On quasi hemi-slant submanifolds of nearly Kaehler manifolds

R. Prasad, P. K. Singh and A. K. Rai

Abstract. In the present paper, we study the quasi hemi-slant submanifolds of nearly Kaehler manifolds. These submanifolds are generalization of hemi-slant submanifolds and semi-slant submanifolds. We also study the geometry of leaves of distributions which are involved in the definition of quasi hemi-slant submanifolds. Some results are worked out on quasi hemi-slant submanifolds of nearly Kaehler manifolds to be totally geodesic. Moreover, we also obtain the necessary and sufficient conditions for a quasi hemi-slant submanifold to be quasi hemi-slant product manifold. We provide examples of quasi hemi-slant submanifolds of nearly Kaehler manifolds.


Key words: Quasi hemi-slant submanifolds; nearly Kaehler manifold; Riemannian maps; totally geodesic.

1 Introduction

In differential geometry, there are numerous important applications of theory of submanifolds in mathematics and in physics. The properties of this theory became an interesting subject in differential geometry, both in complex geometry and in contact geometry.

In 1900, the theory of slant submanifolds was initiated by B. Y. Chen, as a generalization of holomorphic and totally real submanifolds in complex geometry ([6],[8]). This work is the generalization of CR-submanifolds introduced by Bejancu [1], in 1986. The theory of slant submanifolds became a very rich area of research for geometers and many mathematicians have studied these submanifolds. Slant submanifolds were also studied by different geometers in different ambient spaces [18]. In particular, N. Papaghiuc [16] introduced the notion of semi-slant submanifolds as a generalization of slant submanifolds in 1994. A. Lotta ([14],[15]) defined and studied slant submanifolds in contact geometry; J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez studied slant, semi-slant and bi-slant submanifold in contact geometry ([3],[4],[5]). The hemi-slant submanifolds are one of the classes of bi-slant submanifolds. Several geometers studied these submanifolds in different kind of structures and also other context related to it ([9],[11],[17],[19],[22],[23]).
In 1981, Chen [7] defined the canonical de Rham cohomology class for closed CR-submanifolds in a Kaehler manifold and in 1982, Deshmukh [10] proved the same result for nearly Kaehler manifolds. After some time, in 2014, Sahin [21] studied the de Rham cohomology class of hemi-slant submanifolds of Kaehler manifolds. Motivated from above interesting and significant studies, we define quasi hemi-slant submanifolds of nearly Kahler manifolds which is generalization of anti-invariant, semi-invariant, slant, semi-slant and hemi-slant submanifolds.

The paper is organized as follows: In the second section, we collect some basic definitions and properties of some complex structures. In the third section, we define quasi hemi-slant submanifolds of nearly Kaehler manifolds. In the fourth section, we obtain necessary and sufficient conditions for such submanifolds to be integrable. In the fifth section, we study the geometry of leaves of distributions which are involved in the definition of quasi hemi-slant submanifolds and give the necessary and sufficient conditions for such submanifolds to be totally geodesic. In the last section, we provide some examples of such submanifolds.

2 Preliminaries

Let \( \mathcal{N} \) be a Riemannian manifold with an almost complex structure \( J \) and \( g \) be the Riemannian metric. The manifold \( N \) with almost complex structure \( J \) and Riemannian metric \( g \) is said to be a nearly Kaehler manifold if,

\[
(\nabla X J)Y + (\nabla Y J)X = 0,
\]

for any vector field \( X, Y \) in \( T\mathcal{N} \). The above condition is equivalent to the following equation

\[
(\nabla X J)X = 0,
\]

for \( X \) in \( T\mathcal{N} \). For a nearly Kaehler manifold, we also have the following facts:

\[
J^2 = -I \quad \text{and} \quad g(JX, JY) = g(X, Y),
\]

for all vector fields \( X, Y \in T\mathcal{N}[24] \).

We know that any Kaehler manifold is always a nearly Kaehler manifold, but the converse is not always true.

Let \( \gamma : \mathcal{M} \rightarrow \mathcal{N} \) be an isometrical immersion of a Riemannian manifolds \( \mathcal{M} \) into \( \mathcal{N} \). The Riemannian metric for \( \mathcal{M} \) and \( \mathcal{N} \) is denoted by the same symbol \( g \). The Lie algebra of vector fields and the set of all normal fields on \( \mathcal{M} \) are denoted by \( T\mathcal{M} \) and \( T^\perp\mathcal{M} \), respectively. The Levi-Civita connection on \( T\mathcal{M} \) and on \( T^\perp\mathcal{M} \) are denoted by \( \nabla \) and \( \nabla^\perp \), respectively. The Gauss and Weingarten equations are given as

\[
\nabla_X Y = \nabla_X Y + \Omega(X, Y),
\]

\[
\nabla_X V = -SV_X + \nabla^\perp_X V,
\]

for any \( X, Y \) in \( T\mathcal{M} \) and \( V \) in \( T^\perp\mathcal{M} \), where \( \Omega \) is the second fundamental form and \( S \) is the shape operator. The shape operator and the second fundamental form are related by the following equation

\[
g(SV_X, Y) = g(\Omega(X, Y), V).
\]
On quasi hemi-slant submanifolds

The mean curvature vector $\mathcal{H}$ on a submanifold $\mathcal{M}$ is defined as

\[(2.7) \quad \mathcal{H} = \frac{1}{l} \text{trace}(\Omega) = \frac{1}{l} \sum_{i=1}^{l} \Omega(e_i, e_i),\]

where $l$ denotes the dimension of submanifold $\mathcal{M}$ and $\{e_1, e_2, \ldots, e_l\}$ is the local orthonormal frame. The submanifold $\mathcal{M}$ is said to be a totally umbilical submanifold of a Riemannian manifold $\mathcal{N}$, if for the mean curvature vector $\mathcal{H}$, the following relation holds

\[(2.8) \quad \Omega(X, Y) = g(X, Y)\mathcal{H}.\]

Using (2.8), the Gauss and Weingarten equation takes the form

\[(2.9) \quad \nabla_X Y = \nabla_X Y + g(X, Y)\mathcal{H},\]

and

\[(2.10) \quad \nabla_X V = -Xg(\mathcal{H}, V) + \nabla_X^\perp V,\]

for $X, Y \in T\mathcal{M}$ and $V \in T^{\perp}\mathcal{M}$.

A submanifold $\mathcal{M}$ is said to be totally geodesic if $\Omega(X, Y) = 0$, for any $X, Y \in T\mathcal{M}$ and if $\mathcal{H} = 0$, the submanifold $\mathcal{M}$ is minimal.

Now, for a submanifold $\mathcal{M}$ of Riemannian manifold $\mathcal{N}$, we can write $X \in T\mathcal{M}$ as

\[(2.11) \quad JX = TX + NX,\]

where $TX$ and $NX$ are the tangential and normal parts of $JX$ respectively. Similarly, for any $V \in T^{\perp}\mathcal{M}$, we have

\[(2.12) \quad JV = tV + nV,\]

where $tV$ and $nV$ are the tangential and normal components of $JV$ respectively.

Now, denote by $P_X Y$ and $Q_X Y$ the tangential and normal parts of $(\nabla_X J)Y$ respectively, ie.

\[(2.13) \quad (\nabla_X J)Y = P_X Y + Q_X Y,\]

for $X, Y \in T\mathcal{M}$.

From equations (2.4), (2.5), (2.11) and (2.12), the following equations may easily be obtained

\[(2.14) \quad P_X Y = (\nabla_X T)Y - S_{NY} X - t\Omega(X, Y),\]

\[(2.15) \quad Q_X Y = (\nabla_X N)Y + \Omega(X, TY) - n\Omega(X, Y).\]

Similarly, for any $V \in T^{\perp}\mathcal{M}$, denote the tangential and normal parts of $(\nabla_X J)Y$ by $P_X V$ and $Q_X V$ respectively, we obtain

\[(2.16) \quad P_X V = (\nabla_X t)V + TS_V X - S_n V X,\]

\[(2.17) \quad Q_X V = (\nabla_X n)V + \Omega(tV, X) + NS_V X,\]
The covariant derivative of tangential and normal parts of (2.11) and (2.12) is given as

\begin{align*}
(\nabla_X T)Y &= \nabla_X TY - T\nabla_X Y, \\
(\nabla_X N)Y &= \nabla_X^\perp NY - N\nabla_X Y, \\
(\nabla_X t)V &= \nabla_X tV - t\nabla_X^\perp V,
\end{align*}

and

\begin{equation}
(\nabla_X n)V = \nabla_X^\perp nV - n\nabla_X^\perp V,
\end{equation}

for any $X, Y \in T\mathcal{M}$ and $V \in T^\perp \mathcal{M}$. It is straightforward to verify the following properties of $\mathcal{P} - \mathcal{Q}$, which we enlist here for later use:

\begin{enumerate}
  \item[(a1)] (i) $g(\mathcal{P}X Y, W) = -g(Y, \mathcal{P}X W)$, (ii) $g(\mathcal{Q}X Y, V) = -g(Y, \mathcal{Q}X V)$,
  \item[(a2)] $\mathcal{P}X JY + \mathcal{Q}X JY = -J(\mathcal{P}X Y + \mathcal{Q}X Y),$
\end{enumerate}

for all $X, Y \in T\mathcal{M}$ and $V \in T^\perp \mathcal{M}$.

On a Riemannian submanifold $\mathcal{M}$ of a nearly Kaehler manifold $\mathcal{N}$, from equations (2.2) and (2.13), we have

\begin{equation}
(a) \ \mathcal{P}X Y + \mathcal{P}Y X = 0, \quad (b) \ \mathcal{Q}X Y + \mathcal{Q}Y X = 0
\end{equation}

for any $X, Y \in T\mathcal{M}$.

The covariant derivative of the complex structure $J$ is defined as

\begin{equation}
(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y,
\end{equation}

or

\begin{equation}
\nabla_X JX = J\nabla_X X.
\end{equation}

Let $\mathcal{N}$ be a nearly Kaehler manifold and $\mathcal{M}$ be a submanifold of $\mathcal{N}$. There are certain important classes of submanifolds viz totally real submanifolds, holomorphic submanifolds and CR-submanifolds as a generalization of totally real and holomorphic submanifolds. Chen generalized CR-submanifolds as slant submanifolds and Sahin introduced hemi-slant submanifolds from it. The quasi hemi-slant submanifolds is a generalization of hemi-slant submanifolds.

### 3 Quasi hemi-slant submanifolds of nearly Kaehler manifolds

In this section of the paper, we give the definition of quasi hemi-slant submanifolds of nearly Kaehler manifolds and obtain the necessary and sufficient conditions for the distribution to be integrable for such manifolds.

**Definition 3.1.** [2] Let $\mathcal{M}$ be a Riemannian manifold isometrically immersed in almost complex manifold $\mathcal{N}$. A submanifold $\mathcal{M}$ of an almost complex manifold $\mathcal{N}$ is said to be invariant, if $J(T_x \mathcal{M}) \subseteq T_x \mathcal{M}$, for every point $x \in \mathcal{M}$. 

**Definition 3.2.** [12] A submanifold \( \mathcal{M} \) of an almost Hermitian manifold \( \mathcal{N} \) is said to be anti-invariant, if \( J(T_x\mathcal{M}) \subseteq T^\perp_x\mathcal{M} \), for every point \( x \in \mathcal{M} \).

**Definition 3.3.** [3] A submanifold \( \mathcal{M} \) of an almost Hermitian manifold \( \mathcal{N} \) is said to be slant, if for each non-zero vector \( X \) tangent to \( \mathcal{M} \) at \( x \in \mathcal{M} \), the angle \( \theta(X) \) between \( JX \) and \( T_x\mathcal{M} \) is constant, i.e., it does not depend on the choice of the point \( x \in \mathcal{M} \) and \( X \in T_x\mathcal{M} \). In this case, the angle \( \theta \) is called the slant angle of the submanifold. A slant submanifold \( \mathcal{M} \) is called proper slant submanifold if neither \( \theta = 0 \) nor \( \theta = \pi/2 \).

We note that on a slant submanifold \( \mathcal{M} \), if \( \theta = 0 \), then it is an invariant submanifold and if \( \theta = \pi/2 \), then it is an anti-invariant submanifold. This means slant submanifold is a generalization of invariant and anti-invariant submanifolds.

**Definition 3.4.** [13] Let \( \mathcal{M} \) be a submanifold of nearly Kaehler manifold \( \mathcal{N} \), then \( \mathcal{M} \) is said to be a hemi-slant submanifold if there exist two orthogonal distributions \( D^\theta \) and \( D^\perp \) on \( \mathcal{M} \) such that

1. \( TM = D^\theta \oplus D^\perp \),
2. \( D^\theta \) is a slant distribution with slant angle \( \theta \neq \pi/2 \),
3. \( D^\perp \) is a totally real i.e. \( JD^\perp \subseteq T^\perp\mathcal{M} \),

it is clear from the above that CR-submanifolds and slant submanifolds are hemi-slant submanifolds with slant angle \( \theta = \pi/2 \) and \( D^\theta = 0 \), respectively.

Now, we introduce the notion of quasi hemi-slant submanifolds of nearly Kaehler manifolds which have great role in theory of submanifolds.

**Definition 3.5.** Let \( \mathcal{M} \) be a submanifold of nearly Kaehler manifold \( \mathcal{N} \), then \( \mathcal{M} \) is said to be a quasi hemi-slant submanifold if there exist three orthogonal complementary distributions \( D, D^\theta \) and \( D^\perp \) such that

1. \( TM \) admits the orthogonal direct decomposition
\[
TM = D \oplus D^\theta \oplus D^\perp,
\]
2. The distribution \( D \) is invariant i.e. \( JD = D \),
3. The distribution \( D^\theta \) is slant and the angle \( \theta \) between \( JX \) and space \( (D^\theta)_q \) is constant and independent of choice of point \( q \in \mathcal{M} \).
4. The distribution \( D^\perp \) is \( J \) anti-invariant, i.e. \( JD^\perp \subseteq T^\perp\mathcal{M} \),

in this case, we call \( \theta \) the quasi hemi-slant angle of \( \mathcal{M} \).

Suppose the dimension of distributions \( D, D^\theta \) and \( D^\perp \) are \( n_1, n_2 \) and \( n_3 \) respectively. Then we can easily see the following particular cases:

1. If \( n_1 = 0 \), then \( \mathcal{M} \) is hemi-slant submanifold.
2. If \( n_2 = 0 \), then \( \mathcal{M} \) is semi invariant submanifold.
3. If \( n_3 = 0 \), then \( \mathcal{M} \) is semi-slant submanifold.

We say that the quasi hemi-slant submanifold \( \mathcal{M} \) is proper if \( D \neq \{0\} \), \( D^\perp \neq \{0\} \) and \( \theta \neq 0, \pi/2 \).

This means quasi hemi-slant submanifold is a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant, semi-slant submanifolds and also they are the examples of the quasi hemi-slant submanifolds.
Let $M$ be a quasi hemi-slant submanifold of a nearly Kaehler manifold $N$. We denote the projections of $X \in TM$ on the distributions $D$, $D^\theta$ and $D^\perp$ by $P$, $Q$ and $R$ respectively. Then we can write for any $X \in TM$,

\begin{equation}
X = PX + QX + RX.
\end{equation}

Now, from equation (2.11), we know that

\[ JX = TX + NX, \]

where $TX$ and $NX$ are tangential and normal components of $JX$ on $M$, respectively. Using (2.11) and (3.2), we obtain

\begin{equation}
JX = TPX + NPX + TQX + NQX + TRX + NRX.
\end{equation}

Since $JD = D$ and $JD^\perp \subseteq T^\perp M$, we have $NPX = 0$ and $TRX = 0$. Therefore, we get

\begin{equation}
JX = TPX + TQX + NQX + NRX.
\end{equation}

This means that

\begin{equation}
J(TM) = D \oplus TD^\theta \oplus ND^\theta \oplus JD^\perp.
\end{equation}

We also have,

\begin{equation}
T^\perp M = ND^\theta \oplus JD^\perp \oplus \mu,
\end{equation}

where $\mu$ is the invariant distribution of $T^\perp M$ with respect to $J$.

For any non-zero vector field $V \in T^\perp M$, we have from equation (2.12),

\[ JV = tV + nV, \]

where $tV$ and $nV$ are tangential and normal components of $V$ respectively.

**Lemma 3.1.** Let $M$ be the quasi hemi-slant submanifold of a nearly Kaehler manifold $N$. Then we obtain

\begin{equation}
TD = D, \quad TD^\theta = D^\theta, \quad TD^\perp = \{0\}, \quad tND^\theta = D^\theta, \quad tND^\perp = D^\perp.
\end{equation}

Now, using equations (2.3), (2.11) and (2.12) and (3.2), then on comparing the tangential and normal components, we have the following:

**Lemma 3.2.** Let $M$ be the quasi hemi-slant submanifolds of a nearly Kaehler manifold $N$. Then the endomorphism $T$ and projection morphisms $N$, $t$ and $n$ in the tangent bundle of $M$, satisfy the following identities:

(i) $T^2 + tN = -I$ on $TM$,
(ii) $NT + nN = 0$ on $TM$,
(iii) $Nt + nt = 0$ on $T^\perp M$,
(iv) $Tt + tn = 0$ on $T^\perp M$,

where $I$ is the identity operator.
**Lemma 3.3.** Let $M$ be the quasi hemi-slant submanifold of a nearly Kaehler manifold $\mathcal{N}$. Then

(i) \[ T^2X = -(\cos^2 \theta)X, \]

(ii) \[ g(TX, TY) = (\cos^2 \theta)g(X, Y), \]

(iii) \[ g(NX, NY) = (\sin^2 \theta)g(X, Y), \]

for any $X, Y \in D^\theta$.

*Proof.* The proof is same as one can found in [20]. □

**Theorem 3.4.** Let $M$ be the quasi hemi-slant submanifold of a nearly Kaehler manifold $\mathcal{N}$, then for any $X, Y \in TM$, we have the following

\[
\nabla_X TY - S_{NY}X - T\nabla_X Y - 2t\Omega(X, Y) + \nabla_Y TX - S_{NX}Y - T\nabla_Y X = 0, \]

and

\[
\Omega(X, TY) + \nabla_X NY - N\nabla_X Y - 2n\Omega(X, Y) + \Omega(Y, TX) + \nabla_Y NX - N\nabla_Y X = 0. \]

*Proof.* For a nearly Kaehler manifold, we know that

\[
(\nabla_X J)Y + (\nabla_Y J)X = 0, \]

which implies that

\[
\nabla_X Y - J\nabla_X Y + \nabla_Y JX - J\nabla_Y X = 0. \]

Using equations (2.4) and (2.11), we have

\[
\nabla_X TY + \nabla_X NY - J\nabla_X Y - J\Omega(X, Y) + \nabla_Y TX + \nabla_Y NX - J\nabla_Y X - J\Omega(Y, X) = 0. \]

Now, using equations (2.4), (2.5), (2.11) and (2.12), we get

\[
\nabla_X TY + \Omega(X, TY) - S_{NY}X + \nabla_X^\perp NY - T\nabla_X Y - N\nabla_X Y - t\Omega(X, Y) - n\Omega(X, Y) + \nabla_Y TX + \Omega(Y, TX) - S_{NX}Y + \nabla_Y^\perp NX - T\nabla_Y X - N\nabla_Y X - t\Omega(Y, X) - n\Omega(Y, X) = 0. \]

On comparing the tangential and normal components, we have the required results. □

---

### 4 Integrability of distributions

slant distribution ($D^\theta$), invariant distribution ($D$) and anti-invariant distribution ($D^\perp$).
Theorem 4.1. Let $M$ be a proper quasi hemi-slant submanifold of a nearly Kaehler manifold $N$. Then, the slant distribution $D^\theta$ is integrable if and only if

\[ g(S_{NW}Z - S_{NZ}W, TX) = g(S_{NTW}Z - S_{NTZ}W, X), \]

\[ -g(Q_W X, NZ) + g(Q_Z X, NW). \]

for any $Z, W \in D^\theta$ and $X \in D \oplus D^\perp$.

Proof. We know that the slant distribution $D^\theta$ is integrable if and only if $[Z, W] \in D^\theta$ for any $Z, W \in D^\theta$.

Let us consider $Z, W \in D^\theta$ and $X = PX + RX \in D \oplus D^\perp$, we have

\[ g([Z, W], X) = g(J\nabla_W Z, JX) - g(J\nabla_W Z, JX) \]

Now, we compute the both terms in right hand side of (4.2) as follows

\[ g(J\nabla_W Z, JX) = g(\nabla_W JZ, JX) - g(\nabla_W JZ, JX). \]

Using equations (2.3), (2.13) and (3.4) and Lemma 3.3, we get

\[ g(J\nabla_W Z, JX) = g(\nabla_W JZ, JX) - g(\nabla_W JZ, JX). \]

Interchanging $Z$ and $W$ in (4.3), we get

\[ g(J\nabla_W Z, JX) = \cos^2 \theta g(\nabla_W Z, JX) + g(S_{NTZ}W, X) + g(P_Z T W, X) - g(S_{NW}Z, JX) - g(P_Z W, JX) \]

Then from (4.3) - (4.4) and using (4.2), we obtain

\[ \sin^2 \theta g([Z, W], X) = g(S_{NTW}Z - S_{NTZ}W, X) - g(S_NWZ - S_{NZ}W, JX) \]

\[ + g(P_Z T W - P_Z T W, X) - g(P_Z T W - P_Z T W, JX). \]

Using $P - Q$ properties (a2) and (a1)(ii),

\[ g(P_Z W, JX) - g(P_Z W, JX) = g(P_Z T W, X) - g(Q_W X, NZ) \]

\[ - g(P_Z T W, X) + g(Q_Z X, NW). \]

Using equations (4.5) and (4.6), we get

\[ \sin^2 \theta g([Z, W], X) = g(S_{NTW}Z - S_{NTZ}W, X) - g(S_{NW}Z - S_{NZ}W, TX) \]

\[ - g(Q_W X, NZ) + g(Q_Z X, NW). \]

from above equation, we obtain that $[Z, W] \in D^\theta$ if and only if equation (4.7) satisfies.

This proves the theorem completely. \qed
In the following theorem, we give the sufficient condition for slant distribution $D^\theta$ to be integrable with the help of above theorem:

**Theorem 4.2.** Let $M$ be a proper quasi hemi-slant submanifold of a nearly Kaehler manifold $N$, then the slant distribution $D^\theta$ is integrable, if

$$
g(QZ, NW) - g(QW, NZ) \in ND^\theta \oplus \mu,
S_{NT} Z - S_{NT} W \in D^\theta \quad \text{and}
S_{NW} Z - S_{NW} W \in D^\perp \oplus D^\theta,
$$

(4.8)

for any $Z, W \in D^\theta$.

**Theorem 4.3.** Let $M$ be a proper quasi hemi-slant submanifold of a nearly Kaehler manifold $N$. Then the anti-invariant distribution $D^\perp$ is integrable if and only if

$$
g(S_{jZ} W - S_{jW} Z, TX) = -g(\nabla_Z J W - \nabla_W J Z - 2Q_Z W, NX)
$$

(4.9)

for any $Z, W \in D^\perp$ and $X \in D \oplus D^\theta$.

**Proof.** We know that the anti-invariant distribution $D^\perp$ is integrable if and only if $[Z, W] \in D^\perp$ for any $Z, W \in D^\perp$.

Let us consider $Z, W \in D^\perp$ and $X \in D \oplus D^\theta$, we have

$$
g([Z, W], X) = g(J\nabla_Z W, JX) - g(J\nabla_W Z, JX)
$$

$$
= g(\nabla_Z J W, JX) - g((\nabla_Z J)W, JX) - g(\nabla_W J Z, JX) + g((\nabla_W J)Z, JX),
$$

using equations (2.4), (2.5) and (3.4), we get

$$
g([Z, W], X) = g(S_{jZ} W - S_{jW} Z, TX) + g(\nabla_Z J W - \nabla_W J Z, NX)
$$

$$
- g(Q_Z W - Q_W Z, NX),
$$

using equation (2.22), we get

$$
g([Z, W], X) = g(S_{jZ} W - S_{jW} Z, TX) + g(\nabla_Z J W - \nabla_W J Z - 2Q_Z W, NX)
$$

(4.10)

from above equation, we obtain that $[Z, W] \in D^\perp$ if and only if equation (4.10) satisfies. This proves the theorem completely.

Similarly, for any $Z, W \in D$ and $X \in D^\theta \oplus D^\perp$, we have

$$
g([Z, W], X) = g(J\nabla_Z W, JX) - g(J\nabla_W Z, JX),
$$

$$
= g(\nabla_Z TW, JX) - g(P_Z W, JX) - g(\nabla_W TZ, JX) + g(P_W Z, JX),
$$

using equations (2.4) and (3.4), we get

$$
g([Z, W], X) = g(\nabla_Z TW - \nabla_W TZ, TQX) + g(\Omega(Z, TW) - \Omega(W, TZ) - 2P_Z W, JX).
$$

(4.11)

Hence, we have the following theorem:

**Theorem 4.4.** Let $M$ be a proper quasi hemi-slant submanifold of a nearly Kaehler manifold $N$. Then the invariant distribution $D$ is integrable if and only if

$$
g(\nabla_Z TW - \nabla_W TZ, TQX) = g(\Omega(Z, TW) - \Omega(W, TZ) - 2P_Z W, JX),
$$

for any $Z, W \in D$ and $X \in D^\theta \oplus D^\perp$. 
5 Totally geodesic quasi hemi-slant submanifolds and decomposition theorems

As we know that geodesicness and foliations are significant great notions. In the present section, we deal with the totally geodesic quasi hemi-slant submanifolds of nearly Kaehler manifolds. We investigate the geometry of leaves of slant, anti-invariant and invariant distributions and obtain necessary and sufficient conditions for quasi hemi-slant submanifolds \( M \) to be totally geodesic.

**Theorem 5.1.** Let \( M \) be a proper quasi hemi-slant submanifold of a nearly Kaehler manifold \( N \). Then the slant distribution \( D^\theta \) defines a totally geodesic foliation on \( M \) if and only if

\[
g(\nabla X Y, Z) = g(\nabla X Y, JZ) - g(\nabla X Y, JZ)
\]

and

\[
g(\nabla X Y, t_\xi) = g(\nabla X Y, t_\xi) - \Omega(X, TY), n_\xi),
\]

for any \( X, Y \in D^\theta \), \( Z \in D \oplus D^\bot \) and \( \xi \in TM^\bot \).

**Proof.** For any \( X, Y \in D^\theta \), \( Z = PZ + RZ \in D \oplus D^\bot \) and using equations (2.13), (2.23) and (3.2), we have

\[
g(\nabla X Y, Z) = g(\nabla X JY, JZ) - g(\nabla X JY, JZ)
= g(\nabla X TY, Z) - g(\nabla X NTY, Z)
+ g(\nabla X NY, TPZ + NRZ) - g(PX Y, TPZ).
\]

Using equations (2.5), (3.4) and Lemma 3.3, we get

\[
g(\nabla X Y, Z) = \cos^2 \theta g(\nabla X Y, Z) + g(\nabla X TY, Z) - g(\nabla X JY, JZ)
- g(S_{NYX}, TPZ) + g(\nabla X NY, NRZ) - g(PX Y, TPZ),
\]

\[
sin^2 \theta g(\nabla X Y, Z) = g(\nabla X TY, Z) - g(\nabla X JY, JZ)
+ g(\nabla X NY, TPZ + NRZ) - g(PX Y, TPZ).
\]

which is the first required result (5.1) of theorem.

Now, for any \( X, Y \in D^\theta \) and \( \xi \in T^\bot M \),

using equations (2.4), (2.5), (2.12) and (3.4), we have

\[
g(\nabla X Y, \xi) = g(\nabla X JY, J\xi) - g(\nabla X JY, J\xi)
= g(\nabla X TY, \xi) + g(\nabla X JY, J\xi) - g(PX Y, t_\xi)
= g(\nabla X TY, t_\xi) + g(\Omega(X, TY), n_\xi) - g(\nabla X JY, J\xi)
+ g(\nabla X NY, n_\xi) - g(PX Y, t_\xi).
\]

which is the second required result (5.2) of theorem.

Hence, the theorem is proved completely. \( \square \)
Theorem 5.2. Let $M$ be a proper quasi hemi-slant submanifold of a nearly Kaehler manifold $N$. Then the anti-invariant distribution $D^\perp$ defines a totally geodesic foliation on $M$ if and only if

\[(5.4) \quad g(\Omega(X,Y), NTQZ) = g(\nabla^\perp_X NY - Q_X Y, NQZ)\]

and

\[(5.5) \quad g(S_{NY}X, t\xi) = g(\nabla^\perp_X NY, n\xi) - g(Q_X Y, J\xi),\]

for any $X, Y \in D^\perp$, $Z \in D \oplus D^\theta$ and $\xi \in TM^\perp$.

Proof. For any $X, Y \in D^\perp$, $Z = PZ + QZ \in D \oplus D^\theta$ and using equations (2.13), (2.23) and (3.2), we have

\[
g(\nabla_X Y, Z) = g(\nabla_X JY, JZ) - g((\nabla_X J)Y, JZ) = g(\nabla_X JY, JPZ) + g(\nabla_X JY, TQZ) + g(\nabla_X JY, NQZ) - g(Q_X Y, JZ) = g(\nabla_X Y, PZ) - g(\nabla_X Y, T^2QZ) - g(\nabla_X Y, NTQZ) + g(\nabla_X NY, NQZ) - g(Q_X Y, NQZ).
\]

(5.6)

As we know that

\[(5.7) \quad g(\nabla_X Y, Z) = g(\nabla_X Y, PZ) + g(\nabla_X Y, QZ),\]

using equations (2.4), (2.5), (3.4) and Lemma 3.3, we get

\[g(\nabla_X Y, \sin^2 \theta QZ) = -g(\Omega(X,Y), NTQZ) + g(\nabla^\perp_X NY, NQZ) - g(Q_X Y, NQZ),\]

which is the first required result (5.4) of theorem.

Now, for any $X, Y \in D^\perp$ and $\xi \in T^\perp M$,

using equations (2.5), (2.12), (2.13) and (3.4), we have

\[
g(\nabla_X Y, \xi) = g(\nabla_X JY, J\xi) - g((\nabla_X J)Y, J\xi) = g(\nabla_X NY, t\xi) + g(\nabla_X NY, n\xi) - g(Q_X Y, J\xi) = -g(S_{NY}X, t\xi) + g(\nabla^\perp_X NY, n\xi) - g(Q_X Y, J\xi),
\]

which is the second required result (5.5) of theorem.

Hence, the theorem is proved completely. \qed

Theorem 5.3. Let $M$ be a proper quasi hemi-slant submanifold of a nearly Kaehler manifold $N$. Then the invariant distribution $D$ defines a totally geodesic foliation on $M$ if and only if

\[(5.8) \quad g(\nabla_X TY, TQZ) = -g(\Omega(X, TY), N\xi) + g(P_X Y, TQZ)\]

and

\[(5.9) \quad g(\nabla_X TY - P_X Y, t\xi) = -g(\Omega(X, TY), n\xi),\]

for any $X, Y \in D$, $Z \in D^\theta \oplus D^\perp$ and $\xi \in TM^\perp$. 

Proof. For any $X, Y \in D$, $Z = QZ + RZ \in D^\theta \oplus D^\perp$ and using equations (2.4), (2.23) and (3.4), we have

$$g(\nabla_X Y, Z) = g(\nabla_X JY, JZ) - g((\nabla_X J)Y, JZ)$$
$$= g(\nabla_X TY, JZ) - g(P_X Y, JZ)$$
$$= g(\nabla_X TY, TQZ) + g(\Omega(X, TY), JZ) - g(P_X TQZ),$$

which is the first required result (5.8) of theorem.

Now, for any $X, Y \in D$ and $\xi \in T^\perp M$,

using equations (2.4), (2.12) and (3.4), we have

$$g(\nabla_X Y, \xi) = g(\nabla_X JY, J\xi) - g((\nabla_X J)Y, J\xi)$$
$$= g(\nabla_X TY, J\xi) + g(\Omega(X, TY), n\xi) - g(P_X J\xi),$$

which is the second required result (5.9) of theorem.

Hence, the theorem is proved completely. \(\square\)

From Theorem 5.1, 5.2 and 5.3, we have following decomposition theorem:

**Theorem 5.4.** Let $M$ be a proper quasi hemi-slant submanifold of a nearly Kaehler manifold $N$. Then the fiber of $M$ is a local product Riemannian manifold of the form $\mathcal{M}_{D^\theta} \times \mathcal{M}_{D^\perp}$, where $\mathcal{M}_{D^\theta}$ and $\mathcal{M}_{D^\perp}$ are leaves of $D^\theta$ and $D^\perp$ respectively, if and only if the conditions (5.1), (5.2), (5.4), (5.5), (5.8) and (5.9) holds.

**Theorem 5.5.** Let $M$ be a proper quasi hemi-slant submanifold of a nearly Kaehler manifold $N$. Then $M$ is a totally geodesic submanifold of a nearly Kaehler manifold $N$ if and only if

$$g(\Omega(X, PY) + \cos^2 \theta \Omega(X, QY), \xi) = g(\nabla_X^\perp NTQY, \xi) + g(S_{NQY}X + S_{NRY}X + P_X Y, J\xi)$$
$$- g(\nabla_X^\perp NY - QY, n\xi),$$

for any $X, Y \in TM$ and $\xi \in T^\perp M$.

**Proof.** For any $X, Y \in TM$, and $\xi \in T^\perp M$, using equations (2.2), (2.23) and (3.2), we have

$$g(\nabla_X Y, \xi) = g(\nabla_X P Y, \xi) + g(\nabla_X Q Y, \xi) + g(\nabla_X R Y, \xi)$$
$$= g(\nabla_X JPY, J\xi) + g(\nabla_X TQY, J\xi) + g(\nabla_X NQY, J\xi)$$
$$+ g(\nabla_X JRY, J\xi) - g((\nabla_X J)Y, J\xi).$$

Using equations (2.4), (2.5) and Lemma 3.3, we have

$$g(\nabla_X Y, \xi) = g(\nabla_X P Y, \xi) - g(\nabla_X T^2 Q Y, \xi) - g(\nabla_X NTQY, \xi)$$
$$+ g(\nabla_X NQY, J\xi) + g(\nabla_X NRY, J\xi) - g((\nabla_X J)Y, J\xi)$$
$$= g(\Omega(X, PY), \xi) + \cos^2 \theta g(\Omega(X, QY), \xi) - g(\nabla_X NTQY, \xi)$$
$$+ g(-S_{NQY}X + \nabla_X^\perp NQY, J\xi) + g(-S_{NRY}X + \nabla_X^\perp NRY, J\xi)$$
$$- g((\nabla_X J)Y, J\xi),$$
using equations (2.11), (2.12) and (2.13), we have

$$g(\nabla_X Y, \xi) = g(\Omega(X, PY) + \cos^2 \theta \Omega(X, QY), \xi) - g(\nabla_X^\perp NTQY, \xi)$$

$$- g(S_{NQY} X + S_{NY} X + \mathcal{P}_X Y, t\xi) + g(\nabla_X^\perp NY - Q_X Y, n\xi).$$

Hence the proof.

\[ \square \]

### 6 Examples

Let \((x_i, y_i)\) be cartesian coordinates on an Euclidean space \(R^{2n}\) for \(i = 1, 2, \ldots, n\). We can canonically choose an almost complex structure \(J\) on \(R^{2n}\) as follows:

$$J(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3} + a_4 \frac{\partial}{\partial x_4} + \ldots + a_{2n-1} \frac{\partial}{\partial x_{2n-1}} + a_{2n} \frac{\partial}{\partial x_{2n}})$$

$$= (a_1 \frac{\partial}{\partial x_2} - a_2 \frac{\partial}{\partial x_1} + a_3 \frac{\partial}{\partial x_4} - a_4 \frac{\partial}{\partial x_3} + \ldots + a_{2n-1} \frac{\partial}{\partial x_{2n-1}} - a_{2n} \frac{\partial}{\partial x_{2n}}),$$

where \(a_1, a_2, a_3, \ldots, a_{2n-1}, a_{2n}\) are \(C^\infty\) functions defined on \(R^{2n}\).

In this section, we will use this notation.

**Example 6.1.** Consider a submanifold \(M\) of \(R^{14}\) defined by

$$f(x_1, x_2, \ldots, x_8) = \left(\frac{x_1 + \sqrt{3}x_2}{2}, \frac{x_1 - \sqrt{3}x_2}{2}, x_3 \cos \theta, x_4, x_5 \sin \theta, 0, x_5 \cos \theta, 0, -x_5 \sin \theta, 0, \frac{\sqrt{5}x_7 + 2x_8}{2}, \frac{\sqrt{5}x_7 - 2x_8}{2}, x_6, 0\right),$$

where \(\theta\) is constant.

By direct computation, it is easy to check that the tangent bundle of \(M\) is spanned by the set \(\{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8\}\), where

- \(X_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, X_2 = \sqrt{3} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right), X_3 = \frac{\partial}{\partial x_3} \cos \theta + \frac{\partial}{\partial x_4} \sin \theta, X_4 = \frac{\partial}{\partial x_4} \cos \theta + \frac{\partial}{\partial x_3} \sin \theta, X_5 = \frac{\partial}{\partial x_5} \cos \theta - \frac{\partial}{\partial x_6} \sin \theta, X_6 = \frac{\partial}{\partial x_6} \cos \theta + \frac{\partial}{\partial x_5} \sin \theta, X_7 = \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8}, X_8 = \frac{\partial}{\partial x_8} - \frac{\partial}{\partial x_7} x_2\).

Then using the canonical complex structure of \(R^{14}\), we have

- \(JX_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, JX_2 = -\sqrt{3} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right), JX_3 = -\left(\frac{\partial}{\partial x_4} \cos \theta + \frac{\partial}{\partial x_3} \sin \theta\right), JX_4 = \frac{\partial}{\partial x_4} \cos \theta + \frac{\partial}{\partial x_3} \sin \theta, JX_5 = -\left(\frac{\partial}{\partial x_5} \cos \theta - \frac{\partial}{\partial x_6} \sin \theta\right), JX_6 = \frac{\partial}{\partial x_6} \cos \theta + \frac{\partial}{\partial x_5} \sin \theta, JX_7 = \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8}, JX_8 = -\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\).

It is easy to see that \(D = \text{span}\{X_1, X_2, X_7, X_8\}, D^\perp = \text{span}\{X_3, X_4\}\) and \(D^\perp = \text{span}\{X_5, X_6\}\) are invariant, slant and anti-invariant distributions respectively.

**Example 6.2.** Consider a submanifold \(M\) of \(R^{12}\) defined by

$$f(x_1, x_2, \ldots, x_6) = \left(\frac{x_1 + \sqrt{3}x_2}{2}, \frac{x_1 - \sqrt{3}x_2}{2}, \frac{x_3}{\sqrt{2}}, x_4, \frac{x_3}{\sqrt{2}}, 0, \frac{x_5}{\sqrt{2}}, 0, -\frac{x_5}{\sqrt{2}}, 0, x_6, 0\right),$$

where \(\theta = \frac{\pi}{4}\).
By direct computation, it is easy to check that the tangent bundle of $M$ is spanned by the set $\{X_1, X_2, X_3, X_4, X_5, X_6\}$, where
\[
X_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \quad X_2 = \sqrt{3}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}), \quad X_3 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}), \quad X_4 = \frac{\partial}{\partial x_4}, \quad X_5 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_8} - \frac{\partial}{\partial x_10}), \quad X_6 = -\frac{\partial}{\partial x_12}.
\]

Then using the canonical complex structure of $\mathbb{R}^{12}$, we have
\[
JX_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \quad JX_2 = -\sqrt{3}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), \quad JX_3 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}), \quad JX_4 = \frac{\partial}{\partial x_4}, \quad JX_5 = -\frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_8} - \frac{\partial}{\partial x_10}), \quad JX_6 = -\frac{\partial}{\partial x_12}.
\]

It is easy to see that $D^1 = \text{span}\{X_1, X_2\}$, $D^\theta = \text{span}\{X_3, X_4\}$ and $D^\bot = \text{span}\{X_5, X_6\}$ are invariant, slant and anti-invariant distributions respectively.

References


Authors’ addresses:

Rajendra Prasad, Punit Kumar Singh
Department of Mathematics and Astronomy,
University of Lucknow, Lucknow, India.
E-mail: rp.manpur@rediffmail.com , singhpunit1993@gmail.com

Amit Kumar Rai
University School of Basic and Applied Sciences,
Guru Gobind Singh Indraprastha University, New Delhi, India.
E-mail: rai.amit08au@gmail.com