangle,$$

where $[w_i, w_j]_{\mathfrak{k}}$ denotes the projection of $[w_i, w_j]$ to $\mathfrak{k}$.
Take

$$\sum_{t=1}^{n} C^t_{ij} C^t_{ml} = C^t_{ij} C^t_{ml},$$

and

$$\left\langle [w_i, w_j]_{\mathfrak{k}}, Y \right\rangle = C^0_{ij}$$

for any non-zero $Y \in \mathfrak{k}$.
Define

$$f(i, j) = \begin{cases} 1, & \text{if } i < j, \\ 0, & \text{if } i \ge j. \end{cases}$$

With above notations, let us recall following lemmas for later use:

Lemma 3.1. *[12] At the origin $o = eH$, we have:*

(i) $\Gamma^m_{ij} = f(i,j)C^m_{ij} + \langle \nabla_{\hat{w}_i}\hat{w}_j, \hat{w}_m\rangle$,

(ii) $\langle \nabla_{\hat{w}_i}\hat{w}_j, \hat{w}_m\rangle = -\frac{1}{2}\left(C^i_{jm} + C^j_{im} + C^m_{ij}\right)$,

(iii) $\langle \nabla_{\hat{w}_i}\hat{w}_j, \hat{w}_m\rangle \hat{w}_l = \frac{1}{2}\left(C^i_{lt}C^t_{jm} + C^j_{lt}C^t_{im} + C^m_{lt}C^t_{ij} + C^t_{jm}C^t_{li} + C^t_{im}C^t_{lj} + C^t_{ij}C^t_{lm}\right)$,

where $w_i, w_j, w_m, w_l \in \mathfrak{k}$.

Lemma 3.2. *[12] At the origin $o = eH$, we have:*

$$
\begin{aligned}
b_i &= c\delta_{ni},\\
s_{ij} &= \frac{c}{2}C^n_{ij},\\
s_i &= \frac{c^2}{2}C^n_{ni},\\
r_{ij} &= \frac{c}{2}\left(C^j_{in} + C^i_{jn}\right),\\
s_{ij;m} &= \frac{c}{2}\left\{-C^t_{ij}C^n_{mt} + \frac{1}{2}C^n_{it}\left(-C^t_{jm} + C^m_{jt} + C^j_{mt}\right) + \frac{1}{2}C^n_{tj}\left(-C^t_{im} + C^m_{it} + C^i_{mt}\right)\right\},\\
s_{i;j} &= c\left(s_{ni;j} + \frac{c}{2}C^n_{ti}\Gamma^t_{nj}\right),\\
b_{i;j;m} &= c\Big(-\Gamma^i_{nt}\langle \nabla_{\hat{w}_m}\hat{w}_j, \hat{w}_t\rangle - \Gamma^t_{nj}\langle \nabla_{\hat{w}_m}\hat{w}_i, \hat{w}_t\rangle + C^t_{mn}\langle \nabla_{\hat{w}_t}\hat{w}_j, \hat{w}_i\rangle + \hat{w}_m\langle \nabla_{\hat{w}_n}\hat{w}_j, \hat{w}_i\rangle\Big),\\
r_{ij;m} &= s_{ij;m} + b_{j;i;m}.
\end{aligned}
$$

Using above two lemmas, we calculate the quantities used in Theorem 2.2, at the origin of a homogeneous Finsler space:

$$
\begin{aligned}
r_{00} &= cC^0_{0n},\\
s_i &= b^l s_{li} = cs_{ni},\\
s_0 &= cs_{n0} = \frac{c^2}{2}C^n_{n0},\\
r_i &= b^l r_{li} = cr_{ni},\\
r_0 &= cr_{n0} = -\frac{c^2}{2}\left(C^n_{n0} + C^0_{nn}\right) = \frac{c^2}{2}C^n_{0n},\\
r_{00;0} &= cC^0_{0i}\left(C^0_{in} + C^i_{0n}\right),\\
r &= r_{ij}b^i b^j = c^2 r_{nn} = 0,\\
r^l_l &= a^{li}r_{il} = -\frac{c}{2}\delta^{li}\left(C^i_{nl} + C^l_{ni}\right) = -\frac{c}{2}\left(C^l_{nl} + C^l_{nl}\right) = cC^l_{ln},\\
r^l_0 &= a^{li}r_{i0} = \delta^{li}r_{i0} = r_{l0},\\
r_{0l}r^l_0 &= r_{0l}r_{l0} = \left(-\frac{c}{2}\right)^2\left(C^0_{nl} + C^l_{n0}\right)\left(C^l_{n0} + C^0_{nl}\right) = \frac{c^2}{4}\left(C^0_{nl} + C^l_{n0}\right)^2,\\
r_{00;l}b^l &= r_{00;n}b^n = cr_{00;n} = c\left(s_{00;n} + b_{0;0;n}\right) = \frac{c^2}{2}\left(C^t_{0n} + C^0_{tn}\right)\left(C^n_{0t} + C^0_{nt} + C^t_{0n}\right),
\end{aligned}
$$

$$
\begin{aligned}
r_{0l;0}b^l &= r_{0n;0}b^n = cr_{0n;0} = c\left(s_{0n;0} + b_{n;0;0}\right)\\
&= \frac{c^2}{4}\left\{2\, C^n_{nt}C^0_{t0} + \left(C^n_{t0} + C^0_{tn} + C^t_{0n}\right)\left(C^0_{nt} + C^t_{n0}\right)\right\},\\
s^l_0 &= a^{li}s_{i0} = \delta^{li}s_{i0} = s_{l0},\\
r_{0l}s^l_0 &= r_{0l}s_{l0} = \frac{c^2}{4}\left(C^0_{nl}C^n_{0l} + C^l_{n0}C^n_{0l}\right),\\
s_{0;0} &= cs_{n0;0} + \frac{c^2}{2}C^n_{l0}\Gamma^l_{n0} = \frac{c^2}{2}C^n_{nt}C^0_{0t},\\
s_{0l}s^l_0 &= s_{0l}s_{l0} = -\frac{c^2}{4}\left(C^n_{0l}\right)^2,\\
s_l s^l_0 &= s_l s_{l0} = \frac{c^3}{4}C^n_{nl}C^n_{l0},\\
s_l r^l_0 &= s_l r_{l0} = -\frac{c^3}{4}C^n_{nl}\left(C^0_{nl} + C^l_{n0}\right),\\
r_l s^l_0 &= r_l s_{l0} = \frac{c^3}{4}C^n_{ln}C^n_{l0},\\
s_{0;l}b^l &= cs_{0;n} = c\left(cs_{n0;n} + \frac{c^2}{2}C^n_{t0}\Gamma^t_{nn}\right) = \frac{c^3}{4}C^n_{nt}\left(C^t_{0n} + C^0_{nt} + C^n_{0t}\right),\\
s_{l;0}b^l &= cs_{n;0} = c\left(cs_{nn;0} + \frac{c^2}{2}C^n_{tn}\Gamma^t_{n0}\right) = -\frac{c^3}{4}C^n_{tn}\left(C^n_{0t} + C^0_{nt} + C^t_{n0}\right),\\
s^l_{0;l} &= a^{li}s_{i0;l} = \delta^{li}s_{i0;l} = s_{l0;l} = \frac{c}{4}\left\{2C^n_{t0}C^l_{lt} + C^n_{lt}\left(C^t_{0l} + C^l_{0t} + C^0_{lt}\right)\right\},\\
s_l s^l &= s_l s_l = \frac{c^4}{4}\left(C^n_{nl}\right)^2,\\
s^i_l s^l_i &= -\frac{c^2}{4}\left(C^n_{il}\right)^2.
\end{aligned}
$$

Now, we calculate ζ_1 to ζ_{26} for Randers change of square metric:

$$
\zeta_1 = \frac{3\left(-40\,s^8 - 192\,s^7 + (16\,B - 235)\,s^6 + (72\,B + 96)\,s^5 + (235 + 90\,B)\,s^4 - (73 + 20\,B)\,s^2 + 24\,sB + 18\,B + 9\right)}{4\,\left(1 - 3\,s^2 + 2\,B\right)^3\left(1 + 3\,s + s^2\right)^2},
$$

$$
\zeta_2 = \frac{-6\left(9\,s^6 + (-13\,B - 2)\,s^4 + \left(2\,B^2 + 4\,B - 3\right)s^2 + B + 2\,B^2\right)}{\left(1 - 3\,s^2 + 2\,B\right)^4},
$$

$$
\begin{aligned}
\zeta_3 = &\frac{4\,(2\,s + 3)\left(12\,s^6 + 54\,s^5 + (73 + 8\,B)\,s^4 + (48\,B - 12)\,s^3 + (-42 + 78\,B)\,s^2 + (6 + 48\,B)\,s + 18\,B + 9\right)\left(B - s^2\right)}{\left(1 - s^2\right)\left(1 + 3\,s + s^2\right)^2\left(1 - 3\,s^2 + 2\,B\right)^3}\\
&- \frac{4\left(3 - 9\,s^2 - 4\,s^3\right)\left(B - s^2\right)}{\left(1 - s^2\right)^2\left(1 - 3\,s^2 + 2\,B\right)^2}
\end{aligned}
$$

$$- \frac{(2\,s+3)\left(40\,s^6+180\,s^5+(227+16\,B)\,s^4+(96\,B-48)\,s^3 - (138-156\,B)\,s^2+(12+96\,B)\,s+36\,B+27\right)}{(1+3\,s+s^2)^2\,(1-3\,s^2+2\,B)^2\,(1-s^2)}$$
$$- \frac{2\,(2\,s+3)\left(-3+9\,s^2+4\,s^3\right)s}{(1-s^2)\,(1+3\,s+s^2)\,(1-3\,s^2+2\,B)^2}$$
$$- \frac{2\,\left(-3+9\,s^2+4\,s^3\right)\left(2-4\,s^2+2\,B+3\,s-9\,s^3+6\,sB-2\,s^4+2\,s^2B\right)}{(1-s^2)^2\,(1+3\,s+s^2)\,(1-3\,s^2+2\,B)^2},$$

$$\zeta_4 = -\,24\left(\frac{(3+2\,s)\left(18\,s^2+2+4\,B+s-3\,s^3+2\,Bs\right)}{(1-3\,s^2+2\,B)^4\,(1-s^2)} + \frac{12\,\left(6\,s+2\,s^3+9\,s^2+3\right)s^2}{(1-s^2)^3\,(1-3\,s^2+2\,B)^5}\right)\left(B-s^2\right)^2$$
$$+ 4\left(\frac{10\,s^3-3+27\,s^2+6\,s}{(1-s^2)^2\,(1-3\,s^2+2\,B)^2} + \frac{2\,\left(6\,s+2\,s^3+9\,s^2+3\right)}{(1-s^2)^3\,(1-3\,s^2+2\,B)} + \frac{6\,(2\,s+3)\left(14\,s^2+1+2\,B\right)}{(1-3\,s^2+2\,B)^3\,(1-s^2)} + \frac{12\,s}{(1-3\,s^2+2\,B)^3}\right)\left(B-s^2\right)$$
$$- \frac{2\,\left(4\,s^3-3+9\,s^2\right)}{(1-3\,s^2+2\,B)\,(1-s^2)^2} - \frac{12\,s\left(2+8\,s^2+15\,s\right)}{(1-3\,s^2+2\,B)^2\,(1-s^2)} - \frac{2\,\left(6\,s+2\,s^3+9\,s^2+3\right)}{(1-s^2)^3},$$

$$\zeta_5 = \frac{-4\,\left(-3+9\,s^2+4\,s^3\right)}{(1+3\,s+s^2)\,(1-3\,s^2+2\,B)^2},$$

$$\zeta_6 = \frac{12\,s\left(4\,B-3\,s^2-1\right)}{(1-3\,s^2+2\,B)^3},$$

$$\zeta_7 = \frac{-3+9\,s^2+4\,s^3}{2\,(1+3\,s+s^2)\,(1-3\,s^2+2\,B)},$$

$$\zeta_8 = \frac{-6\,s\Big(B-s^2\Big)}{(1-3\,s^2+2\,B)^2},$$

$$\zeta_9 = \frac{8\,(3+2\,s)\left(-3+9\,s^2+4\,s^3\right)\left(B-s^2\right)}{(1-s^2)^3\,(1-3\,s^2+2\,B)^2}$$
$$- \frac{4\,(3+2\,s)^2\left(12\,s^6+54\,s^5+(73+8\,B)\,s^4+(48\,B-12)\,s^3 + (-42+78\,B)\,s^2+(6+48\,B)\,s+18\,B+9\right)\left(B-s^2\right)}{(1-s^2)^2\,(1+3\,s+s^2)^2\,(1-3\,s^2+2\,B)^3}$$
$$+ \frac{(3+2\,s)^2\left(3-9\,s^2-4\,s^3\right)^2}{(1-s^2)^2\,(1+3\,s+s^2)^2\,(1-3\,s^2+2\,B)^2}$$
$$+ \frac{2\,(3+2\,s)^2\left(12\,s^6+54\,s^5+(73+8\,B)\,s^4+(48\,B-12)\,s^3 + (-42+78\,B)\,s^2+(6+48\,B)\,s+18\,B+9\right)}{(1-s^2)^2\,(1+3\,s+s^2)^2\,(1-3\,s^2+2\,B)^2}$$
$$- \frac{4\,(3+2\,s)\left(3-9\,s^2-4\,s^3\right)}{(1-s^2)\,(1+3\,s+s^2)\,(1-3\,s^2+2\,B)^2} - \frac{8\,(3+2\,s)\left(1+3\,s+s^2\right)}{(1-s^2)^3},$$

$$\zeta_{10} = \Bigg\{ \frac{48\,(3+2\,s)\left(-24\,s^5-45\,s^4+12\,s^3+(6\,B+30)\,s^2+(4+8\,B)\,s+3+6\,B\right)}{(1-3\,s^2+2\,B)^4\,(1-s^2)^3}$$

$$+\frac{16\left(18\,s+28\,s^2+18\,s^3+3\,s^4+8\right)}{(1-s^2)^4\,(1-3\,s^2+2\,B)^2} - \frac{144\,s^2\,(3+2\,s)^2}{(1-3\,s^2+2\,B)^4\,(1-s^2)^2}\Bigg\}(B-s^2)^2$$

$$-\Bigg\{\frac{8\,(3+2\,s)\left(-108\,s^5-207\,s^4+(8\,B+52)\,s^3+(42\,B+138)\,s^2+(16+32\,B)\,s+9+18\,B\right)}{(1-s^2)^3\,(1-3\,s^2+2\,B)^3}$$

$$+\frac{16\left(18\,s+28\,s^2+18\,s^3+3\,s^4+8\right)}{(1-s^2)^4\,(1-3\,s^2+2\,B)}$$

$$+\frac{16\left(-15\,s^4-27\,s^3+(2\,B+10)\,s^2+(6\,B+21)\,s+1+2\,B\right)}{(1-3\,s^2+2\,B)^3\,(1-s^2)^2}\Bigg\}(B-s^2)$$

$$+\frac{8\,\left(2+4\,s^2+9\,s\right)\left(-9\,s^4-18\,s^3+(2\,B+4)\,s^2+(6\,B+12)\,s+1+2\,B\right)}{(1-s^2)^3\,(1-3\,s^2+2\,B)^2}$$

$$-\frac{4\,(3+2\,s)\left(-12\,s^3-15\,s^2+(6+8\,B)\,s+6+12\,B\right)}{(1-3\,s^2+2\,B)^2\,(1-s^2)^2}$$

$$+\frac{4\,18\,s+28\,s^2+18\,s^3+3\,s^4+8}{(1-s^2)^4},$$

$$\zeta_{11} = \frac{-4}{(1-3\,s^2+2\,B)^2},$$

$$\zeta_{12} = \frac{8\,(3+2\,s)\left(-3+9\,s^2+4\,s^3\right)}{(1+3\,s+s^2)\,(1-3\,s^2+2\,B)^2\,(1-s^2)},$$

$$\zeta_{13} = \frac{8\left(21\,s^4+27\,s^3-(26\,B-2)\,s^2-(36\,B-9)\,s+1+2\,B\right)}{(1-3\,s^2+2\,B)^3\,(1-s^2)},$$

$$\zeta_{14} = \frac{2}{1-3\,s^2+2\,B},$$

$$\zeta_{15} = \frac{2\,(2\,s+3)\left(3-9\,s^2-4\,s^3\right)}{(1-s^2)\,(1+3\,s+s^2)\,(1-3\,s^2+2\,B)},$$

$$\zeta_{16} = \frac{2\left(-21\,s^4-36\,s^3+(22\,B-4)\,s^2+36\,sB+2\,B+1\right)}{(1-3\,s^2+2\,B)^2\,(1-s^2)},$$

$$\zeta_{17} = \frac{(2\,s+3)\left(3-9\,s^2-4\,s^3\right)}{(1-s^2)\,(1+3\,s+s^2)\,(1-3\,s^2+2\,B)},$$

$$\zeta_{18} = \frac{-2\left(15\,s^4+18\,s^3-(14\,B+4)\,s^2-18\,Bs+1+2\,B\right)}{(1-s^2)\,(1-3\,s^2+2\,B)^2},$$

$$\zeta_{19} = \frac{2\left(-7+27\,s^2+24\,s^3+6\,s^4\right)}{(1-s^2)^3},$$

(3.1)

$$\zeta_{20} = \frac{8\ (2\,s+3)}{\left(1-3\,s^2+2\,B\right)^2 (1-s^2)},$$

$$\zeta_{21} = \frac{2\ (2\,s+3)^2 \left(-3+9\,s^2+4\,s^3\right)}{\left(1-s^2\right)^2 (1+3\,s+s^2)\,(1-3\,s^2+2\,B)},$$

$$\zeta_{22} = \frac{-2\ (2\,s+3)\left(-27\,s^4-36\,s^3+(26\,B+4)\,s^2+36\,Bs-1-2\,B\right)}{\left(1-3\,s^2+2\,B\right)^2 \left(1-s^2\right)^2},$$

$$\zeta_{23} = \frac{2\ (2\,s+3)}{(1-s^2)\,(1-3\,s^2+2\,B)},$$

$$\zeta_{24} = \frac{2\ (2\,s+3)}{1-s^2},$$

$$\zeta_{25} = \frac{-4\ \left(9+4\,s^2+12\,s\right)}{\left(1-s^2\right)^2 (1-3\,s^2+2\,B)},$$

$$\zeta_{26} = \frac{-4\ \left(9+4\,s^2+12\,s\right)}{\left(1-s^2\right)^2}.$$

Theorem 3.3. *Let $F = \dfrac{(\alpha+\beta)^2}{\alpha} + \beta$ be Randers change of square metric on homogeneous Finsler space G/H. Then, the Ricci curvature is given by*

$$\begin{aligned}
Ric(Z) = {}^{\alpha}Ric(Z) &+ \frac{c^2\left(C^0_{0n}\right)^2}{\alpha^2(Z)}\Big\{\,(n-1)\,\zeta_1+\zeta_2\Big\} \\
&+ \frac{c^3 C^0_{0n} C^n_{n0}}{2\alpha(Z)}\Big\{\,(n-1)\,(\zeta_3-\zeta_5)+\zeta_4-\zeta_6\Big\} \\
&- \frac{c C^0_{0l}\left(C^0_{nl}+C^l_{n0}\right)}{\alpha(Z)}\Big\{\,(n-1)\,\zeta_7+\zeta_8\Big\} \\
&+ \frac{c^4\left(C^n_{n0}\right)^2}{4}\Big\{\,(n-1)\,(\zeta_9-\zeta_{12})+\zeta_{10}-\zeta_{11}-\zeta_{13}\Big\} \\
&- \frac{c^2}{4}\Big\{4C^0_{0n}C^l_{nl}+\left(C^l_{n0}+C^0_{nl}\right)\left(2C^0_{nl}+C^n_{0l}+2C^l_{0n}\right)+2C^n_{nl}C^0_{l0}\Big\}\zeta_{14} \\
&+ \frac{c^2}{4}C^n_{0l}\left(C^l_{n0}+C^0_{nl}\right)\Big\{\,(n-1)\,\zeta_{15}+\zeta_{16}\Big\} \\
&+ \frac{c^2}{2}C^n_{nl}C^0_{0l}\Big\{\,(n-1)\,\zeta_{17}+\zeta_{18}\Big\} \\
&- \frac{c^2}{4}\left(C^n_{l0}\right)^2\zeta_{19}+\frac{c^3}{4}\alpha(Z)C^n_{l0}C^n_{nl}\Big\{\,(n-1)\,\zeta_{21}+\zeta_{22}\Big\} \\
&+ \frac{c^3}{4}\alpha(Z)\Big\{4C^n_{n0}C^l_{nl}-C^n_{nl}\left(4C^0_{nl}-C^n_{0l}\right)\Big\}\zeta_{23} \\
&+ \frac{c}{4}\alpha(Z)\Big\{2C^n_{t0}C^l_{lt}+C^n_{lt}C^0_{lt}\Big\}\zeta_{24} \\
&+ \frac{c^2}{4}\alpha^2(Z)\Big\{c^2\left(C^n_{nl}\right)^2\zeta_{25}-\left(C^n_{li}\right)^2\zeta_{26}\Big\},
\end{aligned}$$

where Z is a non-zero vector in $\mathfrak{k}$, and ζ_1 to ζ_{26} for are given by (3.1).

4 Ricci curvature of a homogeneous Finsler space with Randers change of square metric having vanishing S-curvature

In [26], we have given the formula for S-curvature of homogeneous Finsler space with G-invariant Randers change of square metric. Here, we give the equivalent condition for that space to have vanishing S-curvature.

Theorem 4.1. *Let G/H be a compact homogeneous Finsler space with G-invariant Randers change of square metric F. Then $(G/H, F)$ has vanishing S-curvature if and only if $\langle [w, Y]_{\mathfrak{k}}, Y\rangle = 0 \ \forall\ Y \in \mathfrak{k}$ and $w \in \mathfrak{k}$ corresponds to the 1-form β.*

Proof. We know that F has vanishing S-curvature if and only if

$$r_{ij} = 0, \text{ and } s_i = 0 \ \ \forall\ 1 \leq i,\ j \leq n. \tag{4.1}$$

Therefore, we have to prove that the conditions in equation (4.1) are equivalent to $\langle [w, Y]_{\mathfrak{k}}, Y\rangle = 0 \ \forall\ Y \in \mathfrak{k}$.
Firstly, suppose that $r_{ij} = 0$, and $s_i = 0 \ \ \forall\ 1 \leq i,\ j \leq n$. Using Lemma 3.2, we have

$$\frac{c}{2}\left(C^j_{in} + C^i_{jn}\right) = 0, \tag{4.2}$$

and

$$\frac{c^2}{2} C^n_{ni} = 0. \tag{4.3}$$

For $i = j$, (4.3) gives us

$$\langle [w, w_i]_{\mathfrak{k}}, w_i\rangle = 0.$$

Since $\left\{w_1, w_2, w_3, ..., w_n = \dfrac{w}{c}\right\}$ is the orthonormal basis of $\mathfrak{k}$,

$$\langle [w, Y]_{\mathfrak{k}}, Y\rangle = 0 \ \forall\ Y \in \mathfrak{k}.$$

For the second part, suppose

$$\langle [w, Y]_{\mathfrak{k}}, Y\rangle = 0 \ \forall\ Y \in \mathfrak{k}.$$

Therefore,

$$\langle [w, w_i]_{\mathfrak{k}}, w_i\rangle = 0 \ \forall\ 1 \leq i \leq n, \tag{4.4}$$

$$\left\langle [w, w_i + w_j]_{\mathfrak{k}}, w_i + w_j\right\rangle = 0 \ \forall\ 1 \leq i,\ j \leq n, \tag{4.5}$$

$$\langle [w, w + w_i]_{\mathfrak{k}}, w + w_i\rangle = 0 \ \forall\ 1 \leq i \leq n. \tag{4.6}$$

From (4.4) and (4.5), we get $C^i_{nj} + C^j_{ni} = 0$, i.e., $r_{ij} = 0$.
Further, from (4.4) and (4.6), we get $\langle [w, w_i]_{\mathfrak{k}}, w\rangle = 0$, i.e., $s_i = 0$. □

Theorem 4.2. *Let G/H be a compact homogeneous Finsler space with G-invariant Randers change of square metric F. If $(G/H, F)$ has vanishing S-curvature, then Ricci curvature is given by*

$$Ric(Z) = Ric^{\alpha}(Z) - \frac{c^2}{4}\left(C^n_{l0}\right)^2 \zeta_{19} + \frac{c}{4}\alpha(Z)\left(2\, C^n_{t0} C^l_{lt} + C^n_{lt} C^0_{lt}\right)\zeta_{24} - \frac{c^2}{4}\alpha^2(Z)\left(C^n_{ik}\right)^2 \zeta_{26},$$

where Z is a non-zero vector in $\mathfrak{k}$, and $\zeta_{19},\ \zeta_{24},\ \zeta_{26}$ are same as given in (3.3).

Proof. Since $(G/H, F)$ has vanishing S-curvature, by Theorem 4.1,

$$\langle [w, Y]_{\mathfrak{k}}\, , Y\rangle = 0 \ \forall\ Y \in \mathfrak{k}.$$

Therefore,

$$\begin{aligned} C^n_{nl} &= \langle [w_n, w_l]_{\mathfrak{k}}\, , w_n\rangle \\ &= \frac{1}{c^2}\left\{ \langle [w, w_l]_{\mathfrak{k}}\, , w\rangle + \langle [w, w_l]_{\mathfrak{k}}\, , w_l\rangle \right\} \\ &= \frac{1}{c^2}\langle [w, w_l]_{\mathfrak{k}}\, , w + w_l\rangle \\ &= \frac{1}{c^2}\langle [w, w]_{\mathfrak{k}} + [w, w_l]_{\mathfrak{k}}\, , w + w_l\rangle \\ &= \frac{1}{c^2}\langle [w, w + w_l]_{\mathfrak{k}}\, , w + w_l\rangle \\ &= 0,\ \ l = 1, ..., n. \end{aligned}$$

Further, for non-zero $X \in \mathfrak{k}$, we have

$$\begin{aligned} C^n_{n0} &= \langle [w_n, X]_{\mathfrak{k}}\, , w_n\rangle \\ &= \frac{1}{c^2}\left\{ \langle [w, X]_{\mathfrak{k}}\, , w\rangle + \langle [w, X]_{\mathfrak{k}}\, , X\rangle \right\} \\ &= \frac{1}{c^2}\langle [w, X]_{\mathfrak{k}}\, , w + X\rangle \\ &= \frac{1}{c^2}\langle [w, w + X]_{\mathfrak{k}}\, , w + X\rangle \\ &= 0, \\ C^0_{0n} &= \langle [X, w_n]_{\mathfrak{k}}\, , X\rangle \\ &= \left\langle \left[X, \frac{w}{c}\right]_{\mathfrak{k}}, X\right\rangle = 0, \end{aligned}$$

and

$$\begin{aligned} C_{nl}^0 + C_{n0}^l &= \langle [w_n, w_l]_{\mathfrak{k}}, X\rangle + \langle [w_n, X]_{\mathfrak{k}}, w_l\rangle \\ &= \frac{1}{c}\Big\{ \langle [w, w_l]_{\mathfrak{k}}, X\rangle + \langle [w, X]_{\mathfrak{k}}, w_l\rangle \Big\} \\ &= \frac{1}{c}\Big\{ \langle [w, w_l]_{\mathfrak{k}}, X\rangle + \langle [w, X]_{\mathfrak{k}}, w_l\rangle + \langle [w, w_l]_{\mathfrak{k}}, w_l\rangle + \langle [w, X]_{\mathfrak{k}}, X\rangle \Big\} \\ &= \frac{1}{c}\Big\{ \langle [w, w_l]_{\mathfrak{k}}, X + w_l\rangle + \langle [w, X]_{\mathfrak{k}}, X + w_l\rangle \Big\} \\ &= \frac{1}{c}\Big\{ \langle [w, w_l]_{\mathfrak{k}} + [w, X]_{\mathfrak{k}}, X + w_l\rangle \Big\} \\ &= \frac{1}{c}\, \langle [w, X + w_l]_{\mathfrak{k}}, X + w_l\rangle \\ &= 0. \end{aligned}$$

Using all above values in Theorem 3.3, we get the required result. □

Corollary 4.3. *Let G/H be a homogeneous Finsler space with Randers change of square metric F. If $(G/H, F)$ has vanishing S-curvature and negative Ricci curvature, then it must be Riemannian.*

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