$*$-Ricci soliton and gradient $*-$Ricci soliton on three dimensional $N(k)$-contact metric manifolds

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Abstract. The aim of the present paper is to initiate the study of $*$-Ricci soliton and gradient $*$-Ricci soliton on a three dimensional $N(k)$-contact metric manifold.

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1 Introduction

A Ricci soliton is a generalization of an Einstein metric. We reminisce the notion of Ricci soliton according to [8]. On a manifold M, a Ricci soliton is a triple (g, V, λ) with a Riemannian metric g, a vector field V, called potential vector field and a real scalar λ, such that

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{1.1}$$

where $\pounds$ denotes the Lie derivative along V and S denotes the Ricci tensor. The Ricci soliton is a special self similar solution of the Hamilton's [15] Ricci flow: $\frac{\delta}{\delta t} g(t) = -2S(t)$ with the initial condition $g(0) = g$. It is said to be shrinking, steady and expanding accordingly, as λ is positive, zero and negative respectively. If the vector field V is the gradient of a smooth function f on M, that is, $V = \nabla f$, then we say that Ricci soliton is gradient and f is potential function. For a gradient Ricci soliton, Eq. (1.1) takes the form:

$$Hessf + S + \lambda g = 0, \tag{1.2}$$

where $Hess$ denotes the Hessian operator ∇^2 (∇ denotes the Riemannian connection of g). A Ricci soliton on a compact manifold has constant curvature in dimension 2(Hamilton [15]) and also in dimension 3 (Ivey[24]). We also recall the significant result of Perelman [20]: A Ricci soliton on a compact manifold is a gradient Ricci soliton. Equations (1.1) and (1.2) are respectively called almost Ricci soliton and almost gradient Ricci soliton, if λ is a variable smooth function on M. Consider [9] for more details about the Ricci flow and Ricci soliton. In the context of contact geometry,

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Ricci solitons were first initiated by Sharma in [21]. Then, these are extensively studied by Wang ([25],[26]) and many others. In [22], Tachibana first introduced the notion of $*$-Ricci tensor on almost Hermitian manifolds and Hamada [13] apply this notion of $*$-Ricci tensor to almost contact manifolds defined by

$$S^* = \frac{1}{2} trace(Z \to R(X, \phi Y)\phi Z),$$

for any $X, Y, Z \in TM$(where TM is the Lie algebra of all vector fields on M). In 2014, Kaimakamis and Panagiotidou[17] introduced the concept of $*$-Ricci solitons within the framework of real hypersurfaces of a complex space form, where they essentially modified the definition of Ricci soliton by replacing the Ricci tensor S in (1.1) with the $*$-Ricci tensor S^*. A Riemannian metric g on a manifold M is called a $*$-Ricci soliton if there exist a constant λ and a vector field V, such that

$$(\pounds_V g)(X, Y) + 2Ric^*(X, Y) + 2\lambda g(X, Y) = 0, \tag{1.3}$$

for all vector fields X, $Y \in TM$. The $*$-Ricci soliton is said to be shrinking, steady and expanding accordingly, as λ is positive, zero and negative respectively. Moreover, if the vector field V is a gradient of a smooth function f, then we say that $*$-Ricci soliton is gradient and equation (1.3) takes the form

$$Hessf + S^* + \lambda g = 0. \tag{1.4}$$

Note that a $*$-Ricci soliton is trivial if the vector field V is killing and in this case the manifold becomes $*$-Einstein. In 2018, De et. al. [18] first undertook the study of $*$-Ricci solitons and gradient $*$-Ricci solitons on three dimensional Sasakian manifold. In the same year Ghosh and Patra [14] also studied the $*$-Ricci soliton on almost contact metric manifolds.

Inspired by the above-mentioned works, in this paper we consider $*$-Ricci soliton and gradient $*$-Ricci soliton on three dimensional $N(k)$-contact metric manifold. After preliminary, in section 3 we have the $*$-Ricci tensor S^* and in section 4 we prove that if a three dimensional $N(k)$-contact metric manifold admits a $*$-Ricci soliton, then either the manifold is flat or the manifold is an $*$-Einstein and we have also shown that the $*-$Ricci soliton is shrinking. In the next section we prove that if the metric g of a three dimensional $N(k)$-contact manifold is a gradient $*$-Ricci soliton, then the manifold is flat or the manifold is an $*$-Einstein manifold. Finally the last section ends with an example of a $*$-Ricci soliton on three dimensional $N(k)$-contact manifold.

2 Contact metric manifolds

A $(2n+1)$-dimensional manifold M^{2n+1} is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$(a)\ \phi^2 = -I + \eta \otimes \xi, \ \ (b)\ \eta(\xi) = 1, \ \ (c)\ \phi\xi = 0, \ \ (d)\ \eta \circ \phi = 0. \tag{2.1}$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M^{2n+1} \times R$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable, where X is tangent to M, t is the co-ordinate of R

and f is a smooth function on $M \times R$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2.1) it can be easily seen that

$$(a) \quad g(X, \phi Y) = -g(\phi X, Y), (b) \quad g(X, \xi) = \eta(X),$$

for all vector fields X, Y. An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y),$$

for all vector fields X, Y. The 1-form η is then a contact form and ξ is its characteristic vector field. We define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie-differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. We have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also,

$$\nabla_X \xi = -\phi X - \phi h X,$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \ \ \forall X, Y \in TM,$$

where ∇ is the Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a killing vector is said to be a K-contact manifold. Every Sasakian manifold is K-contact but not conversely. However, a 3-dimensional K-contact manifold is Sasakian [16]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ ([1]). On the other hand, on a Sasakian manifold the following holds:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

As a generalization of both $R(X, Y)\xi = 0$ and the Sasakian case; D. Blair, T. Koufogiorgos and B. J. Papantoniou [5] considered the (k, μ)-nullity condition on a contact metric manifold and gave several reasons for studying it. The (k, μ)-nullity distribution $N(k, \mu)$ ([5], [19]) of a contact metric manifold M is defined by $N(k, \mu) : p \to N_p(k, \mu) \subset T_pM$, where

$$N_p(k, \mu) = \{W \in T_pM : R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\},$$

for all $X, Y \in TM$, where $(k, \mu) \in R^2$. A contact metric manifold M^{2n+1} with $\xi \in N(k, \mu)$ is called a (k, μ)-manifold. In particular on a (k, μ)-manifold, we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY].$$

On a (k, μ)-manifold $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and μ is non-determined) and if $k < 1$, then the (k, μ)-nullity condition determines the curvature of

M^{2n+1} completely [5]. In fact, for a (k,μ)-manifold, the condition of being a Sasakian manifold, a K-contact manifold, $k=1$ and $h=0$ are all equivalent.

The k-nullity distribution $N(k)$ of a Riemannian manifold M [23] is defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then the contact metric manifold is called an $N(k)$-contact metric manifold [7]. If $k=1$, then the $N(k)$-contact metric manifold is Sasakian and if $k=0$, then $N(k)$-contact metric manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n>1$ and flat for $n=1$[2]. If $k<1$, then the scalar curvature is $r=2n(2n-2+k)$. If $\mu=0$, then the (k,μ)-contact metric manifold reduces to an $N(k)$-contact metric manifold.

In [6] Blair et al. proved that in a three dimensional contact metric manifold with ξ belonging to the $k-$nullity distribution, the following conditions hold:

$$QX = \left(\frac{r}{2} - k\right)X + \left(3k - \frac{r}{2}\right)\eta(X)\xi,$$

$$Q\phi = \phi Q,$$

$$\nabla_X \xi = -(1+\alpha)\phi X, \tag{2.2}$$

where $\alpha = \pm\sqrt{1-k}$. Contact metric manifolds have also been studied by several authors such as [11], [12] and many others. In [4], $N(k)$-contact metric manifold were studied in some detail. For more details we refer to [6], [3]. In a three dimensional $N(k)$-contact metric manifold the following relations hold:

$$h^2 = (k-1)\phi^2, \quad k \leq 1, \tag{2.3}$$

$$(\nabla_X \phi)(Y) = g(X+hX,Y)\xi - \eta(Y)(X+hX),$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X], \tag{2.4}$$

$$S(X,\xi) = 2k\eta(X),$$

$$S(X,Y) = 2k\eta(X)\eta(Y), \tag{2.5}$$

$$S(\phi X,\phi Y) = S(X,Y) - 2k\eta(X)\eta(Y),$$

$$(\nabla_X \eta)(Y) = g(X+hX,\phi Y), \tag{2.6}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y],$$

$$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].$$

3 $*-$Ricci tensor on 3-dimensional $N(k)-$contact manifolds

Lemma 3.1 *In 3-dimensional $N(k)$-contact manifolds, the $*$-Ricci tensor S^* is given by*

$$S^*(X,Y) = S(X,Y) - kg(X,Y) + (5k-6)\eta(X)\eta(Y).$$

Proof. For $N(k)-$contact metric manifold the following relation holds:

$$\begin{aligned} R(Z,W,\phi X,\phi Y) &= R(Z,W,X,Y) + k\eta(Y)(\eta(W)g(Z,X) - \eta(Z)g(W,X)) \\ &= -g(W+hW,X)(g(Z,Y) - \eta(Z)\eta(Y) + g(hZ,Y)) \\ &\quad +g(Z+hZ,X)(g(W,Y) - \eta(W)\eta(Y) + g(hW,Y)) \\ &\quad +g(X,\phi Z+\phi hZ)(g(W+hW,\phi Y) \\ &\quad -g(X,\phi W+\phi hW)(g(Z+hZ,\phi Y) \\ &\quad -\eta(X)\{(1-k)(\eta(Z)g(W,Y) - \eta(W)g(Z,Y)) \\ &\quad +\eta(Z)g(hW,Y) - \eta(W)g(hZ,Y)\}. \end{aligned} \tag{3.1}$$

Let $\{e_i\}, i = 1,2,3$ be a local orthogonal basis of vector fields in M. Substituting $Z = Y = e_i$ in (3.1) and summing over i we obtain

$$\begin{aligned} S^*(X,W) &= S(X,W) - kg(W,X) + 5(k-1)\eta(X)\eta(W) \\ &\quad +g(X,h^2W) - g(\phi^2X,W) + g(\phi^2hX,W) \\ &\quad -g(X,\phi^2hW) + g(hX,\phi^2hW). \end{aligned} \tag{3.2}$$

Using (2.1) and (2.3) in (3.2) we obtain

$$S^*(X,W) = S(X,W) - kg(X,W) + (5k-6)\eta(X)\eta(W), \tag{3.3}$$

for any $X,Y \in TM$. Hence the proof.
Using (2.5), from the above Lemma 3.1, the $*$-Ricci operator Q^* and the scalar curvature r^* are given by

$$Q^*X = -kX + (7k-6)\eta(X)\xi \tag{3.4}$$

and

$$r^* = 4k-6.$$

4 $*$-Ricci soliton on three dimensional $N(k)$-contact manifolds

In this section we study $*$-Ricci solitons on three dimensional $N(k)$-contact manifolds. Applying (3.3) in (1.3) we obtain

$$(\pounds_V g)(X,Y) = 2(k-\lambda)g(X,Y) - 2(7k-6)\eta(X)\eta(Y). \tag{4.1}$$

Taking covariant differentiation of (4.1) with respect to Z we get

$$\begin{aligned}(\nabla_Z \pounds_V g)(X,Y) = \quad & -2(7k-6)\{g(Z,\phi X)\eta(Y)+g(Z,\phi Y)\eta(X) \\ & +g(Z,h\phi X)\eta(Y)+g(Z,h\phi Y)\eta(X)\}.\end{aligned}$$

In [27], Yano proved that

$$\begin{aligned}&(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[X,Y]} g)(Y,Z) = \\ &\quad -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y).\end{aligned}$$

Since $(\nabla g)=0$, the above formula reduces to

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y). \tag{4.2}$$

As $\pounds_V \nabla$ is a $(1,2)$ type symmetric tensor, then from above equation we obtain

$$\begin{aligned}2g((\pounds_V \nabla)(X,Y),Z) \quad = \quad & (\nabla_X \pounds_V g)(Y,Z) \\ & +(\nabla_Y \pounds_V g)(X,Z) - (\nabla_Z \pounds_V g)(X,Y).\end{aligned} \tag{4.3}$$

Using (4.2) in (4.3) we get

$$\begin{aligned}2g((\pounds_V \nabla)(X,Y),Z) \quad = \quad & -4(7k-6)\{g(Y,\phi Z)\eta(X) \\ & +g(X,\phi Z)\eta(Y)+g(X,h\phi Y)\eta(Z)\}.\end{aligned} \tag{4.4}$$

From equation (4.4) we have

$$(\pounds_V \nabla)(X,Y) = 2(7k-6)\{\phi Y\eta(X) - \phi X\eta(Y) + g(X,h\phi Y)\xi\}.$$

Substituting $Y=\xi$ in the above equation we obtain

$$(\pounds_V \nabla)(X,\xi) = -2(7k-6)\phi X. \tag{4.5}$$

Differentiating (4.5) covariantly along any vector field Y, it yields

$$(\nabla_Y \pounds_V \nabla)(X,\xi) = -2(7k-6)(g(Y+hY,X)\xi - \eta(X)(Y+hY)). \tag{4.6}$$

Again from the identity [27]

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z), \tag{4.7}$$

using (4.7) in (4.6) we have

$$(\pounds_V R)(X,\xi)\xi = 2(7k-6)\{X + hX - \eta(X)\xi\}. \tag{4.8}$$

Also from (4.1) we get

$$(\pounds_V g)(X,\xi) = -2(6k-6+\lambda)\eta(X). \tag{4.9}$$

Now Lie-differentiating of the equation $\eta(X)=g(X,\xi)$ along V and by virtue of above equation (4.9) we have

$$(\pounds_V \eta)(X) - g(X,\pounds_V \xi) + 2(6k-6+\lambda)\eta(X) = 0. \tag{4.10}$$

Putting $X = \xi$ in the above equation yields

$$\eta(\pounds_V \xi) = 6k - 6 + \lambda.$$

Also three dimensional $N(k)$-contact metric manifold satisfies

$$R(X,\xi)\xi = k[X - \eta(X)\xi]. \tag{4.11}$$

Taking Lie-derivative of (4.11) along the vector field V we obtain

$$(\pounds_V R)(X,\xi)\xi = -k\{(\pounds_V \eta)(X)\xi - g(\pounds_V \xi, X)\xi\}. \tag{4.12}$$

Now from (4.10), we obtain

$$g(\pounds_V \xi, X) - (\pounds_V \eta)X = 2(6(k-1) + \lambda)\eta(X). \tag{4.13}$$

Using (4.13) in (4.12), we get

$$(\pounds_V R)(X,\xi)\xi = 2k(6(k-1) + \lambda)\eta(X)\xi. \tag{4.14}$$

Equating (4.8) and (4.14) we have

$$2(7k-6)\{X + hX - \eta(X)\xi\} = 2k(6(k-1) + \lambda)\eta(X)\xi. \tag{4.15}$$

Taking the inner product of (4.15) with ξ, we obtain

$$2k(6(k-1) + \lambda) = 0.$$

Then either $k = 0$, or $\lambda = 6(1-k)$.

Case I: If $k = 0$, then the manifold is flat [2].

Case II: If $\lambda = 6(1-k) > 0$, then the $*-$Ricci soliton is shrinking.

Also from (4.8), we infer:

$$2(7k-6)\{X + hX - \eta(X)\xi\} = 0,$$

which gives $k = \frac{6}{7}$, so from Lemma 3.1 we conclude that the manifold is $*$-Ricci Einstein.

Hence we can state the following theorems:

Theorem 4.1. *If a three dimensional $N(k)$-contact metric manifold admits a $*$-Ricci soliton, then either the manifold is flat or the manifold is $*$-Einstein.*

and

Theorem 4.2. *In a non-flat three dimensional $N(k)$-contact metric manifold the $*$-Ricci soliton is shrinking.*

5 Gradient $*$-Ricci solitons on 3-dimensional $N(k)$-contact manifolds

Let M be a three dimensional $N(k)$-contact manifold and g is a gradient $*$-Ricci soliton. Then the equation (1.4) can be written as

$$\nabla_Y Df = Q^*Y + \lambda Y, \tag{5.1}$$

for all vector fields Y in TM, where D denotes the gradient operator of g. From (5.1) it follows that

$$R(X,Y)Df = (\nabla_X Q^*)Y - (\nabla_Y Q^*)X. \tag{5.2}$$

Taking the covariant differentiation of (3.4) and using (2.2) and (2.6), we obtain

$$\begin{aligned}(\nabla_X Q^*)Y &= (7k-6)\{(\nabla_X\eta)(Y)\xi + \eta(Y)(\nabla_X\xi)\}\\ &= (7k-6)\{g(X,\phi Y)\xi + g(hX,\phi Y)\xi - (1+\alpha)\eta(Y)\phi X\}.\end{aligned} \tag{5.3}$$

Using (5.3) in (5.2), we have

$$\begin{aligned}R(X,Y)Df = \ & (7k-6)[2g(X,\phi Y)\xi + 2g(X,h\phi Y)\xi\\ & -(1+\alpha)\{\eta(Y)\phi X - \eta(X)\phi Y\}],\end{aligned}$$

which gives

$$R(\xi,Y)Df = (7k-6)(\alpha+1)\phi Y. \tag{5.4}$$

Using (5.4) and (2.4), we infer

$$k\{g(Y,Df)\xi - \eta(Df)Y\} = (7k-6)(\alpha+1)\phi Y.$$

From the above equation we also have

$$k\{g(Y,Df) - \eta(Df)\eta(Y)\} = 0.$$

This implies either $k=0$, or $g(Y,Df) - \eta(Df)\eta(Y) = 0$.

Case I: If $k=0$, then the manifold is flat [2].

Case II: We have $g(Y,Df) - \eta(Df)\eta(Y) = 0$, which implies

$$Df = (\xi f)\xi.$$

This leads to

$$\nabla_Y Df = Y(\xi f)\xi + (\xi f)\nabla_Y \xi.$$

Taking the inner product with X and using (2.2), we have

$$g(\nabla_Y Df, X) = Y(\xi f)\eta(X) - (1+\alpha)(\xi f)g(X,\phi Y). \tag{5.5}$$

Using (5.5) from (5.1), we get

$$S^*(X,Y) + \lambda g(X,Y) = Y(\xi f)\eta(X) - (1+\alpha)(\xi f)g(X,\phi Y). \tag{5.6}$$

Putting $X = \xi$ in the above equation and using (3.4), we obtain

$$Y(\xi f) = (6(k-1) + \lambda)\eta(Y). \tag{5.7}$$

From (5.6) and (5.7), we have

$$S^*(X,Y) + \lambda g(X,Y) = (6(k-1) + \lambda)\eta(Y)\eta(X) - (1+\alpha)(\xi f)g(X, \phi Y). \tag{5.8}$$

By interchanging X and Y in the above equation yields

$$S^*(X,Y) + \lambda g(X,Y) = (6(k-1) + \lambda)\eta(Y)\eta(X) - (1+\alpha)(\xi f)g(Y, \phi X). \tag{5.9}$$

Equations (5.8) and (5.9) imply $(1+\alpha)(\xi f)g(X,\phi Y) = 0$, that is, $(1+\alpha)(\xi f)d\eta(X,Y) = 0$. Since $d\eta \neq 0$, we get $(1+\alpha)\xi f = 0$, which implies either $(1+\alpha) = 0$, or $\xi f = 0$. Consequently either $\alpha = -1$, which implies that the manifold is flat [2], or $\xi f = 0$, which implies that M is an $*$-Einstein manifold. So we have the following:

Theorem 5.1. *If the metric g of a three dimensional $N(k)$-contact manifold is a gradient $*$-Ricci soliton then, either the manifold is flat or the manifold is an $*$-Einstein manifold.*

6 An example

In this section we illustrate verify Theorem 4.1 by an example. In [10], De constructed an example of a three dimensional $N(k)$-contact metric manifold. Here we consider the three dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3\}$ and the vector fields e_1, e_2 *and* e_3, which satisfy

$$[e_1, e_2] = (1+\alpha)e_3, \quad [e_2, e_3] = 2e_1, \quad [e_3, e_1] = (1-\alpha)e_2,$$

where $\alpha \neq \pm 1$ is a real number. We define ξ, g, η, ϕ as $\xi = e_1$, $g(e_i, e_j) = \delta_{ij}(i,j = 1,2,3)$ and $\phi e_1 = 0, \phi e_2 = e_3, \phi e_3 = -e_2$ and $\eta(X) = g(X, e_1)$ for any $X \in TM$. Then it is shown in [10] that (M, ϕ, ξ, η, g) is a three dimensional $N(k)$-contact metric manifold. Using Koszul's formula we get the following

$$\begin{array}{ccc} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0 \\ \nabla_{e_2} e_1 = -(1+\alpha)e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = (1+\alpha)e_1 \\ \nabla_{e_3} e_1 = (1-\alpha)e_2, & \nabla_{e_3} e_2 = -(1-\alpha)e_1, & \nabla_{e_3} e_3 = 0. \end{array}$$

Let $X = a_1e_1 + a_2e_2 + a_3e_3$ and $Y = b_1e_1 + b_2e_2 + b_3e_3$ be any two vector fields and $a_i, b_i \in \mathbb{R}, i = 1,2,3$.

Now if we consider $v = e_1$, then

$$\begin{aligned} (\pounds_V g)(X,Y) &= g(\nabla_X V, Y) + g(X, \nabla_Y V) \\ &= -2\alpha(b_2a_3 + a_2b_3) \end{aligned}$$

and

$$\begin{aligned} (\pounds_V g)(X,Y) &= 2(k-\lambda)g(X,Y) - 2(7k-6)\eta(X)\eta(Y) \\ &= 2(k-\lambda)a_1b_1 + a_2b_2 + a_3b_3 + 2(7k-6)a_1b_1. \end{aligned}$$

If we choose $\alpha = 0$ and $\lambda = \frac{6}{7}$, then the metric of the manifold satisfies $*$-solitons and the manifold becomes a $*$-Einstein manifold. Thus Theorem 4.1 is verified.

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