Yamabe and gradient Yamabe solitons on 3-dimensional hyperbolic Kenmotsu manifolds

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Abstract. The main goal of this manuscript is to study the properties of 3-dimensional hyperbolic Kenmotsu manifolds endowed with Yamabe and gradient Yamabe metrics.

M.S.C. 2010: 53C25, 53C21, 53C44.
Key words: Yamabe soliton; gradient Yamabe soliton; hyperbolic Kenmotsu manifold; conformal vector field; Einstein manifold.

1 Introduction

The notion of Yamabe flow was introduced by Richard Hamilton [17] as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics. A time dependent metric $g(\cdot, t)$ on an n-dimensional Riemannian or pseudo-Riemannian manifold M_n is said to evolve by the Yamabe flow if the metric g satisfies

$$\frac{\partial g(t)}{\partial t} = -\tau \, g(t), \quad g(0) = g_0,$$

where τ is the scalar curvature of M_n and 't' is the time. Yamabe flow is a perfect match to the fast diffusion case of plasma equation in mathematical physics. Yamabe flow that moves by one parameter family of diffeomorphism φ_t is characterized as Yamabe soliton. Here φ_t is generated by a fixed vector field $\mathcal{V}$ on M_n and homotheries, that is, $g(\cdot, t) = \sigma(t)\varphi^*(t)g_0$. A (semi-)Riemannian manifold M_n is derived as a Yamabe soliton [17] if there exists a vector field $\mathcal{V}$ such that

$$\mathcal{L}_{\mathcal{V}} g = (\lambda - \tau)g \tag{1.1}$$

for some real constant $\lambda \in \mathbb{R}$. The smooth vector field $\mathcal{V}$ on M_n is called the soliton or potential vector field of Yamabe soliton and $\mathcal{L}_{\mathcal{V}}$ denotes the Lie-derivative operator of g along the potential vector field $\mathcal{V}$. Here $\mathbb{R}$ represents the set of real numbers. A Yamabe soliton is characterized as *shrinking, steady* or *expanding* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. A Yamabe soliton is said to be a gradient Yamabe soliton, if there exists a smooth function $f : M \to \mathbb{R}$ such that $\mathcal{V} = Df$, where D denotes the gradient operator of g. In this case, equation (1.1) reduces to

$$2Hess\, f = (\lambda - \tau)g.$$

Differential Geometry - Dynamical Systems, Vol.23, 2021, pp. 175-188.

Here f is called the potential function of the gradient Yamabe soliton and $Hess$ represents Hessian. In view of $Hess\, f(\mathcal{X},\mathcal{Y}) = g(\nabla_{\mathcal{X}} Df, \mathcal{Y})$, the above equation can be re-written as:

$$\nabla_{\mathcal{X}} Df = \frac{(\lambda - \tau)}{2}\mathcal{X} \tag{1.2}$$

for arbitrary vector fields $\mathcal{X}$ and $\mathcal{Y}$ on M_n. Here ∇ denotes the Levi-Civita connection of g. For more details about the Yamabe and gradient Yamabe solitons, we cite [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 24, 25, 27, 29, 30, 31] and the references therein. In [21], Ma and Cheng proved that a gradient Yamabe soliton on a complete non-compact manifold has warped product structure in the region $\Sigma := \{|\nabla f| \neq 0\}$. For more details, we refer [16, 18]. In two-dimension, the Yamabe flow is equivalent to the Ricci flow. But in higher dimension ($n > 2$), Yamabe and Ricci flows are not equivalent, since Yamabe flow preserves the conformal class of metric but the Ricci flows do not preserve in general. A Ricci soliton [17] is considered as a natural generalization of Einstein metric. A Ricci soliton $(g, \mathcal{V}, \mu)$ on a semi-Riemannian manifold (M_n, g) is elucidated by

$$\mathcal{L}_{\mathcal{V}} g + 2S = 2\mu g, \tag{1.3}$$

where S denotes the Ricci tensor of M_n and μ is a real constant. A Ricci soliton is said to be *shrinking, steady* or *expanding* according as $\mu > 0$, $\mu = 0$ or $\mu < 0$, respectively.

On the other hand, the notion of almost contact hyperbolic (f, g, η, ξ)-structure was introduced by Upadhyay and Dube [28]. Further, it was studied by number of authors [1, 19, 23, 26]. A non-zero vector field $\nu \in T_p(M)$ is said to be timelike (resp., null, space-like, and non-space-like) if it satisfies $g_p(\nu, \nu) < 0$ (resp., $= 0$, > 0, and ≤ 0) [22], where $T_p(M)$ denotes the tangent space of M at point p. Let $\{e_1, e_2, \ldots, e_{2n}, e_{2n+1} = \xi\}$ be a local orthonormal basis of vector fields in a $(2n+1)$-dimensional semi-Riemannian manifold. Thus, the Ricci tensor S and the scalar curvature τ of a $(2n+1)$-dimensional almost hyperbolic contact metric manifold M_{2n+1} endowed with the semi-Riemannian metric g are, respectively, defined as follows:

$$\begin{aligned} S(\mathcal{X},\mathcal{Y}) &= \sum_{i=1}^{2n+1} \varepsilon_i g(R(e_i,\mathcal{X})\mathcal{Y}, e_i) \\ &= \sum_{i=1}^{2n} \varepsilon_i g(R(e_i,\mathcal{X})\mathcal{Y}, e_i) - g(R(\xi,\mathcal{X})\mathcal{Y}, \xi), \end{aligned}$$

$$\tau = \sum_{i=1}^{2n+1} \varepsilon_i S(e_i, e_i) = \sum_{i=1}^{2n} \varepsilon_i S(e_i, e_i) - S(\xi, \xi) \tag{1.4}$$

for all vector fields $\mathcal{X}$ and $\mathcal{Y}$, where $\varepsilon_i = g(e_i, e_i)$, ξ is the unit timelike vector field, that is, $g(\xi, \xi) = -1$ and R represents the curvature tensor of M_{2n+1} [22].

In 1972, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions [20], named as Kenmotsu manifold. Kenmotsu proved that a locally Kenmotsu manifold is a warped product $I \times_f M$ of an interval I and a Kaehler manifold M with warping function $f(t) = se^t$; where s is a non-zero constant. This is expressed by the condition

$$(\nabla_{\mathcal{X}}\phi)\mathcal{Y} = g\left(\phi\mathcal{X}, \mathcal{Y}\right)\xi - \eta\left(\mathcal{Y}\right)\phi\mathcal{X}, \tag{1.5}$$

for all vector fields $\mathcal{X}$ and $\mathcal{Y}$.

In this paper, we are going to study the properties of 3-dimensional hyperbolic Kenmotsu manifolds with Yamabe and gradient Yamabe metrics. Also, we give a non-trivial example of 3-dimensional hyperbolic Kenmotsu manifold and validate our some results.

2 Hyperbolic Kenmotsu manifolds

In a $(2n+1)$-dimensional almost hyperbolic contact manifold M_{2n+1} [28] with a fundamental tensor field ϕ of type $(1,1)$, a timelike vector field ξ, a 1-form η, we have

$$\phi^2 = I + \eta\otimes\xi, \quad \eta(\xi) = -1 \implies \phi(\xi) = 0, \quad \eta\circ\phi = 0, \quad rank(\phi) = 2n, \tag{2.1}$$

where I is the identity endomorphism of the tangent bundle of M_{2n+1} and $\otimes$ represents the tensor product. An almost hyperbolic contact manifold M_{2n+1} is said to be an almost hyperbolic contact metric manifold if the semi-Riemannian metric g of M_{2n+1} satisfies

$$\begin{aligned} g(\phi\mathcal{X}, \phi\mathcal{Y}) &= -g(\mathcal{X}, \mathcal{Y}) - \eta(\mathcal{X})\eta(\mathcal{Y}), \\ g(\phi\mathcal{X}, \mathcal{Y}) &= -g(\mathcal{X}, \phi\mathcal{Y}), \quad g(\mathcal{X}, \xi) = \eta(\mathcal{X}) \end{aligned} \tag{2.2}$$

for all vector fields $\mathcal{X}$ and $\mathcal{Y}$. The structure (ϕ, ξ, η, g) on M_{2n+1} is called almost hyperbolic contact metric structure. An almost hyperbolic contact metric manifold M_{2n+1} is called a hyperbolic Kenmotsu manifold [1] if equation (1.5) is satisfied. From equation (1.5), it follows that

$$\nabla_{\mathcal{X}}\xi = -\mathcal{X} - \eta(\mathcal{X})\xi, \tag{2.3}$$

which gives

$$(\nabla_{\mathcal{X}}\eta)\mathcal{Y} = g(\phi\mathcal{X}, \phi\mathcal{Y}) = -g(\mathcal{X}, \mathcal{Y}) - \eta(\mathcal{X})\eta(\mathcal{Y}). \tag{2.4}$$

In line to achieve the purpose of the article, we require the following lemma:

Lemma 2.1. *On a $(2n+1)$-dimensional hyperbolic Kenmotsu manifold M_{2n+1}, we have*

1. $R(\mathcal{X}, \mathcal{Y})\xi = \eta(\mathcal{Y})\mathcal{X} - \eta(\mathcal{X})\mathcal{Y}$,
2. $R(\mathcal{X}, \xi)\xi = -\mathcal{X} - \eta(\mathcal{X})\xi$,
3. $R(\xi, \mathcal{X})\mathcal{Y} = g(\mathcal{X}, \mathcal{Y})\xi - \eta(\mathcal{Y})\mathcal{X}$,
4. $S(\mathcal{X}, \xi) = 2n\eta(\mathcal{X})$, $\quad S(\xi, \xi) = -2n$, *and* $\quad Q\xi = 2n\xi$,

where R, S and Q are the curvature tensor, the Ricci tensor and the Ricci operator of M_{2n+1}, respectively.

Proof. It is well-known that

$$R(\mathcal{X},\mathcal{Y})\xi = \nabla_{\mathcal{X}}\nabla_{\mathcal{Y}}\xi - \nabla_{\mathcal{Y}}\nabla_{\mathcal{X}}\xi - \nabla_{[\mathcal{X},\mathcal{Y}]}\xi. \tag{2.5}$$

Taking account of equations (1.5), (2.3) and (2.5), we get the first result. Second result is straight forward and third can be achieved by using $g(R(\mathcal{X},\mathcal{Y})\xi,\mathcal{Z}) = g(R(\xi,\mathcal{Z})\mathcal{X},\mathcal{Y})$. In view of equation (1.4), first result of Lemma 2.1 and $S(\mathcal{X},\mathcal{Y}) = g(Q\mathcal{X},\mathcal{Y})$, we get the fourth one. □

The Weyl conformal curvature tensor C of type $(1,3)$ on a $(2n+1)$-dimensional manifold M_{2n+1} is defined by

$$\begin{aligned} C(\mathcal{X},\mathcal{Y})\mathcal{Z} =& R(\mathcal{X},\mathcal{Y})\mathcal{Z} - \frac{1}{2n-1}[S(\mathcal{Y},\mathcal{Z})\mathcal{X} - S(\mathcal{X},\mathcal{Z})\mathcal{Y} \\ &+ g(\mathcal{Y},\mathcal{Z})Q\mathcal{X} - g(\mathcal{X},\mathcal{Z})Q\mathcal{Y}] \\ &+ \frac{\tau}{2n(2n-1)}[g(\mathcal{Y},\mathcal{Z})\mathcal{X} - g(\mathcal{X},\mathcal{Z})\mathcal{Y}] \end{aligned} \tag{2.6}$$

for all vector fields $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ on M_{2n+1} [15].

We know that the Weyl conformal curvature tensor vanishes in a 3-dimensional (semi-)Riemannian manifold, therefore from equation (2.6) we have

$$\begin{aligned} R(\mathcal{X},\mathcal{Y})\mathcal{Z} =& g(\mathcal{Y},\mathcal{Z})Q\mathcal{X} - g(\mathcal{X},\mathcal{Z})Q\mathcal{Y} + S(\mathcal{Y},\mathcal{Z})\mathcal{X} \\ &- S(\mathcal{X},\mathcal{Z})\mathcal{Y} - \frac{\tau}{2}(g(\mathcal{Y},\mathcal{Z})\mathcal{X} - g(\mathcal{X},\mathcal{Z})\mathcal{Y}). \end{aligned} \tag{2.7}$$

On taking $\mathcal{X} = \mathcal{Z} = \xi$ in the above equation and using equation (2.3) and Lemma 2.1, we obtain

$$Q\mathcal{X} = \left(\frac{\tau}{2} - 1\right)\mathcal{X} + \left(\frac{\tau}{2} - 3\right)\eta(\mathcal{X})\xi. \tag{2.8}$$

The above equation can be re-written as:

$$S(\mathcal{X},\mathcal{Y}) = \left(\frac{\tau}{2} - 1\right)g(\mathcal{X},\mathcal{Y}) + \left(\frac{\tau}{2} - 3\right)\eta(\mathcal{X})\eta(\mathcal{Y}). \tag{2.9}$$

Hence, we can say that the 3-dimensional hyperbolic Kenmotsu manifolds are η-Einstein. In the theory of relativity and cosmology, equation (2.8) represents the perfect fluid spacetime. Taking covariant differentiation of equation (2.8) along the vector field $\mathcal{Y}$ and using equation (2.3) one obtains:

$$\begin{aligned} (\nabla_{\mathcal{Y}}Q)\mathcal{X} =& \frac{1}{2}\mathcal{Y}(\tau)\mathcal{X} + \frac{1}{2}\mathcal{Y}(\tau)\eta(\mathcal{X})\xi \\ &- \left(\frac{\tau}{2} - 3\right)[g(\mathcal{X},\mathcal{Y})\xi + 2\eta(\mathcal{X})\eta(\mathcal{Y})\xi + \eta(\mathcal{X})\mathcal{Y}]. \end{aligned} \tag{2.10}$$

Contracting the above equation for $\mathcal{Y}$ and making use of the well-known formula $(div\,Q)(\mathcal{X}) = \frac{1}{2}\nabla_{\mathcal{X}}\tau$, we get

$$\xi(\tau)\eta(\mathcal{X}) = 2(\tau - 6)\eta(\mathcal{X}),$$

which leads to the following proposition:

Proposition 2.2. *On a 3-dimensional hyperbolic Kenmotsu manifold M_3, we have*

$$\xi(\tau) = 2(\tau - 6). \tag{2.11}$$

3 Yamabe solitons on 3-dimensional hyperbolic Kenmotsu manifolds

In this section, we study the properties of 3-dimensional hyperbolic Kenmotsu manifolds with Yamabe metric. Let us recall the definition of conformal vector field:

Definition 3.1. A vector field $\mathcal{V}$ on a $(2n+1)$-dimensional semi-Riemannian manifold (M, g) is said to be conformal if

$$\mathcal{L}_{\mathcal{V}} g = 2\rho g$$

holds for smooth function ρ on M. The smooth function ρ is also known as conformal coefficient. In particular, the conformal vector field with a vanishing conformal coefficient reduces to the Killing vector field, and with constant conformal coefficient it becomes homothetic vector field (see, [11, 12, 13]).

A conformal vector field $\mathcal{V}$ on M satisfies

$$(\mathcal{L}_{\mathcal{V}} S)(\mathcal{X}, \mathcal{Y}) = -(2n-1) g(\nabla_{\mathcal{X}} D\rho, \mathcal{Y}) + \Delta\rho g(\mathcal{X}, \mathcal{Y}), \tag{3.1}$$

$$\mathcal{L}_{\mathcal{V}} \tau = -2\rho\tau + 4n\Delta\rho, \tag{3.2}$$

where $\Delta = -divD$ is the Laplacian operator of g and div stands for divergence [32].

In view of Definition 3.1 and equation (1.1) (take $\rho = \frac{\lambda - \tau}{2}$), we can say that the potential vector field $\mathcal{V}$ is conformal [24]. Taking the Lie-derivative of $\eta(\xi) = g(\xi, \xi) = -1$ along the vector field $\mathcal{V}$ and using equations (1.1) and (2.1), we obtain

$$(\mathcal{L}_{\mathcal{V}} \eta)\xi = -\eta(\mathcal{L}_{\mathcal{V}} \xi) = -\frac{1}{2}(\lambda - \tau). \tag{3.3}$$

Since the potential vector field $\mathcal{V}$ is conformal with $\rho = \frac{\lambda - \tau}{2}$, therefore equations (3.1) and (3.2) for a 3-dimensional semi-Riemannian manifold can be re-written as:

$$(\mathcal{L}_{\mathcal{V}} S)(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} g(\nabla_{\mathcal{X}} D\tau, \mathcal{Y}) - \frac{1}{2}\Delta\tau g(\mathcal{X}, \mathcal{Y}), \tag{3.4}$$

$$\mathcal{L}_{\mathcal{V}} \tau = -\tau(\lambda - \tau) - 2\Delta\tau. \tag{3.5}$$

Let us take the Lie-derivative of equation (2.9) in the direction of potential vector field $\mathcal{V}$ and using equations (1.1), (3.4) and (3.5), we get

$$\begin{aligned} g(\nabla_{\mathcal{X}} D\tau, \mathcal{Y}) = &- (\Delta\tau + 2(\lambda - \tau)) g(\mathcal{X}, \mathcal{Y}) - (2\Delta\tau + \tau(\lambda - \tau))\eta(\mathcal{X})\eta(\mathcal{Y}) \\ &+ (\tau - 6)\left[\eta(\mathcal{Y})(\mathcal{L}_{\mathcal{V}}\eta)\mathcal{X} + \eta(\mathcal{X})(\mathcal{L}_{\mathcal{V}}\eta)\mathcal{Y}\right]. \end{aligned} \tag{3.6}$$

Substituting $\mathcal{Y} = \xi$ in the above equation and using equations (2.1), (2.2), (2.11) and (3.3), we infer

$$(\tau - 6)(\mathcal{L}_{\mathcal{V}}\eta)\mathcal{X} = \left[\Delta\tau - 2(\tau - 6) + (\lambda - \tau)\left(\frac{\tau}{2} + 1\right)\right]\eta(\mathcal{X}) - 3\mathcal{X}(\tau). \tag{3.7}$$

Again replacing $\mathcal{X}$ with ξ in equation (3.7) and using equations (2.1), (2.2), (2.11) and (3.3), we obtain

$$\Delta\tau = -4(\lambda - 6). \tag{3.8}$$

In view of equations (3.7) and (3.8), equation (3.6) can be re-written as:

$$\begin{aligned} g(\nabla_{\mathcal{X}} D\tau, \mathcal{Y}) =& 2(\lambda + \tau - 12)g(\mathcal{X}, \mathcal{Y}) + 2(\lambda - 3\tau + 12)\eta(\mathcal{X})\eta(\mathcal{Y}) \\ &- 3\mathcal{X}(\tau)\eta(\mathcal{Y}) - 3\mathcal{Y}(\tau)\eta(\mathcal{X}), \end{aligned} \tag{3.9}$$

which implies

$$\nabla_{\mathcal{X}} D\tau = 2(\lambda + \tau - 12)\mathcal{X} + 2(\lambda - 3\tau + 12)\eta(\mathcal{X})\xi - 3\mathcal{X}(\tau)\xi - 3\eta(\mathcal{X})D\tau. \tag{3.10}$$

Substituting $\mathcal{X}$ with ξ in the above equation and making use of equations (2.1) and (2.11), we have

$$\nabla_{\xi} D\tau = 2(\tau - 6)\xi + +3D\tau.$$

The well-known formula $R(\mathcal{X}, \mathcal{Y})\mathcal{Z} = \nabla_{\mathcal{X}}\nabla_{\mathcal{Y}}\mathcal{Z} - \nabla_{\mathcal{Y}}\nabla_{\mathcal{X}}\mathcal{Z} - \nabla_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z}$ and equations (2.3), (2.4), (3.10) give

$$R(\mathcal{Y}, \mathcal{X})D\tau = \mathcal{X}(\tau)\mathcal{Y} - \mathcal{Y}(\tau)\mathcal{X} + 8(\lambda - 6)\eta(\mathcal{Y})\mathcal{X} - 8(\lambda - 6)\eta(\mathcal{X})\mathcal{Y}.$$

Next, we shall consider a local orthonormal frame $\{e_i : i = 1, 2, 3\}$ on a three-dimensional hyperbolic Kenmotsu manifold M_3. The contraction of the above equation over $\mathcal{Y}$ gives

$$S(\mathcal{X}, D\tau) = 2\mathcal{X}(\tau) - 16(\lambda - 6)\eta(\mathcal{X}).$$

Replacing $\mathcal{X}$ with ξ in the above equation and making use of equations (2.1) and (2.11), we get

$$S(\xi, D\tau) = 4(\tau - 6) + 16(\lambda - 6). \tag{3.11}$$

On the other hand, it follows from equations (2.1), (2.2), (2.9) and (2.11) that

$$S(\xi, D\tau) = 4(\tau - 6). \tag{3.12}$$

Consequently from equations (3.11) and (3.12), we get

$$\lambda = 6. \tag{3.13}$$

Hence, we can state the following theorem:

Theorem 3.1. *Every Yamabe soliton on a three-dimensional hyperbolic Kenmotsu manifold is shrinking with $\lambda = 6$.*

In view of equations (3.8) and (3.13), we can also conclude the following corollary.

Corollary 3.2. *Let a 3-dimensional hyperbolic Kenmotsu manifold M admit a Yamabe soliton, then the scalar curvature of M is harmonic.*

Now, we prove the following theorem:

Theorem 3.3. *Let a 3-dimensional hyperbolic Kenmotsu manifold M admit a Yamabe soliton, then we can state the following:*

1. *The scalar curvature of M is constant,* i.e., $\tau = 6$.
2. *The potential vector field $\mathcal{V}$ of Yamabe soliton is Killing.*
3. *M_3 is Einstein.*
4. *M_3 possesses the constant sectional curvature $K = 1$.*

Proof. The covariant derivative of equation (1.1) along an arbitrary vector field $\mathcal{X}$ gives

$$\nabla_{\mathcal{X}} \mathcal{L}_{\mathcal{V}} g = -\mathcal{X}(\tau) g. \tag{3.14}$$

We have the following well-known formula [32]

$$\begin{aligned}\left(\mathcal{L}_{\mathcal{V}} \nabla_{\mathcal{X}} g - \nabla_{\mathcal{X}} \mathcal{L}_{\mathcal{V}} g - \nabla_{[\mathcal{V},\mathcal{X}]} g\right)(\mathcal{Y}, \mathcal{Z}) &= \\ -g((\mathcal{L}_{\mathcal{V}} \nabla)(\mathcal{X}, \mathcal{Y}), \mathcal{Z}) - g((\mathcal{L}_{\mathcal{V}} \nabla)&(\mathcal{X}, \mathcal{Z}), \mathcal{Y}).\end{aligned}$$

Since the Levi-Civita connection ∇ is metric, the above formula can be re-written as:

$$\left(\nabla_{\mathcal{X}} \mathcal{L}_{\mathcal{V}} g\right)(\mathcal{Y}, \mathcal{Z}) = g((\mathcal{L}_{\mathcal{V}} \nabla)(\mathcal{X}, \mathcal{Y}), \mathcal{Z}) + g((\mathcal{L}_{\mathcal{V}} \nabla)(\mathcal{X}, \mathcal{Z}), \mathcal{Y}). \tag{3.15}$$

As $\nabla_{\mathcal{X}} \mathcal{L}_{\mathcal{V}}$ is a symmetric operator, therefore in view of (3.14) and (3.15) we obtain

$$(\mathcal{L}_{\mathcal{V}} \nabla)(\mathcal{X}, \mathcal{Y}) = \frac{1}{2}\left[g(\mathcal{X}, \mathcal{Y}) D\tau - \mathcal{X}(\tau)\mathcal{Y} - \mathcal{Y}(\tau)\mathcal{X}\right]. \tag{3.16}$$

The covariant derivative of equation (3.16) along an arbitrary vector field $\mathcal{Z}$ gives

$$(\nabla_{\mathcal{Z}} \mathcal{L}_{\mathcal{V}} \nabla)(\mathcal{X}, \mathcal{Y}) = \frac{1}{2}\left[g(\mathcal{X}, \mathcal{Y}) \nabla_{\mathcal{Z}} D\tau - g(\mathcal{X}, \nabla_{\mathcal{Z}} D\tau)\mathcal{Y} - g(\mathcal{Y}, \nabla_{\mathcal{Z}} D\tau)\mathcal{X}\right]. \tag{3.17}$$

In view of $(\mathcal{L}_{\mathcal{V}} R)(\mathcal{X}, \mathcal{Y})\mathcal{Z} = (\nabla_{\mathcal{X}} \mathcal{L}_{\mathcal{V}} \nabla)(\mathcal{Y}, \mathcal{Z}) - (\nabla_{\mathcal{Y}} \mathcal{L}_{\mathcal{V}} \nabla)(\mathcal{X}, \mathcal{Z})$ and equation (3.17), we can write

$$\begin{aligned}(\mathcal{L}_{\mathcal{V}} R)(\mathcal{X}, \mathcal{Y})\mathcal{Z} = {}& \frac{1}{2} g(\mathcal{Y}, \mathcal{Z}) \nabla_{\mathcal{X}} D\tau - \frac{1}{2} g(\mathcal{X}, \mathcal{Z}) \nabla_{\mathcal{Y}} D\tau \\ & + \frac{1}{2} g(\mathcal{Z}, \nabla_{\mathcal{Y}} D\tau)\mathcal{X} - \frac{1}{2} g(\mathcal{Z}, \nabla_{\mathcal{X}} D\tau)\mathcal{Y} \\ & + \frac{1}{2} g(\mathcal{X}, \nabla_{\mathcal{Y}} D\tau)\mathcal{Z} - \frac{1}{2} g(\mathcal{Y}, \nabla_{\mathcal{X}} D\tau)\mathcal{Z}.\end{aligned} \tag{3.18}$$

Replacing $\mathcal{Z}$ by timelike vector field ξ in the above equation and making use of equations (2.1), (2.2), (2.11), (3.9), (3.10) and (3.13), we obtain

$$\begin{aligned}(\mathcal{L}_{\mathcal{V}} R)(\mathcal{X}, \mathcal{Y})\xi = {}& \left\{2(\tau - 6)\eta(\mathcal{Y}) + \frac{3}{2}\mathcal{Y}(\tau)\right\}\mathcal{X} + \frac{3}{2}\mathcal{Y}(\tau)\eta(\mathcal{X})\xi \\ & - \left\{2(\tau - 6)\eta(\mathcal{X}) + \frac{3}{2}\mathcal{X}(\tau)\right\}\mathcal{Y} - \frac{3}{2}\mathcal{X}(\tau)\eta(\mathcal{Y})\xi\end{aligned} \tag{3.19}$$

for any vector fields $\mathcal{X}$ and $\mathcal{Y}$. In view of equations (3.3), (3.13) and Lemma 2.1 (1), we have

$$\begin{aligned}(\mathcal{L}_{\mathcal{V}}R)(\mathcal{X},\mathcal{Y})\xi &= \{g(\mathcal{L}_{\mathcal{V}}\xi,\mathcal{Y})-(\tau-6)\eta(\mathcal{Y})\}\,\mathcal{X}\\ &\quad -\{g(\mathcal{L}_{\mathcal{V}}\xi,\mathcal{X})-(\tau-6)\eta(\mathcal{X})\}\,\mathcal{Y}-R(\mathcal{X},\mathcal{Y})\mathcal{L}_{\mathcal{V}}\xi\end{aligned}\tag{3.20}$$

for any vector fields $\mathcal{X}$ and $\mathcal{Y}$. Equations (3.19) and (3.20) yield

$$\begin{aligned}R(\mathcal{X},\mathcal{Y})\mathcal{L}_{\mathcal{V}}\xi &= \left\{g(\mathcal{L}_{\mathcal{V}}\xi,\mathcal{Y})-3(\tau-6)\eta(\mathcal{Y})-\frac{3}{2}\mathcal{Y}(\tau)\right\}\mathcal{X}+\frac{3}{2}\mathcal{Y}(\tau)\eta(\mathcal{X})\xi\\ &\quad -\left\{g(\mathcal{L}_{\mathcal{V}}\xi,\mathcal{X})-3(\tau-6)\eta(\mathcal{X})-\frac{3}{2}\mathcal{X}(\tau)\right\}\mathcal{Y}-\frac{3}{2}\mathcal{X}(\tau)\eta(\mathcal{Y})\xi.\end{aligned}\tag{3.21}$$

Let $\{e_i : i = 1, 2, 3\}$ be a local orthonormal frame on a 3-dimensional hyperbolic Kenmotsu manifold. Contracting equation (3.21) over $\mathcal{X}$ and making use of equations (2.1), (2.11) and (3.13), we infer

$$S(\mathcal{Y},\mathcal{L}_{\mathcal{V}}\xi)=2g(\mathcal{L}_{\mathcal{V}}\xi,\mathcal{Y})-3(\tau-6)\eta(\mathcal{Y})-\frac{3}{2}\mathcal{Y}(\tau)\tag{3.22}$$

for any vector field $\mathcal{Y}$. Again, using equations (2.9), (3.3) and (3.13), we obtain

$$S(\mathcal{Y},\mathcal{L}_{\mathcal{V}}\xi)=\left(\frac{\tau}{2}-1\right)g(\mathcal{L}_{\mathcal{V}}\xi,\mathcal{Y})-\left(\frac{\tau-6}{2}\right)^2\eta(\mathcal{Y})\tag{3.23}$$

for any vector field $\mathcal{Y}$. The following equation can be obtained by subtracting equation (3.22) from (3.23),

$$(\tau-6)\mathcal{L}_{\mathcal{V}}\xi=\frac{1}{2}(\tau-6)(\tau-18)\xi-3D\tau.\tag{3.24}$$

If we assume $\tau \neq 6$, then equation (3.24) can be re-written as:

$$\mathcal{L}_{\mathcal{V}}\xi=\frac{1}{2}(\tau-18)\xi-\frac{3}{(\tau-6)}D\tau.\tag{3.25}$$

Now we write

$$(\mathcal{L}_{\mathcal{V}}\nabla)(\mathcal{X},\xi)=\mathcal{L}_{\mathcal{V}}\nabla_{\mathcal{X}}\xi-\nabla_{\mathcal{X}}\mathcal{L}_{\mathcal{V}}\xi-\nabla_{[\mathcal{V},\mathcal{X}]}\xi.\tag{3.26}$$

Taking reference of equations (1.1), (2.3), (3.10), (3.13) and (3.26), equation (3.25) can be re-written as:

$$\begin{aligned}(\mathcal{L}_{\mathcal{V}}\nabla)(\mathcal{X},\xi) &= -\frac{6}{(\tau-6)}\eta(\mathcal{X})D\tau-\frac{3}{(\tau-6)^2}\mathcal{X}(\tau)D\tau-\frac{(\tau+6)}{2(\tau-6)}\mathcal{X}(\tau)\xi\\ &\quad +\left(\frac{\tau}{2}-15\right)\eta(\mathcal{X})\xi+\left(\frac{\tau}{2}-3\right)\mathcal{X}\end{aligned}\tag{3.27}$$

for arbitrary vector field $\mathcal{X}$. The above equation together with equation (3.16) assumes the form

$$\begin{aligned}&\frac{(\tau+6)}{2(\tau-6)}\eta(\mathcal{X})D\tau+\frac{3}{(\tau-6)^2}\mathcal{X}(\tau)D\tau-3\left(\frac{\tau}{2}-3\right)\mathcal{X}\\ &\quad +\frac{6}{(\tau-6)}\mathcal{X}(\tau)\xi-\left(\frac{\tau}{2}-15\right)\eta(\mathcal{X})\xi=0\end{aligned}\tag{3.28}$$

for arbitrary vector field $\mathcal{X}$. Replace $\mathcal{X}$ with ξ in the above equation and taking reference of equations (2.1) and (2.11) yield

$$D\tau = -2(\tau - 6)\xi.$$

Substituting the value of $D\tau$ and replacing $\mathcal{X}$ by $\phi\mathcal{X}$ in equation (3.28), we obtain

$$(\tau - 6)\phi\mathcal{X} = 0 \tag{3.29}$$

for arbitrary vector field $\mathcal{X}$. It is obvious from the above equation that $\tau = 6$, this contradict our earlier assumption, *i.e.*, $\tau \neq 6$. Hence, we can say that $\tau = 6$. This complete the part-(1) of the theorem. On taking $\lambda = \tau = 6$ in equation (1.1), we obtain $\mathcal{L}_{\mathcal{V}} g = 0$. Hence, we can say that the potential vector field $\mathcal{V}$ of the Yamabe soliton is Killing, *i.e.*, part-(2) of the theorem. On substituting $\tau = 6$ in equation (2.9) leads to $S(\mathcal{X}, \mathcal{Y}) = 2g(\mathcal{X}, \mathcal{Y})$, which complete part-(3) of the theorem. We also know that, if the scalar curvature of an n-dimensional manifold is constant then it is equal to $n(n-1)$ times the sectional curvature. Hence in view of equation (3.29), we can say that the manifold is of constant sectional curvature $K = 1$. □

Corollary 3.4. *Let a 3-dimensional hyperbolic Kenmotsu manifold admit a Yamabe soliton. Then the Yamabe soliton is shrinking with $\lambda = 6$ if and only if the Ricci soliton is shrinking with $\mu = 2$.*

Proof. As discussed in Theorem 3.3 that the Ricci tensor $S(\mathcal{X}, \mathcal{Y}) = 2g(\mathcal{X}, \mathcal{Y})$ and the potential vector field $\mathcal{V}$ is Killing, *i.e.*, $\mathcal{L}_{\mathcal{V}} g = 0$. Applying these results in the equation (1.3), we obtain

$$2(\mu - 2)g(\mathcal{X}, \mathcal{Y}) = 0$$

for arbitrary vector field $\mathcal{X}$, $\mathcal{Y}$, which implies that $\mu = 2$. Thus, the Ricci soliton is shrinking. The converse part is obvious. □

4 Existence of gradient Yamabe solitons on 3-dimensional hyperbolic Kenmotsu manifolds

This section deals with the existence of gradient Yamabe solitons on a 3-dimensional hyperbolic Kenmotsu manifold M_3.

Theorem 4.1. *There does not exist a non-trivial gradient Yamabe soliton on a 3-dimensional hyperbolic Kenmotsu manifold M_3. In other words, the Yamabe soliton on M_3 is trivial.*

Proof. Taking covariant differentiation of equation (1.2) along an arbitrary vector field $\mathcal{Y}$, we obtain

$$\nabla_{\mathcal{Y}} \nabla_{\mathcal{X}} Df = \frac{(\lambda - \tau)}{2} \nabla_{\mathcal{Y}} \mathcal{X} - \frac{1}{2} \mathcal{Y}(\tau) \mathcal{X}. \tag{4.1}$$

Interchange $\mathcal{X}$ and $\mathcal{Y}$ in (4.1) results

$$\nabla_{\mathcal{X}} \nabla_{\mathcal{Y}} Df = \frac{(\lambda - \tau)}{2} \nabla_{\mathcal{X}} \mathcal{Y} - \frac{1}{2} \mathcal{X}(\tau) \mathcal{Y}. \tag{4.2}$$

By virtue of the well-known formula $R(\mathcal{X},\mathcal{Y})\mathcal{Z} = \nabla_{\mathcal{X}}\nabla_{\mathcal{Y}}\mathcal{Z} - \nabla_{\mathcal{Y}}\nabla_{\mathcal{X}}\mathcal{Z} - \nabla_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z}$ and equations (1.2), (4.1), (4.2), we have

$$R(\mathcal{X},\mathcal{Y})Df = \frac{1}{2}\left[\mathcal{Y}(\tau)\mathcal{X} - \mathcal{X}(\tau)\mathcal{Y}\right]. \tag{4.3}$$

Take a local orthonormal frame $\{e_i : i = 1, 2, 3\}$ on a 3-dimensional hyperbolic Kenmotsu manifold. Contracting equation (4.3) over $\mathcal{X}$, we obtain

$$S(\mathcal{Y}, Df) = \mathcal{Y}(\tau), \tag{4.4}$$

where S is the Ricci tensor of M_3. Substituting $\mathcal{X} = \xi$ in equation (4.3) and recalling Lemma 2.1 provides

$$\mathcal{Y}(\tau)\xi - 2(\tau - 6)\mathcal{Y} = 2\mathcal{Y}(f)\xi - 2\xi(f)\mathcal{Y}.$$

Replacing $\mathcal{X}$ with Df in equation (2.9) and using equation (4.4), we obtain

$$\mathcal{Y}(\tau) = \left(\frac{\tau}{2} - 1\right)\mathcal{Y}(f) + \left(\frac{\tau}{2} - 3\right)\eta(\mathcal{Y})\xi(f). \tag{4.5}$$

Again setting $\mathcal{Y} = \xi$ in equation (4.5) and taking reference of equations (2.1) and (2.11), we have

$$\xi(f) = (\tau - 6). \tag{4.6}$$

In view of equation (4.6), equation (4.4) can be written as:

$$\mathcal{Y}(\tau) = 2\mathcal{Y}(f). \tag{4.7}$$

Recalling equations (4.5), (4.6) and (4.7), one can easily obtain

$$(\tau - 6)\{\mathcal{Y}(f) + \xi(f)\eta(\mathcal{Y})\} = 0, \tag{4.8}$$

where $\mathcal{Y}$ is an arbitrary vector field. The foregoing equation (4.8) results that either $\tau = 6$ or $\mathcal{Y}(f) = -\xi(f)\eta(\mathcal{Y})$. If possible, we suppose that $\tau \neq 6$. Then we have $\mathcal{Y}(f) = -\xi(f)\eta(\mathcal{Y}) \iff Df = -\xi(f)\xi$. This reflects that the gradient of potential function is pointwise collinear with the timelike vector field ξ. The equation (4.8) results

$$Df = -(\tau - 6)\xi. \tag{4.9}$$

Taking covariant differentiation of equation (4.9) along an arbitrary vector field $\mathcal{X}$ and making use of equations (1.2) and (2.3) yields

$$(\lambda - 3\tau + 12)\mathcal{X} = 2(\tau - 6)\eta(\mathcal{X})\xi - 2\mathcal{X}(\tau)\xi.$$

Substituting $\mathcal{X} = \xi$ in the foregoing equation and making use of equations (2.1) and (2.11) leads to

$$(\lambda + 3\tau - 24)\xi = 0 \implies \lambda = -3(\tau - 8). \tag{4.10}$$

This shows that the three-dimensional hyperbolic Kenmotsu manifold admitting a gradient Yamabe soliton possesses the constant scalar curvature. The equations (4.7) and (4.10) infer that $f =$ constant. This shows that the gradient Yamabe soliton on a 3-dimensional hyperbolic Kenmotsu manifold is trivial. This fact together with equation (4.9) lead to $r = 6$, which contradicts our hypothesis. Thus we have $\tau = 6$, which leads to the required result. $\square$

In view of equation (4.10), we can state the following:

Corollary 4.2. *If a 3-dimensional hyperbolic Kenmotsu manifold admits a non-trivial gradient Yamabe soliton, then the gradient Yamabe soliton is* shrinking, steady, *or* expanding *if* $\tau < 8$, $\tau = 8$, *or* $\tau > 8$, *respectively.*

Corollary 4.3. *A three-dimensional hyperbolic Kenmotsu manifold with a gradient Yamabe soliton possesses the constant scalar curvature.*

5 Example of 3-dimensional hyperbolic Kenmotsu manifold

Example 5.1. Consider a 3-dimensional manifold $M_3 = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ with the standard coordinate system (x, y, z) of $\mathbb{R}^3$. Let $e_1 = e^z \frac{\partial}{\partial x}$, $e_2 = e^z \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z} = \xi$ be linearly independent vector fields of M_3, and therefore they form a basis of the tangent space at each point of M_3. Let g be a semi-Riemannian metric of M_3 defined by

$$g(e_i, e_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and ϕ is a $(1, 1)$-tensor field defined by

$$\phi(e_1) = e_2, \ \ \phi(e_2) = e_1, \ \ \phi(e_3) = 0.$$

Using linearity of ϕ and g, we have

$$\eta(e_3) = -1, \quad \phi^2\mathcal{X} = \mathcal{X} + \eta(\mathcal{X})e_3, \quad g(\phi\mathcal{X}, \phi\mathcal{Y}) = -g(\mathcal{X}, \mathcal{Y}) - \eta(\mathcal{X})\eta(\mathcal{Y})$$

for any $\mathcal{X}, \mathcal{Y} \in TM$. Here η is a 1-form on M_3 defined by $\eta(\mathcal{X}) = g(\mathcal{X}, e_3)$ for any $\mathcal{X} \in TM$. Hence for $\xi = e_3$, the structure (ϕ, ξ, η, g) defines an almost hyperbolic contact metric structure on M_3. The Lie bracket can be calculated by using the definition $[\mathcal{X}, \mathcal{Y}]f = \mathcal{X}(\mathcal{Y}f) - \mathcal{Y}(\mathcal{X}f)$. All possible Lie brackets for the example are as follows:

$$\begin{array}{lll} [e_1, e_1] = 0, & [e_1, e_2] = 0, & [e_1, e_3] = -e_1, \\ [e_2, e_1] = 0, & [e_2, e_2] = 0, & [e_2, e_3] = -e_2, \\ [e_3, e_1] = e_1, & [e_3, e_2] = e_2, & [e_3, e_3] = 0\ . \end{array}$$

Let ∇ be a Levi-Civita connection with respect to the semi-Riemannian metric g. Using the Koszul's formula

$$\begin{aligned} 2g(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) \ &= \ \mathcal{X}g(\mathcal{Y}, \mathcal{Z}) + \mathcal{Y}g(\mathcal{Z}, \mathcal{X}) - \mathcal{Z}g(\mathcal{X}, \mathcal{Y}) \\ &\quad + g([\mathcal{X}, \mathcal{Y}], \mathcal{Z}) - g([\mathcal{Y}, \mathcal{Z}], \mathcal{X}) + g([\mathcal{Z}, \mathcal{X}], \mathcal{Y}) \end{aligned}$$

and the semi-Riemannian metric g, we can easily calculate

$$\nabla_{e_1}e_1 = -e_3, \qquad \nabla_{e_1}e_2 = 0, \qquad \nabla_{e_1}e_3 = -e_1,$$
$$\nabla_{e_2}e_1 = 0, \qquad \nabla_{e_2}e_2 = e_3, \qquad \nabla_{e_2}e_3 = -e_2,$$
$$\nabla_{e_3}e_1 = 0, \qquad \nabla_{e_3}e_2 = 0, \qquad \nabla_{e_3}e_3 = 0\ .$$

Now for $\mathcal{X} = \mathcal{X}^1 e_1 + \mathcal{X}^2 e_2 + \mathcal{X}^3 e_3$ and $\xi = e_3$, we have

$$\begin{aligned} \nabla_{\mathcal{X}}\xi &= \nabla_{(\mathcal{X}^1 e_1 + \mathcal{X}^2 e_2 + \mathcal{X}^3 e_3)}e_3 \\ &= \mathcal{X}^1\nabla_{e_1}e_3 + \mathcal{X}^2\nabla_{e_2}e_3 + \mathcal{X}^3\nabla_{e_3}e_3 \\ &= -\mathcal{X}^1 e_1 - \mathcal{X}^2 e_2 \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} -\mathcal{X} - \eta(\mathcal{X})\xi &= -(\mathcal{X}^1 e_1 + \mathcal{X}^2 e_2 + \mathcal{X}^3 e_3) - g(\mathcal{X}^1 e_1 + \mathcal{X}^2 e_2 + \mathcal{X}^3 e_3, e_3)e_3 \\ &= -\mathcal{X}^1 e_1 - \mathcal{X}^2 e_2 - \mathcal{X}^3 e_3 + \mathcal{X}^3 e_3 \\ &= -\mathcal{X}^1 e_1 - \mathcal{X}^2 e_2, \end{aligned} \tag{5.2}$$

where $\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3$ are scalars. In view of equations (5.1) and (5.2), we can say that the structure (ϕ, ξ, η, g) is a hyperbolic Kenmotsu structure on M_3. *Consequently, $M_3(\phi, \xi, \eta, g)$ is a 3-dimensional hyperbolic Kenmotsu manifold.*

The non-zero components of the curvature tensor are as under:

$$R(e_1,e_2)e_1 = -e_2 = -R(e_2,e_1)e_1, \qquad R(e_1,e_3)e_1 = -e_3 = -R(e_3,e_1)e_1,$$
$$R(e_1,e_2)e_2 = -e_1 = -R(e_2,e_1)e_2, \qquad R(e_2,e_3)e_2 = e_3 = -R(e_3,e_2)e_2,$$
$$R(e_1,e_3)e_3 = -e_1 = -R(e_1,e_3)e_3, \qquad R(e_2,e_3)e_3 = -e_2 = -R(e_3,e_2)e_3.$$

In line to find the Ricci tensor, we take the reference of equation (1.4). The Ricci tensor can be written as:

$$\begin{aligned} S(\mathcal{X},\mathcal{Y}) &= \sum_{i=1}^{3}\varepsilon_i g(R(e_i,\mathcal{X})\mathcal{Y}, e_i) \\ &= g(R(e_1,\mathcal{X})\mathcal{Y}, e_1) - g(R(e_2,\mathcal{X})\mathcal{Y}, e_2) - g(R(\xi,\mathcal{X})\mathcal{Y}, \xi). \end{aligned}$$

Hence we have the following:

$$S(e_1,e_1) = 2, \quad S(e_2,e_2) = -2, \quad S(e_3,e_3) = -2.$$

Again, in view of equation (1.4), the scalar curvature of the given hyperbolic Kenmotsu manifold can be calculated as under:

$$\tau = \sum_{i=1}^{3}\varepsilon_i S(e_i,e_i) = S(e_1,e_1) - S(e_2,e_2) - S(e_3,e_3) = 6.$$

Let $\mathcal{V} = a^1 e_1 + a^2 e_2$ be a soliton vector for the Yamabe soliton defined in (1.1). Then we can easily verify that $(\mathcal{L}_{\mathcal{V}} g)(e_i, e_i) = 0$, $\forall\ i = 1, 2, 3$. Here a^1 and a^2 are some scalars. Hence we can say that the hyperbolic Kenmotsu metric g on M_3 is a Yamabe soliton with $\lambda = 6$. Thus, the soliton vector field V under consideration is Killing and therefore from equation (1.1) we infer that $\tau = \lambda = 6$, which verifies Theorem 3.1 and Theorem 3.3.

Acknowledgements. The authors express their sincere thanks to the Editor and anonymous referees for providing their valuable suggestions.

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