On a special type of almost pseudo Ricci symmetric manifold

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Abstract. The object of the present paper is to study almost pseudo M-projective Ricci symmetric manifolds denoted by $A(PMRS)_n$. In the first section, almost pseudo M-projective Ricci symmetric manifolds are denoted. In the second section, some theorems including the properties of $A(PMRS)_n$ are proved. In the section 3, it is shown that the totally geodesic hypersurface of this manifold is also $A(PMRS)_n$. In addition, it is also proved that a necessary and sufficient condition the totally umbilical hypersurface of this manifold be also $A(PMRS)_n$ is that the mean curvature of this hypersurface be zero or constant.


Key words: pseudo Ricci symmetric manifold; M-projective Ricci tensor; Codazzi tensor; cyclic Ricci tensor; totally umbilical; totally geodesic.

1 Introduction

A Riemannian manifold is Ricci symmetric if its Ricci tensor $S$ of type $(0, 2)$ satisfies $\nabla S = 0$, where $\nabla$ denotes the Riemannian connection. The notion of Ricci symmetric manifolds have been weakened by many authors in the last five decades such as Ricci-recurrent manifolds [17], Ricci semi-symmetric manifolds [20], pseudo Ricci symmetric manifolds [1], [6].

A non-flat Riemannian manifold $(M_n, g)$ is said to be pseudo Ricci symmetric if the Ricci tensor $S$ of type $(0, 2)$ satisfies the condition

$$(\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X),$$

where $A$ is a non-zero 1-form,

$$g(X, P) = A(X),$$

for every vector field $A$ and $\nabla$ denotes the operator of covariant differentiation with respect to the metric $g$. This manifold is denoted by $(PRS)_n$. The name ”Pseudo Ricci symmetric” is chosen, because if the 1-form $A$ in the above equation is taken

as zero, then the equation assumes the form $(\nabla_X S)(Y, Z) = 0$ and the manifold thus becomes Ricci symmetric.

As an extended class of pseudo Ricci symmetric manifolds introduced by Chaki [1], Chaki and Kawaguchi [2] introduced the notion of almost pseudo Ricci symmetric manifolds.

A Riemannian manifold $(M_n, g)$ is called an almost pseudo Ricci symmetric manifold if its Ricci tensor $S$ is not identically zero and satisfies the condition

$$ (\nabla_X S)(Y, Z) = (A(X) + B(X))S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X), $$

where $\nabla$ denotes the operator of covariant differentiation with respect to the metric $g$ and $A, B$ are nowhere vanishing 1-forms such that

$$ g(X, \rho) = A(X), \quad g(X, \mu) = B(X) $$

for all $X$ and $\rho, \mu$ are called the basic vector fields of the manifold. Here, $A$ and $B$ are called the associated 1-forms and an $n$-dimensional manifold of this kind is denoted by $A_{PRS}^n$. If, in particular $B = A$, then (1.1) reduces to a pseudo Ricci symmetric manifold, [1].

The almost pseudo Ricci symmetric manifolds have been studied by some authors ([5], [7], [19], etc.).

Let $(M_n, g)$ be an $n$-dimensional differentiable manifold of class $C^\infty$ with the metric tensor $g$ and the Riemannian connection $\nabla$. The M-projective curvature tensor of the manifold defined by G.P. Pokhariyal and R.S. Mishra in 1971, [18], is the following form

$$ M(X, Y)Z = R(X, Y)Z - \frac{1}{2(n - 1)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY], $$

where $R(X, Y)Z$ and $S(X, Y)$ are the curvature tensor and the Ricci tensor of $M_n$, respectively. Some authors studied the properties of this tensor ([3], [8], [9], [10], [12], [13], [14], [15], [16]).

In this paper, extending the notion of almost pseudo Ricci symmetric manifold, the author introduces a type of non-flat Riemannian manifold $(M_n, g)$, $(n > 3)$ whose M-projective Ricci tensor $\tilde{M}$ of type $(0, 2)$ satisfies the condition

$$ (\nabla_X \tilde{M})(Y, Z) = (A(X) + B(X))\tilde{M}(Y, Z) + A(Y)\tilde{M}(X, Z) + A(Z)\tilde{M}(Y, X), $$

where $A, B$ and $\nabla$ have the meanings already mentioned. Such a manifold shall be called almost pseudo M-projective Ricci symmetric manifold. $A$ and $B$ shall be its associated 1-forms and an $n$-dimensional manifold of this kind shall be denoted by $A_{PMRS}^n$.

The object of the present paper is to study $A_{PMRS}^n$. The paper is organized as follows:

Section 2 is devoted to the study of some properties of $A_{PMRS}^n$. It is shown that a necessary and sufficient condition the M-projective curvature tensor of a Riemannian
manifold be conservative is that the Ricci tensor of this manifold be Codazzi type. After, it is proved that in an \( A(PMRS)_n \), if the M-projective curvature tensor of this manifold is conservative then \((r - n + 1)\) is an eigenvalue of the Ricci tensor \( S \) corresponding to the eigenvector \( \mu \). In addition, it is shown that in an \( A(PMRS)_n \) with cyclic parallel M-projective Ricci tensor, the associated 1-forms \( A \) and \( B \) of this manifold are related by \( B(X) = -3A(X) \). In this section, it is also proved that if the M-projective curvature tensor of this manifold is conservative then \( \mu \) and \( \xi \) are co-directional where \( \xi \) is defined by \( B(QX) = g(QX, \mu) = D(X) = g(X, \mu) \).

Section 3 deals with the hypersurface of \( A(PMRS)_n \). It is proved that the totally geodesic hypersurface of this manifold is also \( A(PMRS)_n \). Again, in this section, it is found that a necessary and sufficient condition the totally umbilical hypersurface of this manifold be also \( A(PMRS)_n \) is that the mean curvature be zero or constant.

2 Almost pseudo M-projective Ricci symmetric manifolds

In this section, let us consider the M-projective curvature tensor given by (1.3). Then, we obtain the M-projective Ricci tensor \( \tilde{M} \) of type \((0,2)\) as

\[
\tilde{M}(X,Y) = \frac{1}{2}(S(X,Y) - g(X,Y)).
\]

We assume that our manifold \((M_n, g)\) is \( A(PMRS)_n \). Thus, we have the following theorem:

**Theorem 2.1.** A necessary and sufficient condition the M-projective curvature tensor of a Riemannian manifold be conservative is that the Ricci tensor of this manifold be Codazzi type.

**Proof.** From (1.3), we obtain

\[
(divM)(X,Y)Z = (divR)(X,Y)Z - \frac{1}{2(n-1)}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)].
\]

It is known that in a Riemannian manifold, we have

\[
\]

Hence, by putting (2.3) in (2.2), we find

\[
(divM)(X,Y)Z = \frac{2n-3}{2n-2}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)].
\]

The equation (2.4) shows that if the M-projective curvature tensor of a Riemannian manifold is conservative then the Ricci tensor of this manifold is Codazzi type. The converse statement of this theorem is also true.

**Theorem 2.2.** In an \( A(PMRS)_n \), if the M-projective curvature tensor is conservative then \((r - n + 1)\) is an eigenvalue of the Ricci tensor \( S \) corresponding to the eigenvector \( \mu \).
Proof. We assume that our manifold is an $A(\text{PMRS})_n$. From (1.4) and (2.1), we get

$$\nabla_X S(Y, Z) = (A(X) + B(X))(S(Y, Z) - g(Y, Z)) + A(Y)(S(Z, X) - g(Z, X)) + A(Z)(S(Y, X) - g(Y, X))$$

Thus, by changing the indices in the equation (2.5), it can be easily seen that

$$\nabla_X S(Y, Z) - \nabla_Z S(Y, X) = B(X)(S(Y, Z) - g(Y, Z)) - B(Z)(S(Y, X) - g(Y, X)).$$

If the M-projective curvature tensor of an $A(\text{PMRS})_n$ is conservative then from Theorem 2.1, the Ricci tensor of this manifold is Codazzi type. In this case, it follows from (2.6) that

$$B(X)(S(Y, Z) - g(Y, Z)) = B(Z)(S(Y, X) - g(Y, X)).$$

Setting $Y = Z = e_i$ in (2.7) and taking summation over $i$, $1 \leq i \leq n$, we get that

$$B(QX) = (r - n + 1)B(X).$$

Thus, the proof is completed. □

Theorem 2.3. In an $A(\text{PMRS})_n$ with cyclic parallel M-projective Ricci tensor, the associated 1-forms of this manifold are related by the equation $B(X) = -3A(X)$.

Proof. Considering the M-projective Ricci tensor of an $A(\text{PMRS})_n$ be cyclic parallel, from (1.4), we obtain

$$3A(X) + B(X))\tilde{M}(Y, Z) + (3A(Y) + B(Y))\tilde{M}(Z, X) + (3A(Z) + B(Z))\tilde{M}(X, Y) = 0.$$

By taking $\lambda(X) = (3A(X) + B(X))$, then (2.8) reduces to

$$\lambda(X)\tilde{M}(Y, Z) + \lambda(Y)\tilde{M}(Z, X) + \lambda(Z)\tilde{M}(X, Y) = 0.$$

In Walker’s Lemma, [21], it is said that if $a(X, Y)$ and $b(X)$ are the numbers satisfying $a(X, Y) = a(Y, X)$ and

$$b(X)a(Y, Z) + b(Y)a(Z, X) + b(Z)a(X, Y) = 0.$$

for all $X, Y, Z$. Then either all the $a(X, Y)$ are zero or all the $b(X)$ are zero. Hence, by the above Lemma, from (2.9) and (2.10), it must be either $\lambda(X) = 3A(X) + B(X) = 0$ or $\tilde{M}(Y, Z) = 0$. Since $\tilde{M}(Y, Z) \neq 0$ then it must be $\lambda(X) = 0$, i.e.,

$$B(X) = -3A(X).$$

Thus, from (2.11), (1.4) reduces to

$$\nabla_X \tilde{M}(Y, Z) = -2A(X)\tilde{M}(Y, Z) + A(Y)\tilde{M}(X, Z) + A(Z)\tilde{M}(Y, X),$$

This completes the proof. □
Theorem 2.4. In an \( A(PMRS)_n \) with cyclic parallel M-projective Ricci tensor and constant scalar curvature, \((r - n + 1)\) is an eigenvalue of the Ricci tensor \( S \) corresponding to the eigenvector \( \rho \).

Proof. Since our manifold \( A(PMRS)_n \) is of cyclic parallel M-projective Ricci tensor, from (2.11) and (2.12), it can be found that

\[
(\nabla_X S)(Y, Z) = -2A(X)(S(Y, Z) - g(Y, Z)) + A(Y)(S(X, Z) - g(X, Z)) + A(Z)(S(Y, X) - g(Y, X)).
\]

Contracting on \( Y \) and \( Z \) in (2.13) on \( Y \) and \( Z \), we get

\[
dr(X) = -2(r - n + 1)A(X) + 2A(QX).
\]

If the scalar curvature of this manifold is constant then by the aid of (2.14), it is seen that

\[ A(QX) = (r - n + 1)A(X). \]

Thus, the proof is completed. \( \square \)

Theorem 2.5. If the M-projective curvature tensor of an \( A(PMRS)_n \) is conservative then the vector fields \( \mu \) and \( \xi \) are co-directional where \( B(QX) = g(QX, \mu) \) and \( D(X) = g(X, \mu) \) for all \( X \).

Proof. In an \( A(PMRS)_n \), if we assume that the M-projective curvature tensor is conservative then from Theorem 2.1, the Ricci tensor of this manifold is Codazzi type. Thus, we have the equation (2.7). Let \( B(QX) = g(QX, \mu) = D(X) = g(X, \mu) \) for all \( X \). Then, by putting \( Y = \mu \) in (2.6) and using (1.2), it can be obtained that

\[ B(X)D(Z) = B(Z)D(X). \]

In this case, we can say that \( \mu \) and \( \xi \) are co-directional. This completes the proof. \( \square \)

3 On the hypersurface of an \( A(PMRS)_n \)

Let \((\bar{V}, \bar{g})\) be an \((n + 1)\)-dimensional Riemannian manifold covered by a system of coordinate neighborhoods \( \{U, y^\alpha\} \). Let \((V, g)\) be a hypersurface of \((\bar{V}, \bar{g})\) defined in a locally coordinate system by means of a system of parametric equation \( y^\alpha = y^\alpha(x^i) \) where Greek indices take values 1, 2, ..., \( n \) and Latin indices take values 1, 2, ..., \((n+1)\). Let \( n^\alpha \) be the components of a local unit normal to \((V, g)\). Then we have

\[
g_{ij} = \bar{g}_{\alpha\beta} y^\alpha_i y^\beta_j.
\]

\[
\bar{g}_{\alpha\beta} n^\alpha y^\beta_j = 0, \quad \bar{g}_{\alpha\beta} n^\alpha n^\beta = e = 1.
\]

\[
g^i_j y^\beta_j g^{ij} = g^{\alpha\beta}, \quad g_{\alpha\beta} n^\alpha y^\beta_j = 0, \quad g^\alpha = \frac{\partial y^\alpha}{\partial x^i}.
\]
The hypersurface \((V, g)\) is called a totally umbilical ([4], [11]) of \((\bar{V}, \bar{g})\) if its second fundamental form \(\Omega_{ij}\) satisfies

\[
\Omega_{ij} = H g_{ij}, \quad \gamma_{ij}^\alpha = g_{ij} H n^\alpha
\]

where the scalar function \(H\) is called the mean curvature of \((V, g)\) given by the equation

\[
H = \frac{1}{n} \sum g^{ij} \Omega_{ij}.
\]

If, in particular, \(H = 0\), i.e.,

\[
\Omega_{ij} = 0
\]

then the totally umbilical hypersurface is called a totally geodesic hypersurface of \((\bar{V}, \bar{g})\).

The equation of Weingarten for \((V, g)\) can be written as \(n_{ij}^\alpha = -\frac{H}{n} y_{ij}^\alpha\). The structure equations of Gauss and Codazzi ([4], [11]) for \((V, g)\) and \((\bar{V}, \bar{g})\) are respectively given by

\[
R_{ijkl} = \bar{R}_{\alpha\beta\gamma\delta} B^{\alpha\beta\gamma\delta}_{ijkl} + H^2 G_{ijkl}
\]

\[
\bar{R}_{\alpha\beta\gamma\delta} B^{\alpha\beta\gamma\delta}_{ijkl} n_i^\delta = H_{i} g_{jk} - H_{j} g_{ik}
\]

where \(R_{ijkl}\) and \(\bar{R}_{\alpha\beta\gamma\delta}\) are the curvature tensors of \((V, g)\) and \((\bar{V}, \bar{g})\), respectively, and

\[
B^{\alpha\beta\gamma\delta}_{ijkl} = B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta, \quad B_i^\alpha = y_i^\alpha, \quad G_{ijkl} = g_{ij} g_{kl} - g_{ik} g_{jl}
\]

Also we have ([4], [11])

\[
\bar{S}_{\alpha\beta} B_i^\alpha B_j^\beta = S_{ij} - H^2 g_{ij}
\]

\[
\bar{S}_{\alpha\beta} n^\alpha B_i^\beta = (n - 1) H_i
\]

\[
\bar{r} = r - n(n - 1) H^2
\]

where \(S_{ij}\) and \(\bar{S}_{\alpha\beta}\) are the Ricci tensors and \(r\) and \(\bar{r}\) are the scalar curvatures of \((V, g)\) and \((\bar{V}, \bar{g})\), respectively.

Thus we can state the following theorem:

**Theorem 3.1.** The totally geodesic hypersurface of an \(A(\text{PMRS})_n\) is also \(A(\text{PMRS})_n\).

**Proof.** Let us consider the totally geodesic hypersurface of an \(A(\text{PMRS})_n\). With the help of (3.5) and (3.9), we have

\[
S_{ij} = \bar{S}_{\alpha\beta} B_i^\alpha B_j^\beta.
\]

By putting (3.1) and (3.12) in (2.1), we find

\[
\bar{M}_{ij} = \bar{M}_{\alpha\beta} B_i^\alpha B_j^\beta.
\]
Since \((\bar{V}, \bar{g})\) be an \(A(PMRS)_n\), then we get from (1.4)

\[ M_{\alpha\beta,\gamma} = (A_\gamma + B_\gamma)M_{\alpha\beta} + A_{\alpha}M_{\gamma\beta} + A_{\beta}M_{\gamma\alpha}. \]  

Multiplying both sides of (3.14) by \(B_{ijk}^{\alpha\beta\gamma}\) and using (3.13), we obtain

\[ \bar{M}_{ij,k} = (A_k + B_k)\bar{M}_{ij} + A_i\bar{M}_{jk} + A_j\bar{M}_{ik}. \]

Hence, the proof is completed. \(\Box\)

**Theorem 3.2.** A necessary and sufficient condition the totally umbilical hypersurface of an \(A(PMRS)_n\) be also \(A(PMRS)_n\) is that the mean curvature be zero or constant.

*Proof.* We assume that our manifold is \(A(PMRS)_n\). Thus, we have the equation (3.14). Multiplying (3.14) by \(B_{ijk}^{\alpha\beta\gamma}\), we find

\[ B_{ijk}^{\alpha\beta\gamma}M_{\alpha\beta,\gamma} = (A_k + B_k)M_{ij} + A_iM_{jk} + A_jM_{ik}. \]

From (2.1), for \(A(PMRS)_n\), we obtain

\[ M_{\alpha\beta,\gamma} = \frac{1}{2}S_{\alpha\beta,\gamma}. \]

Combining the equations (3.15) and (3.16), we get

\[ \frac{1}{2}S_{\alpha\beta,\gamma}B_{ijk}^{\alpha\beta\gamma} = (A_k + B_k)M_{ij} + A_iM_{jk} + A_jM_{ik}. \]

If we consider the hypersurface is totally umbilical then by taking the covariant derivative of (3.9), it can be seen that

\[ S_{\alpha\beta,\gamma}B_{ijk}^{\alpha\beta\gamma} = S_{ij,k} - 2(n - 1)HH_{,k}g_{ij}. \]

We conclude from (3.17) and (3.18) that

\[ \frac{1}{2}S_{ij,k} - (n - 1)HH_{,k}g_{ij} = (A_k + B_k)M_{ij} + A_iM_{jk} + A_jM_{ik}. \]

If this hypersurface of \(A(PMRS)_n\) is also \(A(PMRS)_n\) then by the aid of (1.4), (2.1) and (3.19), it can be found that

\[ HH_{,k}g_{ij} = 0. \]

This means that either \(H = 0\) or \(H_{,k} = 0\), i.e., \(H\) is constant. Conversely, if \(H = 0\) or \(H\) is constant from (2.1) and (3.19), we get (1.4). In this case, the totally umbilical hypersurface of this manifold is also \(A(PMRS)_n\). \(\Box\)
References


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