Some curves on three-dimensional trans-Sasakian manifolds

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Abstract. In the present paper we study biharmonic magnetic curves in three-dimensional trans-Sasakian manifolds with respect to Levi-Civita connection. It is shown that for biharmonic magnetic curves in a three-dimensional trans-Saakian manifold the structure function β is zero. We study locally ϕ-symmetric Legendre curves in a three-dimensional trans-Sasakian manifold. Also we characterized non-geodesic Legendre curves in a three-dimensional trans-Sasakian manifold for the Reeb vector field parallel to principal normal and binormal vector.

M.S.C. 2010: 53C15, 53C25.
Key words: magnetic curve; biharmonic curve; Legendre curve; trans-Sasakian manifold; locally ϕ-symmetric.

1 Introduction

Let (M,g) and (N,h) be two Riemannian manifolds and $\Psi : (M,g) \to (N,h)$ a smooth map. The energy functional of Ψ is defined by $E(\Psi) = \frac{1}{2}\int_M |d\Psi|^2 v_g$. Critical points of the energy functional are called harmonic maps and the Euler-Lagrange equation for the energy is $\tau(\Psi) = trace\nabla d\Psi = 0$, where ∇ denotes Levi-Civita connection on M. Biharmonic maps which can considered a natural generalization of harmonic maps are defined as critical points of the bienergy functional given by $E(\Psi) = \frac{1}{2}\int_M |\tau(|\Psi)|^2 v_g$. The first variation formula for the bienergy was derived by G. Y. Jiang [15] and it was proved that the Euler-Lagrange equation for bienergy is

$$\tau_2(\Psi) = -J(\tau(\Psi)) = -\nabla\tau(\Psi) - traceR^N(d\Psi, \tau(\Psi))d\Psi = 0,$$

where J is the Jacobi operator, $\nabla = -trace(\nabla^\Psi\nabla^\Psi - \nabla^\Psi_\nabla)$ is the rough Laplacian on the sections of pull bundle $\Psi^{-1}TN$, ∇^Ψ is the pull-back connection [10] and R^N is the curvature operation on N. One can easily see that harmonic maps are always biharmonic. But the converse is not true. Nonharmonic biharmonic maps are said to be proper. It is well known that proper biharmonic maps into R, that is, biharmonic function play an important role in elasticity and hydrodynamics. For the study of biharmonic maps we may refer to ([1], [2], [11]). Also biharmonic curves have been studied in the papers ([12], [14], [20]).

Differential Geometry - Dynamical Systems, Vol.23, 2021, pp. 155-166.

The notion of contact magnetic fields and flow lines were introduced by Cabrerizo and collaborators [8]. In a contact manifold the fundamental 2-form is closed. A magnetic field on a manifold M is a closed 2-form. The fundamental 2-form of a three-dimensional trans-Sasakian is also closed. So, we can consider the fundamental 2-form of three-dimensional trans-Sasakian manifold as a magnetic field. A curve γ is called a magnetic curve in a three-dimensional trans-Sasakian manifold if $\nabla_{\dot{\gamma}}\dot{\gamma} = \phi\dot{\gamma}$. The Legendre curves play an important role in the study of contact manifolds. In a 3-dimensional Sasakian manifold, the Legendre curves are studied by C. Baikoussis and D. E. Blair who gave the Frenet 3-frame in this space [5]. Baikoussis and Hirica studied Legendre curves in Riemannian and Lorentzian Sasakian spaces [3]. On the other hand, J. Welyzcko [25] studied Legendre curves on three-dimensional normal almost contact metric manifolds. Also, Legendre curves in α-Sasakian spaces are studied by Özgür and Tripathi [19]. In [14], the authors have introduced a 1-parameter family of linear connections on three-dimensional almost contact metric manifolds to study biharmonic curves on almost contact manifolds. The author of the present paper has studied some curves on three-dimensional trans-Sasakian manifolds with semi-symmetric metric connection [21]. The author has also studied some curves on α-Sasakian manifolds with indefinite metric [23].

The present paper is organized as follows: After the introduction, we give some required preliminaries in Section 2. Section 3, we study biharmonic magnetic and biharmonic helix in a three-dimensional trans-Sasakian manifold. Next section we study locally ϕ-symmetric Legendre curves in a three-dimensional trans-Sasakian manifold and a non-geodesic Legendre curves in a three-dimensional trans-Sasakian manifold for the Reeb vector field parallel to principal normal and binormal vector.

2 Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$ tensor field, ξ is a vector field, η is an 1-form and g is compatible Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in T(M)$ [9].

The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y), \tag{2.4}$$

for $X, Y \in T(M)$.

An almost contact metric manifold is normal if $[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0$.

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called trans-Sasakian structure [18] if $(M \times R, J, G)$ belongs to the class W_4 [13], where J is the almost complex structure on $M \times R$ defined by $J(X, fd/dt) = (\phi X - f\xi, \eta(X)d/dt)$, for all vector fields X on M, a smooth function f on $M \times R$ and the product metric G on $M \times R$. This may be expressed by the condition [4]

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.5}$$

for smooth functions α and β on M. Here ∇ is Levi-Civita connection on M. We say M as the trans-Sasakian manifold of type (α, β). From (2.5) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.6}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.7}$$

A trans-Sasakian manifold is said to be
- cosymplectic or co Kaehler manifold if $\alpha = \beta = 0$,
- quasi-Sasakian manifold if $\beta = 0$ and $\xi(\alpha) = 0$,
- α-Sasakian manifold if α is a non-zero constant and $\beta = 0$,
- β-Kenmotsu manifold if $\alpha = 0$ and β is a non-zero constant.

Therefore, trans-Sasakian manifold generalizes a large class of almost contact manifolds.

In a three-dimensional trans-Sasakian manifold following relations hold ([3], [19]):

$$2\alpha\beta + \xi\alpha = 0, \tag{2.8}$$

$$\begin{aligned} S(X,Y) = {} & \{\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\}g(X,Y) \\ & - \{\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\}\eta(X)\eta(Y) - \{Y\beta + (\phi X)\alpha\}\eta(Y), \end{aligned} \tag{2.9}$$

$$\begin{aligned} R(X,Y)Z = {} & (\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))(g(Y,Z)X - g(X,Z)Y) \\ & - g(Y,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\xi \\ & - \eta(X)(\phi \mathrm{grad}\alpha - \mathrm{grad}\beta) + (X\beta + (\phi X)\alpha)\xi] \\ & + g(X,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\xi \\ & - \eta(Y)(\phi \mathrm{grad}\alpha - \mathrm{grad}\beta) + (Y\beta + (\phi Y)\alpha)\xi] \\ & - [(Z\beta + (\phi Z)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z) \\ & + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\eta(Z)]X \\ & + [(Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z) \\ & + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Z)]Y, \end{aligned} \tag{2.10}$$

where S is the Ricci tensor of type $(0, 2)$, and r is the scalar curvature of the manifold M with respect to Levi-Civita connection.

From here after we consider α and β are constant, then the above relations become

$$\begin{aligned} R(X,Y)Z = &\{\frac{r}{2}-(\alpha^2-\beta^2)\}[g(Y,Z)X-g(X,Z)Y] \\ &+\{\frac{r}{2}-(\alpha^2-\beta^2)\}[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]\xi \\ &+\{\frac{r}{2}-3(\alpha^2-\beta^2)\}[\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X], \end{aligned} \tag{2.11}$$

$$\begin{aligned} S(X,Y) = &\{\frac{r}{2}-(\alpha^2-\beta^2)\}g(X,Y) \\ &-\{\frac{r}{2}-3(\alpha^2-\beta^2)\}\eta(X)\eta(Y), \end{aligned} \tag{2.12}$$

$$S(X,\xi) = 2(\alpha^2-\beta^2)\eta(X), \tag{2.13}$$

$$QX = \{\frac{r}{2}-(\alpha^2-\beta^2)\}X-\{\frac{r}{2}-3(\alpha^2-\beta^2)\}\eta(X)\xi, \tag{2.14}$$

$$R(X,Y)\xi = (\alpha^2-\beta^2)\{\eta(Y)X-\eta(X)Y\}, \tag{2.15}$$

$$R(\xi,X)Y = 2(\alpha^2-\beta^2)\{g(X,Y)\xi-\eta(Y)X\}. \tag{2.16}$$

From (2.8) it follows that if α and β are constant, then the manifold is either α-Sasakian or β-Kenmotsu or cosymplectic.

A smooth curve $\gamma : I \subseteq R \to M$ on a three-dimensional trans-Sasakian manifold is called magnetic curve if satisfies ([17], [22])

$$\nabla_T T = \phi T, \tag{2.17}$$

where $T = \dot{\gamma}$.

A magnetic curve is called normal if $g(\dot{\gamma},\dot{\gamma}) = 1$.

Let M be a three-dimensional Riemannian manifold. Let $\gamma : I \to M$, I being an interval, be a curve in M which is parameterized by arc length, and let $\nabla_{\dot{\gamma}}$ denote the covariant differentiation along γ with respect to the Levi-Civita connection on M. It is said that γ is a Frenet curve if one of the following three cases hold:

(a) γ is of osculating order 1, i.e., $\nabla_T T = 0$ (geodesic), $T = \dot{\gamma}$. Here, "." denotes differentiation with respect to the arc parameter.

(b) γ is of osculating order 2, i.e., there exist two orthonormal vector fields $T(=\dot{\gamma})$, N and a non-negative function κ (curvature) along γ such that $\nabla_T T = \kappa N$, $\nabla_T N = -\kappa T$.

(c) γ is of osculating order 3, i.e., there exist three orthonormal vectors $T(=\dot{\gamma})$, N, B and two non-negative functions κ(curvature) and τ(torsion) along γ such that

$$\nabla_T T = \kappa N, \tag{2.18}$$

$$\nabla_T N = -\kappa T + \tau B, \tag{2.19}$$

$$\nabla_T B = -\tau N. \tag{2.20}$$

With respect to Levi-Civita connection, a Frenet curve of osculating order 3 for which κ is a positive constant and $\tau = 0$ is called a circle in M; a Frenet curve of osculating order 3 is called a helix in M if κ and τ both are positive constants and the curve is called a generalized helix if $\frac{\kappa}{\tau}$ is a constant.

A Frenet curve is called a slant curve if it makes a constant angle with Reeb vector field ξ. If a curve γ on an almost contact metric manifold is a slant curve then $\eta(\dot{\gamma}) = \cos\theta$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$ where θ is a constant and is called slant angle. In particular if the angle is $\frac{\pi}{2}$, the curve becomes a Legendre curve. A slant curve is called proper if it is neither parallel nor perpendicular to the Reeb vector ξ. For more details we refer ([5], [6], [7], [16], [25]).

It is to be mentioned that in the paper [14], curves satisfying the above properties on almost contact manifolds have been termed as almost contact curve while Welyczko [25] has termed such curves on almost contact manifolds as Legendre curves. Henceforth by Legendre curves on almost contact manifolds we shall mean almost contact curves.

3 Biharmonic curves in a three-dimensional trans-Sasakian manifold

In the section we study biharmonic magnetic curves and biharmonic helix in a three-dimensional trans-Sasakian manifold.

Definition 3.1. A curve γ in a three-dimensional trans-Sasakian manifold is called biharmonic with respect to Levi-Civita connection if it satisfies the equation [14]

$$\nabla_T^3 T + R(\nabla_T T, T)T = 0, \tag{3.1}$$

where $\dot{\gamma} = T$ is a tangent vector field of the curve and R is the curvature tensor of type (1,3).

Proposition 3.1. *A magnetic curve γ in a three-dimensional trans-Sasakian manifold M with $(\eta(T)^2 - 1) \neq 0$, is a slant curve if and only if M is α-Sasakian or cosymplectic manifold.*

Proof. Let γ be a unit speed magnetic curve in a three-dimensional trans-Sasakian manifold.

Differentiating $\eta(T) = g(T, \xi)$ along a magnetic curve γ, and using (2.6), (2.17) we have

$$\begin{aligned} \frac{d}{dt} g(T, \xi) &= g(\nabla_T T, \xi) + g(T, \nabla_T \xi) \\ &= \beta\{(1 - \eta(T)^2)\}. \end{aligned} \tag{3.2}$$

This completes the proof. □

Theorem 3.2. *Let M be a three-dimensional trans-Sasakian manifold with α, β=constant and $\gamma : I \to M$ be a non-null magnetic curve. Then γ is a biharmonic Legendre magnetic curve if and only if one of the following holds:*
(i) $\alpha = \pm 1$ and $\beta = 0$ or,
(ii) $\alpha = -1$ and $\beta = \pm 2$.

Proof. Differentiating (2.17) with respect to T and using (2.1) and (2.5) we have

$$\nabla_T^2 T = -[\alpha\eta(T) + 1]T - \beta\eta(T)\phi T + [\alpha + \eta(T)]\xi. \tag{3.3}$$

Again, differentiating (3.3) with respect to T, we get

$$\begin{aligned}\nabla_T^3 T = {} & [\alpha\beta\eta(T)^2 + 2\beta\eta(T) + \alpha\beta]T \\ & + [\beta^2\eta(T)^2 - 2\alpha\eta(T) - \alpha^2 - 1]\phi T \\ & - 2\beta[\alpha\eta(T) + \eta(T)^2]\xi.\end{aligned} \tag{3.4}$$

From (2.11) and (2.17) we get

$$R(\nabla_T T, T)T = 2(\alpha^2 - \beta^2)\phi T. \tag{3.5}$$

Now using (3.4) and (3.5) in (3.1) we get

$$\begin{aligned}\nabla_T^3 T + R(\nabla_T T, T)T = {} & [\alpha\beta\eta(T)^2 + 2\beta\eta(T) + \alpha\beta]T \\ & + [\beta^2\eta(T)^2 - 2\alpha\eta(T) + \alpha^2 - 2\beta^2 - 1]\phi T \\ & - 2\beta[\alpha\eta(T) + \eta(T)^2]\xi \\ = {} & 0.\end{aligned} \tag{3.6}$$

Hence the theorem is proved. □

From the Theorem 3.2 we have the following Corollary:

Corollary 3.3. *There does not exist a biharmonic Legendre magnetic curve in a three-dimensional cosymplectic manifold.*

Now we study biharmonic helix in a three-dimensional trans-Sasakian manifold.

Theorem 3.4. *Let γ be a biharmonic helix in a three-dimensional trans-Sasakian manifold with α, β =constant. Then $\kappa^2 + \tau^2 = 2(\alpha^2 - \beta^2)$, where κ and τ are curvature and torsion respectively.*

Proof. Let γ be a biharmonic helix in a three-dimensional trans-Sasakian manifold with α, β= constant. Then

$$\nabla_T^3 T + R(\nabla_T T, T)T = 0, \tag{3.7}$$

where $\dot{\gamma} = T$, is tangent vector, and the curvature κ and the torsion τ are constant.

Let N and B be principal normal and binormal respectively. Then the Serret-Frenet equation are

$$\nabla_T T = \kappa N, \tag{3.8}$$

$$\nabla_T N = -\kappa T + \tau B, \tag{3.9}$$

$$\nabla_T B = -\tau N. \tag{3.10}$$

Now differentiating (3.8) with respect to T, we get

$$\begin{aligned} \nabla_T^2 T &= \nabla_T(\kappa N) \\ &= \kappa \nabla_T N \\ &= \kappa(-\kappa T + \tau B) \\ &= -\kappa^2 T + \kappa\tau B. \end{aligned} \tag{3.11}$$

Again differentiating (3.11) with respect to T, we get

$$\begin{aligned} \nabla_T^3 T &= \nabla_T(-\kappa^2 T + \kappa\tau B) \\ &= -\kappa^2(\kappa N) + \kappa\tau(-\tau N) \\ &= -\kappa^3 N - \kappa\tau^2 N. \end{aligned} \tag{3.12}$$

Therefore

$$\begin{aligned} R(\nabla_T T, T)T = &\{\frac{r}{2} - (\alpha^2 - \beta^2)\}[g(T,T)\nabla_T T - g(\nabla_T T, T)T] \\ &+ \{\frac{r}{2} - (\alpha^2 - \beta^2)\}[g(\nabla_T T, T)\eta(T) - g(T,T)\eta(\nabla_T T)]\xi \\ &+ \{\frac{r}{2} - 3(\alpha^2 - \beta^2)\}[\eta(\nabla_T T)\eta(T)T - \eta(T)\eta(T)\nabla_T T] \\ = &\{\frac{r}{2} - (\alpha^2 - \beta^2)\}[\kappa N - 0] \\ &+ \{\frac{r}{2} - (\alpha^2 - \beta^2)\}[0 - \kappa\eta(N)]\xi \\ &+ \{\frac{r}{2} - 3(\alpha^2 - \beta^2)\}[\kappa\eta(N)\eta(T)T - \eta(T)^2\kappa N]. \end{aligned} \tag{3.13}$$

Using (3.12) and (3.13) in (3.7) we have

$$\begin{aligned} \nabla_T^3 T + R(\nabla_T T, T)T &= -\kappa^3 N - \kappa\tau^2 N \\ &+ \kappa\{\frac{r}{2} - (\alpha^2 - \beta^2)\}N \\ &- \kappa\{\frac{r}{2} - (\alpha^2 - \beta^2)\}\eta(N)\xi \\ &+ \kappa\{\frac{r}{2} - 3(\alpha^2 - \beta^2)\}[\eta(N)\eta(T)T - \eta(T)^2 N] \\ &= 0. \end{aligned} \tag{3.14}$$

Taking inner product in (3.14) with ξ, we get

$$\begin{aligned} 0 = &-\kappa^3\eta(N) - \kappa\tau^2\eta(N) \\ &+ \kappa\{\frac{r}{2} - (\alpha^2 - \beta^2)\}\eta(N) \\ &- \kappa\{\frac{r}{2} - (\alpha^2 - \beta^2)\}\eta(N). \end{aligned} \tag{3.15}$$

The above equation implies that

$$\kappa\eta(N)[\kappa^2 + \tau^2 - 2(\alpha^2 - \beta^2)] = 0. \tag{3.16}$$

Since κ and $\eta(N)$ are non-zero, we have

$$\kappa^2 + \tau^2 = 2(\alpha^2 - \beta^2). \tag{3.17}$$

Hence the theorem is proved. □

From the Theorem 3.5. we state following result:

Corollary 3.5. *There does not exist a proper biharmonic helix in a three-dimensional trans-Sasakian manifold with constant structure functions α, β, if $\alpha^2 - \beta^2 \leq 0$.*

4 Legendre curves in a three-dimensional trans-Sasakian manifold

The notion of locally ϕ-symmetric manifolds was introduced by T. Takahashi [24] in the context of Sasakian geometry. Since every smooth curve is one-dimensional differentiable manifold we may apply the concept of local ϕ-symmetry on a smooth curve. In [21], locally ϕ-symmetric Legendre curves have been studied.

Definition 4.1. With respect to the Levi-Civita connection a three-dimensional trans-Sasakian manifold will be called locally ϕ-symmetric if it satisfies

$$\phi^2(\nabla_W R)(X, Y)Z = 0, \tag{4.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 4.2. A Legendre curve γ on a three-dimensional trans-Sasakian manifold is said to be locally ϕ-symmetric with respect to the Levi-Civita connection if it satisfies

$$\phi^2(\nabla_T R)(\nabla_T T, T)T = 0, \tag{4.2}$$

where $T = \dot{\gamma}$.

Theorem 4.1. *A necessary and sufficient condition for a Legendre curve on a three-dimensional trans-Sasakian manifold with constant structure functions α, β to be locally ϕ-symmetric with respect to the Levi-Civita connection is either $\tilde{k} = 0$, or scalar curvature r is a constant.*

Proof. By definition of covariant differentiation of the Riemannian curvature tensor R of type $(1, 3)$ we obtain

$$\begin{aligned}(\nabla_T R)(\nabla_T T, T)T = {} & \nabla_T R(\nabla_T T, T)T - R(\nabla_T^2 T, T)T \\ & - R(\nabla_T T, \nabla_T T)T - R(\nabla_T T, T)\nabla_T T.\end{aligned}$$

Using Serret-Frenet formula after simplification we get

$$(\nabla_T R)(\nabla_T T, T)T = \nabla_T R(\nabla_T T, T)T - \kappa\tau R(B,T)T - \kappa' R(N,T)T - \kappa^2 R(N,T)N. \tag{4.3}$$

Let us consider a Legendre curve γ with respect to the Levi-civita connection in a three-dimensional trans-Sasakian manifold. We take $\{T, \phi T, \xi\}$ as right handed system when $\phi T = -N$, $\phi N = T$. Then for Legendre curve $\eta(T) = 0$ and $\eta(N) = 0$ for the Frenet frame.

After some straight forward calculation, the above equation together with (2.11) we have

$$(\nabla_T R)(\nabla_T T, T)T = \kappa\{\frac{r}{2} - (\alpha^2 - \beta^2)\}' N. \tag{4.4}$$

Applying ϕ^2 in both sides of the above equation and using (2.1) we have

$$\phi^2(\nabla_T R)(\nabla_T T, T)T = -\kappa\{\frac{r}{2} - (\alpha^2 - \beta^2)\}' N. \tag{4.5}$$

Hence the above equation (4.5) and (4.2) completes the proof of the theorem. □

Now we characterized a non-geodesic Legendre curves in a three-dimensional trans-Sasakian manifold for the Reeb vector field parallel to principal normal and binormal vector.

Theorem 4.2. *Let γ be a non-geodesic Legendre curve in a three-dimensional trans-Sasakian manifold. If the unit vector ξ is parallel to biharmonic vector B. Then the manifold is a α-Sasakian and the value of the torsion is α.*

Proof. Let γ be a Legendre curve in a three-dimensional trans-Sasakian manifold. If ξ is along the vector B. Then $\{\dot\gamma, \phi\dot\gamma, \xi\}$ are orthonormal vector field along γ and $T = \dot\gamma, N = \phi\dot\gamma, B = \xi$.

Let κ and τ be the curvature and the torsion of the curve γ. Then

$$\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma, \tag{4.6}$$

$$\nabla_{\dot\gamma}\phi\dot\gamma = -\kappa\dot\gamma + \tau\xi, \tag{4.7}$$

$$\nabla_{\dot\gamma}\xi = -\tau\phi\dot\gamma. \tag{4.8}$$

Also using (2.1) and (2.5), we have

$$\begin{aligned}\nabla_{\dot\gamma}\phi\dot\gamma &= (\nabla_{\dot\gamma}\phi)\dot\gamma + \phi(\nabla_{\dot\gamma}\dot\gamma)\\ &= \alpha\{g(\dot\gamma,\dot\gamma)\xi - \eta(\dot\gamma)\dot\gamma\} + \beta\{g(\phi\dot\gamma,\dot\gamma)\xi - \eta(\dot\gamma)\phi\dot\gamma)\} - \phi(\kappa\phi\dot\gamma)\\ &= \alpha\xi - \kappa\dot\gamma,\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\nabla_{\dot\gamma}\xi &= -\alpha\phi\dot\gamma + \beta(\dot\gamma - \eta(\dot\gamma)\xi)\\ &= -\alpha\phi\dot\gamma + \beta\dot\gamma.\end{aligned} \tag{4.10}$$

Now comparing the equation (4.7) with (4.9) and (4.8) with (4.10), we get

$$\tau = \alpha, \tag{4.11}$$

and

$$\beta = 0. \tag{4.12}$$

This completes the proof. □

Theorem 4.3. *Let γ be a non-geodesic Legendre curve in a three-dimensional trans-Sasakian manifold with constant structure functions α, β. If the unit vector ξ is parallel to normal vector N. Then the curve is generalized helix provided $\alpha \neq 0, \beta \neq 0$.*

Proof. Let γ be a Legendre curve in a three-dimensional trans-Sasakian manifold. If ξ is along the vector N. Then $\{\dot{\gamma}, \xi, \phi\dot{\gamma}\}$ are orthonormal vector field along γ and $T = \dot{\gamma}, N = \xi, B = \phi\dot{\gamma}$.

Let κ and τ be the curvature and the torsion of the curve γ. Then

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \kappa\xi, \tag{4.13}$$

$$\nabla_{\dot{\gamma}}\xi = -\kappa\dot{\gamma} + \tau\phi\dot{\gamma}, \tag{4.14}$$

$$\nabla_{\dot{\gamma}}\phi\dot{\gamma} = -\tau\xi. \tag{4.15}$$

Also using (2.1) and (2.5), we have

$$\begin{aligned} \nabla_{\dot{\gamma}}\phi\dot{\gamma} &= (\nabla_{\dot{\gamma}}\phi)\dot{\gamma} + \phi(\nabla_{\dot{\gamma}}\dot{\gamma}) \\ &= \alpha\{g(\dot{\gamma}, \dot{\gamma})\xi - \eta(\dot{\gamma})\dot{\gamma}\} + \beta\{g(\phi\dot{\gamma}, \dot{\gamma})\xi - \eta(\dot{\gamma})\phi\dot{\gamma})\} - \phi(\kappa\xi) \\ &= \alpha\xi, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \nabla_{\dot{\gamma}}\xi &= -\alpha\phi\dot{\gamma} + \beta(\dot{\gamma} - \eta(\dot{\gamma})\xi) \\ &= -\alpha(\phi\dot{\gamma}) + \beta\dot{\gamma}. \end{aligned} \tag{4.17}$$

Now comparing the equation (4.14) with (4.17) and (4.15) with (4.16), we get

$$\tau = -\alpha, \tag{4.18}$$

and

$$\kappa = -\beta. \tag{4.19}$$

Therefore

$$\frac{\kappa}{\tau} = \frac{\beta}{\alpha} = \text{constant}.$$

This completes the proof. □

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