Miao-Tam equation on normal almost contact metric manifolds

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Abstract. In the present paper, we have studied Miao-Tam equation on normal almost contact metric manifolds and obtained several results on the said manifolds. Also we give an example to verify deduced results.

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1 Introduction

The total scalar curvature functional $\omega : A \to \mathbb{R}$ defined by

$$\omega(g) = \int_M r_g dv_g,$$

where A denotes the set of all Riemannian metrics on (M^n, g) of unit volume, r_g is the scalar curvature of g and dv_g the volume form of g. Einstein and Hilbert proved that the critical points of the total scalar curvature functional restricted to the set of smooth Riemannian structures on M^n of unit volume are Einstein [2].

Let $(M^n, g), n>2$ be a compact orientable Riemannian manifold with a smooth boundary metric ∂M. Then g is said to be a critical metric if there exists a smooth function $\lambda : M^n \to \mathbb{R}$ such that

$$-(\Delta_g \lambda)g + \nabla_g^2 \lambda - \lambda S = g \tag{1.1}$$

on M and $\lambda = 0$ on ∂M, where Δ_g, ∇_g^2 are the negative Laplacian, Hessain operator with respect to the metric g and S is the $(0, 2)$ Ricci curvature of g. The function λ is known as the potential function. The equation (1.1) is known as Miao-Tam equation and the metrics which satisfy (1.1) are known as Miao-Tam critical metrics[11].

In [11], Miao-Tam have proved that any Riemannian metric g safisfying the equation (1.1) must have constant scalar curvature. They have also classified Einstein and conformally flat Riemannian manifolds satisfying the equation (1.1) [12]. Many authors such as Barros and Ribeiro [1], Chen [6], Ghosh and Patra {[9], [10]}, Patra

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and Ghosh [13] and Sarkar and Biswas [14] have also studied critical point equation and Miao-Tam equation on several manifolds.

In this paper we would like to study some properties of the Miao-Tam equation on normal almost contact metric manifolds.

The paper is organized as follows: after introduction, we give some preliminaries in the Section 2. In Section 3, we study the normal almost contact metric manifolds satisfying Miao-Tam equation. In the last Section, we give an example to verify deduced results.

2 Preliminaries

Let M be a $(2n+1)$-dimensional smooth manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M such that [7], [8]

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1. \tag{2.1}$$

As a consequence, we get the following:

$$\phi\xi = 0, \quad g(X, \xi) = \eta(X), \quad \eta(\phi X) = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0,$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y),$$

for all vector fields X, $Y \in \chi(M)$. A differentiable manifold M of dimension $(2n+1)$ with almost contact metric structure is called an almost contact metric manifold. We can always define the 2-form Φ by

$$\Phi(X, Y) = g(X, \phi Y),$$

for all X, $Y \in \chi(M)$.

Let $\mathbb{R}$ be the real line and t a coordinate on $\mathbb{R}$. Define an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, \lambda\frac{d}{dt}) = (\phi X - \lambda\xi, \eta(X)\frac{d}{dt}),$$

where the pair $(X, \lambda\frac{d}{dt})$ denotes a tangent vector on $M \times \mathbb{R}$, X and $\lambda\frac{d}{dt}$ being tangent to M and $\mathbb{R}$, respectively.

M and (ϕ, ξ, η, g) are called normal if the structure J is integrable. The necessary and sufficient condition for (ϕ, ξ, η, g) to be normal is

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where the pair $[\phi, \phi]$ is the Nijenhuis tensor of ϕ defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

for any $X, Y \in \chi(M)$.

We say that the contact form η has rank $r = 2s$ if $(d\eta)^s \neq 0$ and $\eta \wedge (d\eta)^s = 0$ and has rank $r = 2s+1$ if $(d\eta)^{(s+1)} = 0$ and $\eta \wedge (d\eta)^s \neq 0$. We say that r is the rank of the structure (ϕ, ξ, η, g).

A normal almost contact metric structure (ϕ, ξ, η, g) satisfying $d\eta = \Phi$ is said to be Sasakian. The rank of such structure is 3. A three dimensional smooth manifold is said to be a contact metric manifold if $\eta \wedge d\eta = 0$ and a normal contact metric manifold is called Sasakian. Moreover a normal almost contact metric structure satisfying the condition $d\Phi = 0$ is said to be quasi-Sasakian.

For a normal almost contact metric structure (ϕ, ξ, η, g), we have [7], [8]

$$(\nabla_X \phi)(Y) = g(\phi \nabla_X \xi, Y) - \eta(Y)\phi \nabla_X \xi,$$

$$\nabla_X \xi = \alpha[X - \eta(X)\xi] - \beta\phi X, \tag{2.2}$$

where $2\alpha = div\xi$ and $2\beta = tr(\phi\nabla\xi)$, $div\xi$ is the divergence of ξ defined by $div\xi = trace\{X \to \nabla_X \xi\}$ and $tr(\phi\nabla\xi) = trace\{X \to \phi\nabla_X \xi\}$. Thus

$$(\nabla_X \phi)(Y) = \alpha[g(\phi X, Y)\xi - \eta(Y)\phi X] + \beta[g(X, Y)\xi - \eta(Y)X].$$

Also in this manifold, the following relations hold

$$\begin{aligned} R(X, Y)\xi =& [Y\alpha + (\alpha^2 - \beta^2)\eta(Y)]\phi^2 X - [X\alpha + (\alpha^2 - \beta^2)\eta(X)]\phi^2 Y \\ &+ [Y\beta + 2\alpha\beta\eta(Y)]\phi X - [X\beta + 2\alpha\beta\eta(X)]\phi Y, \end{aligned} \tag{2.3}$$

$$S(X, \xi) = -X\alpha - (\phi X)\beta - [\xi\alpha + 2(\alpha^2 - \beta^2)]\eta(X), \tag{2.4}$$

$$\xi\beta + 2\alpha\beta = 0,$$

$$(\nabla_X \eta)(Y) = \alpha g(\phi X, \phi Y) - \beta g(\phi X, Y), \tag{2.5}$$

where R denotes the curvature tensor and S is the Ricci tensor of type $(0, 2)$.

On the other hand, the curvature tensor in a three dimensional Riemannian manifold always satisfies

$$\begin{aligned} R(X, Y)Z =& S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &- g(X, Z)QY - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{2.6}$$

where r is the scalar curvature of the manifold.
From (2.3), (2.4) and (2.6), we get

$$S(Y,Z) =(\frac{r}{2}+\xi\alpha+\alpha^2-\beta^2)g(\phi Y,\phi Z)-\eta(Y)(Z\alpha+(\phi Z)\beta)$$
$$-\eta(Z)(Y\alpha+(\phi Y)\beta)-2(\alpha^2-\beta^2)\eta(Y)\eta(Z).$$

For α and β are non-zero constants, we get from above

$$\begin{aligned}R(X,Y)Z =&(\frac{r}{2}+2(\alpha^2-\beta^2))\{g(Y,Z)X-g(X,Z)Y\}\\ &+(\frac{r}{2}+3(\alpha^2-\beta^2))\{g(X,Z)\xi-\eta(Z)X\}\eta(Y)\\ &-(\frac{r}{2}+3(\alpha^2-\beta^2))\{g(Y,Z)\xi-\eta(Z)Y\}\eta(X),\end{aligned} \tag{2.7}$$

$$S(X,Y)=(\frac{r}{2}+(\alpha^2-\beta^2))g(X,Y)-(\frac{r}{2}+3(\alpha^2-\beta^2))\eta(X)\eta(Y), \tag{2.8}$$

$$QX=(\frac{r}{2}+(\alpha^2-\beta^2))X-(\frac{r}{2}+3(\alpha^2-\beta^2))\eta(X)\xi, \tag{2.9}$$

where Q is the Ricci operator defined by $S(X,Y)=g(QX,Y)$.

If α and β are constants, then the manifold is either β-Sasakian if $\alpha=0$ or α-Kenmotsu if $\beta=0$ or cosymplectic if $\alpha=0$, $\beta=0$.

Lemma 2.1. *[9] Let (M^n,g) be a Riemannian manifold of dimension n satisfies the Miao-Tam equation. Then the curvature tensor R can be expressed as*

$$\begin{aligned}R(X,Y)D\lambda =&(X\lambda)QY-(Y\lambda)QX+\lambda((\nabla_X Q)Y-(\nabla_Y Q)X)\\ &+(Xf)Y-(Yf)X,\end{aligned} \tag{2.10}$$

for any vector fields X, Y on M, where $f=-\frac{r\lambda+1}{n-1}$ and D is the gradient operator.

3 Normal almost contact metric manifolds satisfying Miao-Tam equation

In the present section, we study three-dimensional normal almost contact metric manifolds with α, β constants which satisfy Miao-Tam equation.

Taking covariant derivative of (2.9) along the vector field Y and using (2.2) and (2.5), we obtain

$$\begin{aligned}(\nabla_Y Q)X =&-(\frac{r}{2}+3(\alpha^2-\beta^2))[\alpha\{g(X,Y)-\eta(X)\eta(Y)\}+\beta g(\phi X,Y)]\xi\\ &-(\frac{r}{2}+3(\alpha^2-\beta^2))[\alpha\{Y-\eta(Y)\xi\}-\beta\phi Y]\eta(X),\end{aligned} \tag{3.1}$$

since, for a manifold satisfying Miao-Tam equation, the scalar curvature is constant. Interchanging X and Y in (3.1), we get

$$\begin{aligned}(\nabla_X Q)Y =& -(\frac{r}{2}+3(\alpha^2-\beta^2))[\alpha\{g(X,Y)-\eta(X)\eta(Y)\}-\beta g(\phi X,Y)]\xi \\ & -(\frac{r}{2}+3(\alpha^2-\beta^2))[\alpha\{X-\eta(X)\xi\}-\beta\phi X]\eta(Y).\end{aligned} \tag{3.2}$$

Therefore, using (2.9), (3.1) and (3.2) in (2.10), we get

$$\begin{aligned}R(X,Y)D\lambda =&(X\lambda)\{(\frac{r}{2}+(\alpha^2-\beta^2))Y-(\frac{r}{2}+3(\alpha^2-\beta^2))\eta(Y)\xi\} \\ & -(Y\lambda)\{(\frac{r}{2}+(\alpha^2-\beta^2))X-(\frac{r}{2}+3(\alpha^2-\beta^2))\eta(X)\xi\} \\ & +\lambda(\frac{r}{2}+3(\alpha^2-\beta^2))\{2\beta g(\phi X,Y)\xi-\alpha(\eta(Y)X-\eta(X)Y) \\ & +\beta(\eta(Y)\phi X-\eta(X)\phi Y\}+(Xf)Y-(Yf)X.\end{aligned}$$

Since r is constant, $(Xf)=-\frac{r}{2}(X\lambda)$ and $(Yf)=-\frac{r}{2}(Y\lambda)$. Therefore from above equation, we get

$$\begin{aligned}R(X,Y)D\lambda =&(X\lambda)\{(\alpha^2-\beta^2)Y-(\frac{r}{2}+3(\alpha^2-\beta^2))\eta(Y)\xi\} \\ & -(Y\lambda)\{(\alpha^2-\beta^2)X-(\frac{r}{2}+3(\alpha^2-\beta^2))\eta(X)\xi\} \\ & +\lambda(\frac{r}{2}+3(\alpha^2-\beta^2))\{2\beta g(\phi X,Y)\xi-\alpha(\eta(Y)X \\ & -\eta(X)Y)+\beta(\eta(Y)\phi X-\eta(X)\phi Y)\}.\end{aligned} \tag{3.3}$$

Putting $X=\xi$ and then taking inner product with ξ in (3.3), we get

$$g(R(\xi,Y)D\lambda,\xi)=(\frac{r}{2}+2(\alpha^2-\beta^2))\{(Y\lambda)-(\xi\lambda)\eta(Y)\}. \tag{3.4}$$

Also from (2.7), we get

$$\begin{aligned}R(X,Y)D\lambda =&(\frac{r}{2}+2(\alpha^2-\beta^2))\{(Y\lambda)X-(X\lambda)Y\} \\ & +(\frac{r}{2}+3(\alpha^2-\beta^2))\{(X\lambda)\xi-(\xi\lambda)X\}\eta(Y) \\ & -(\frac{r}{2}+3(\alpha^2-\beta^2))\{(Y\lambda)\xi-(\xi\lambda)Y\}\eta(X).\end{aligned} \tag{3.5}$$

Putting $X=\xi$ in (3.5) and then taking inner product with ξ, we get

$$g(R(\xi,Y)D\lambda,\xi)=-(\alpha^2-\beta^2)\{(Y\lambda)-(\xi\lambda)\eta(Y)\}. \tag{3.6}$$

From (3.4) and (3.6), we obtain

$$(\frac{r}{2}+3(\alpha^2-\beta^2))\{(Y\lambda)-(\xi\lambda)\eta(Y)\}=0,$$

which gives, either $(\frac{r}{2}+3(\alpha^2-\beta^2))=0$, or $(Y\lambda)=(\xi\lambda)\eta(Y)$.

Case-I : If $(\frac{r}{2} + 3(\alpha^2 - \beta^2)) = 0$, then we get the scalar curvature of the manifold is $-6(\alpha^2 - \beta^2)$.

Case-II : $(Y\lambda) = (\xi\lambda)\eta(Y)$, which gives

$$D\lambda = (\xi\lambda)\xi. \tag{3.7}$$

Taking differentiation of (3.7) covariantly along the vector field X and using (2.2), we get

$$\nabla_X D\lambda = X(\xi\lambda)\xi + (\xi\lambda)\{\alpha(X - \eta(X)\xi) - \beta\phi X\}. \tag{3.8}$$

Taking inner product of (3.8) with Y, we obtain

$$\begin{aligned} g(\nabla_X D\lambda, Y) =& X(\xi\lambda)\eta(Y) + (\xi\lambda)\{\alpha(g(X,Y) \\ & -\eta(X)\eta(Y)) - \beta g(\phi X, Y)\}. \end{aligned} \tag{3.9}$$

Interchanging X and Y in (3.9), we get

$$\begin{aligned} g(\nabla_Y D\lambda, X) =& Y(\xi\lambda)\eta(X) + (\xi\lambda)\{\alpha(g(X,Y) \\ & -\eta(X)\eta(Y)) + \beta g(\phi X, Y)\}. \end{aligned} \tag{3.10}$$

Using Poincare lemma: $g(\nabla_X D\lambda, Y) = g(\nabla_Y D\lambda, X)$, we get, from (3.9) and (3.10)

$$X(\xi\lambda)\eta(Y) - Y(\xi\lambda)\eta(X) = 2\beta(\xi\lambda)g(\phi X, Y). \tag{3.11}$$

Replacing X by ϕX and Y by ϕY in (3.11), we obtain

$$\beta(\xi\lambda)g(X, \phi Y) = 0,$$

which gives either $\beta = 0$ i.e., the manifold is α-Kenmotsu or $(\xi\lambda) = 0$ i.e., λ is a constant.

Thus we can state the following

Theorem 3.1. *If a normal almost contact metric manifold of dimension three with α, β constants satisfies Miao-Tam equation, then either the scalar curvature is $-6(\alpha^2 - \beta^2)$ or the manifold is α-Kenmotsu or the potential function is constant i.e., the manifold is Einstein.*

4 Example

Let us consider the manifold $M = \{x, y, z \in \mathbb{R}^3 : z \neq 0\}$ of dimension 3, where $\{x, y, z\}$ are standard co-ordinates in $\mathbb{R}^3$. We choose the vector fields

$$e_1 = e^{\frac{z}{2}}\frac{\partial}{\partial x}, \quad e_2 = e^{\frac{z}{2}}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of M. We get the following by direct computations

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\frac{1}{2}e_1, \quad [e_2, e_3] = -\frac{1}{2}e_2.$$

Let the metric tensor g be defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$$

and

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

The 1-form η is defined by $\eta(X) = g(X, e_3)$, for all X on M. Let ϕ be the $(1, 1)$-tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then we find that

$$\eta(e_3) = 1, \quad \phi^2 X = -X + \eta(X)e_3,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M.

Also

$$[\phi, \phi](e_i, e_j) + 2d\eta(e_i, e_j)e_3 = 0,$$

for all $i, j = 1, 2, 3$. Thus (ϕ, e_3, η, g) defines a normal almost contact metric structure.

For the Levi-Civita connection ∇ with respect to the metric g on M, we can write

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is known as Koszul's formula.
By Koszul's formula, we get the following expressions

$$\nabla_{e_1} e_1 = \frac{1}{2}e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\frac{1}{2}e_1,$$

$$\nabla_{e_2} e_2 = \frac{1}{2}e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_3 = -\frac{1}{2}e_2,$$

$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0.$$

From the above expressions of ∇, we conclude that the given manifold is normal almost contact metric manifold with $\alpha = -\frac{1}{2}$ and $\beta = 0$.

Using the formula

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

we get

$$R(e_1, e_2)e_1 = -\frac{1}{4}e_2, \quad R(e_1, e_2)e_2 = -\frac{1}{4}e_1, \quad R(e_2, e_3)e_3 = -\frac{1}{4}e_2,$$

$$R(e_3,e_2)e_2=-\frac{1}{4}e_3,\quad R(e_1,e_3)e_3=-\frac{1}{4}e_1,\quad R(e_3,e_1)e_1=-\frac{1}{4}e_3,$$

$$R(e_1,e_2)e_3=0,\quad R(e_2,e_3)e_1=0,\quad R(e_1,e_3)e_2=0.$$

From the expressions of curvature tensor, we get

$$S(e_1,e_1)=-\frac{1}{2},\quad S(e_2,e_2)=-\frac{1}{2},\quad S(e_3,e_3)=-\frac{1}{2}.$$

Let r be the scalar curvature, then from above

$$r=S(e_1,e_1)+S(e_2,e_2)+S(e_3,e_3)=-\frac{3}{2}.$$

From the expressions of the Ricci tensor S, we conclude that the given manifold is Einstein.

Let $\lambda=ae^z+b$, where a and b are constants.
Therefore,

$$D\lambda=ae^z e_3=(\lambda-b)e_3.$$

Thus we get the following

$$\nabla_{e_1}D\lambda=-\frac{1}{2}(\lambda-b)e_1,\quad \nabla_{e_2}D\lambda=-\frac{1}{2}(\lambda-b)e_2,\quad \nabla_{e_3}D\lambda=(\lambda-b)e_3.$$

From the above expressions, we get

$$\Delta_g\lambda=g(\nabla_{e_1}D\lambda,e_1)+g(\nabla_{e_2}D\lambda,e_2)+g(\nabla_{e_3}D\lambda,e_3)=0.$$

Thus

$$-(\Delta_g\lambda)g(e_1,e_1)+g(\nabla_{e_1}D\lambda,e_1)-\lambda S(e_1,e_1)=\frac{b}{2},$$

$$-(\Delta_g\lambda)g(e_2,e_2)+g(\nabla_{e_2}D\lambda,e_2)-\lambda S(e_2,e_2)=\frac{b}{2},$$

$$-(\Delta_g\lambda)g(e_3,e_3)+g(\nabla_{e_3}D\lambda,e_3)-\lambda S(e_3,e_3)=\frac{3}{2}ae^z+\frac{b}{2}.$$

From the last three expressions, we conclude that the given manifold satisfies the Miao-Tam equation if $a=0$ and $b=2$ and this verifies Theorem 3.1.

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