Volumes of balls in Finsler manifolds without conjugate points

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Abstract. A complete Finsler manifold is said to be without conjugate points if no geodesic contains a pair of mutually conjugate points. In this paper, we estimate the average value of the volume of balls in compact Finsler manifolds without conjugate points. Our method is based on the Riccati equation and the measure entropy in Finsler manifolds and its connections with the global geometry of the manifolds.

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1 Introduction

In a Finsler manifold, two points along a geodesic are conjugate when there exists a non-trivial Jacobi field vanishing at those two points; equivalently, the geodesic ceases to be a locally length-minimizing path past the second point in the conjugate pair. A celebrated theorem of Burago and Ivanov ([5]), first established in dimension two by Hopf, asserts that any Riemannian torus without conjugate points must be flat. For the Finsler version of the this result, there are many nonflat Finsler metrics on a torus which do not have conjugate points (see, for example, [8, 4, 22]). The author asserted that any Finsler torus without focal points is flat by examining the growth rate of the fundamental group ([16]) and the integral of the Ricci curvature on the unit tangent bundle is nonpositive ([17]).

Croke ([11]) showed that the volumes of balls in a simply connected Riemannian manifold M without conjugate points are asymptotically at least as large as those in Euclidean space, with equality in the limit if and only if M is flat. This is proved in ([6, 7]) for a Riemannian metric on $\mathbb{R}^n$ which is invariant under some $\mathbb{Z}^n$ acting by translations. In this paper we discuss an estimate of Croke [11] for the average value of the volumes of balls in a compact Finsler manifold M without conjugate points. We consider the Holmes-Thompson volume $d\mu$ of balls in the universal cover $\widetilde{M}$ of M. We will let $\mu\big(\widetilde{B}(x,r)\big)$ be the volume of a metric ball $\widetilde{B}(x,r)$ of radius r in $\widetilde{M}$ centered at a point $\widetilde{x}$ that projects to x. For any function f on M we will let $\operatorname{ave}[f(x)]$ be the average of f with respect to the volume $d\mu$ on M, i.e., $\operatorname{ave}[f(x)] := \frac{1}{\mu(M)}\int_M f(x)d\mu$.

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We will use h_μ to represent the measure entropy of the geodesic flows on the unit tangent bundle SM. It we shown in [12] that the measure entropy h_μ is the average of the mean curvature, $u(v)$ of the horospheres in $S\widetilde{M}$. For Finsler manifolds without conjugate points it is clear that $h_\mu \geq 0$. Let $\delta := \delta(h_\mu) := -\big(\frac{h_\mu}{n-1}\big)^2$ be such that the n-dimensional simply connected space M^n_δ of constant sectional curvature δ has horospheres with mean curvature h.

Let $\mu\big(B_\delta(r)\big)$ be the volume of a ball $B_\delta(r)$ of radius r in M^n_δ. Our purpose in the paper is to compare the average value of the volume of two sets, $\widetilde{B}(x,r)$ and $B_\delta(r)$. Let S be the S-curvature with respect to $d\mu$ and $S_{max} = \sup_{v \in SM} S(v)$.

Theorem 1.1. *Let M be an n-dimensional compact Finsler manifold without conjugate points and let $\delta := \delta(h_\mu)$, then for every $r > 0$,*

$$\text{ave}\left[\mu\big(\widetilde{B}(x,r)\big)\right] \geq e^{-|S_{max}|}\mu\big(B_\delta(r)\big),$$

with equality holding if and only if M has constant flag curvature δ.

Akbar-Zadeh [1] showed that if a Finsler metric on a compact manifold has constant negative flag curvature, then it is Riemannian, and, if it has zero flag curvature, then it is locally Minkowskian. In the Riemannian case, the S-curvature vanishes identically, i.e., $S_{max} = 0$, and, therefore, the main theorem becomes the comparison theorem of the average volume of balls in Riemannian geometry ([2, 10, 11, 18, 19, 20]).

2 Preliminaries

In this section, we shall recall some well-known facts about Finsler geometry. See [21], for more details. Let M be an n-dimensional smooth manifold and TM denote its tangent bundle. A *Finsler structure* on a manifold is a map $F : TM \to [0, \infty)$ which has the following properties

- F is smooth on $TM \setminus \{0\}$;
- $F(t \cdot y) = t \cdot F(y)$, for all $t > 0$, $y \in T_xM$;
- for each $y \in T_xM \setminus \{0\}$, the following quadratic g_y is an inner product in T_xM,

$$g_y(v,w) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + s \cdot v + t \cdot w)\Big]\Big|_{s=t=0}.$$

A manifold M endowed with a Finsler structure will be called a Finsler manifold. Note that we never require smoothness at the zero section.

For a fixed $v \in T_xM$ let $\gamma_v(t)$ be the geodesic from $\gamma_v(0) = x$ with $\gamma_v'(0) = v$. Along $\gamma_v(t)$, we have the *osculating Riemannian metrics*

$$g_{\gamma_v'(t)} := g\big(\gamma_v(t), \gamma_v'(t)\big)$$

in $T_{\gamma_v(t)}M$. Define the *flag curvature*

$$R_{\gamma_v'(t)} : T_{\gamma_v(t)}M \to T_{\gamma_v(t)}M$$

by

$$R_{\gamma_v'(t)}\big(w(t)\big) := R\big(W(t), V(t)\big)V(t),$$

where R is the curvature tensor, $W(t) = (\gamma_v(t); w(t))$, and $V(t) = (\gamma_v(t); \gamma_v'(t)) \in \pi^*TM$.

Let $\{dx^i, dy^i\}_{i=1}^n$ be the dual basis for $T^*(TM \setminus \{0\})$. The Holmes-Thompson volume form $d\mu$ on M is defined by

$$d\mu(x) = \sigma(x)dx^1 \wedge \cdots \wedge dx^n := \sigma(x)dx,$$

where

$$\sigma(x) = \frac{1}{\mathfrak{c}_{n-1}} \int_{S_xM} \det\big(g_{ij}(x,y)\big)\Big(\sum_{n-1}^{n}(-1)^{i-1}y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n\Big),$$

and $\mathfrak{c}_{n-1}$ is the volume of the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. For a tangent vector $v = (x,y) \in TM \setminus \{0\}$, define the *distortion* τ by

$$\tau(v) := \ln \frac{\sqrt{\det\big(g_{ij}(x,y)\big)}}{\sigma(x)}, \tag{2.1}$$

and the *S-curvature* $S : TM \setminus \{0\} \to \mathbb{R}$ is defined by

$$S(v) := \frac{d}{dt}\Big|_{t=0}\Big[\tau\big(\gamma_v'(t)\big)\Big].$$

An important property is that $S = 0$ for Finsler manifolds modeled on a single Minkowski space.

By the Chern connection, we obtain the decomposition

$$T^*(TM \setminus \{0\}) = \operatorname{span}\{dx^i\} \oplus \operatorname{span}\{\delta y^i\},$$

where δy^i is the vertical component dy^i and is given by $\delta y^i = dy^i + N^i_j dx^j$ for some N^s_l determined by the Chern connection. Then there is a naturally induced Sasaki metric $\hat{g}$ on $TM \setminus \{0\}$ defined by

$$\hat{g}(x,y) = g_{ij}(x,y)dx^i \otimes dx^j \oplus g_{ij}(x,y)\delta y^i \otimes \delta y^j,$$

and the Liouville measure form dV of $\hat{g}$ on $TM \setminus \{0\}$ is given by

$$\begin{aligned} dV(x,y) &:= \sqrt{\det\big(g_{ij}(x,y)\big)}dx^1 \wedge \cdots \wedge dx^n \wedge \sqrt{\det\big(g_{ij}(x,y)\big)}\delta y^1 \wedge \cdots \wedge \delta y^n \\ &= \det\big(g_{ij}(x,y)\big)dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n. \end{aligned}$$

Let $\omega = \frac{\partial F}{\partial y^i}dx^i$ be the Hilbert 1-form on $TM \setminus \{0\}$. In local coordinates, we have $dV = (d\omega)^n/n!$.

For any $v \in SM$, we denote by $\varphi_t(v)$ the *geodesic flow* on SM with $\varphi_0(v) = v$. It is obvious that $\pi \circ \varphi_t(v) = \gamma_v(t)$ and $\gamma_v'(t) = \varphi_t(v)$, where $\pi : SM \to M$ is the bundle projection. Then there is another interpretation of the Liouville measure on tangent space. Let $i : SM \to TM \setminus \{0\}$ the natural embedding, and X_ω the Reeb field of the Hilbert 1-form ω. It is uniquely determined by the conditions $\omega(X_\omega) = 1, i_{X_\omega}(d\omega) = 0$. In particular we have $L_{X_\omega}\omega = 0$ and the geodesic flow of F, i.e., the flow with infinitesimal generator X_ω, consists of contact diffeomorphisms and the Liouville measure form $i^*(dV)$ on SM is $\frac{1}{(n-1)!}\omega \wedge (d\omega)^{n-1}$. Since $L_{X_\omega}\omega = 0$, the Liouville measure form is also invariants under the geodeisc flow φ_t of F. We shall use the same notation dV for the Liouville measure forms of TM and SM, if no confusion is caused. By definition of the Holmes-Thompson volume $d\mu$ on M and the Liouville measure dV on SM, we obtain $dV = \mathfrak{c}_{n-1} \cdot d\mu$.

3 Riccati equations

The associated nonlinear Riccati first order ordinary differential equation has been a useful tool in oscillation theory and related comparison theory for the second order linear Jacobi equation. With the Chern connection, we can define the covariant derivative $D_{\gamma_v'(t)}J_v(t)$ of a vector field $J_v(t)$ along a geodesic $\gamma_v(t)$. A vector field $J(t)$ along $\gamma_v(t)$ is called a *Jacobi field* if it satisfies

$$D_{\gamma_v'(t)}D_{\gamma_v'(t)}J_v(t) + R_{\gamma'(t)}\big(J_v(t)\big) = 0.$$

The point $\gamma(t_0)$ is said to be *conjugate* to $\gamma(0)$ along geodesic γ, if there exists a Jacobi field J along γ, not identically zero, with $J(0) = 0 = J(t_0)$. We have the stable Jacobi tensors of Finsler manifolds without conjugate points as in the case of Riemannian manifolds (see [12]). There are Jacobi tensors, A_v, which satisfy

- The Jacobi equation
$$A_v''(t) + R_v(t) \cdot A_v(t) = 0,$$
where $R_v(t)$ is the linear representation of $R_{\gamma_v'(t)}$ with respect to $g_{\gamma_v'(t)}$;
- $A_v'(t) \cdot A_v^{-1}(t)$ is symmetric whenever A_v^{-1} exists;
- $A_v(t)$ is nondegenerate in the sense that the $\ker A_v(t) \cap \ker A_v'(t) = \varnothing$. Furthermore, it is not hard to check that
$$A_v(t) = J_v(t) \int_0^t J_v^{-1}(s) J_v^{-1^*}(s) ds \tag{3.1}$$
defines the solution with $J_v(0) = \mathrm{Id}$ as long as $J_v(s)$ is nonsingular for $s \in [0, t]$ and called the *stable Jacobi tensor*.

Making the change of variables $U_v(t) := \big(\ln J_v(t)\big)' = J_v'(t) \cdot J_v^{-1}(t)$ for t values for which det $\big(J_v(t)\big) \neq 0$. In fact, if $\gamma_v(t)$ has no points conjugate to $\gamma(0)$ on $(0, \infty)$, then $U_v(t)$ is defined for all $t \in (0, \infty)$. Then the tensor U defined on the unit tangent bundle SM such that for every $v \in SM$, $U_v(t)$ is a self-adjoint linear operator on

$$\gamma_v'(t)^\perp := \{w \in T_{\gamma_v(t)}M \mid g_{\gamma_v'(t)}(\gamma_v'(t), w) = 0\}$$

and satisfies the *Riccati equation*

$$U_v'(t) + U_v^2(t) + R_v(t) = 0. \tag{3.2}$$

We will let $u(v) := \operatorname{tr}\big(U_v(0)\big)$ be the *mean curvaure* (the trace of the second fundamental form) of the stable horosphere through v in the universal covering space. We are in the position to state the result which is required to prove the main theorem.

Theorem 3.1. *([12, Théorème 2.5]) The measure entropy h of a compact Finsler manifold M without conjugate points satisfies the following equality*

$$h = \int_{SM} u(v)\, dV.$$

This theorem was proved for Riemannian metrics by Friere and Mañé [13], mechanical Lagrangians by Innami [14], and convex Hamiltonians by Contreras and Iturriaga [9].

Since $J_v'(t)J_v^{-1}(t) = U_v(t)$, we have

$$\frac{d}{dt}\ln\big[\det\big(J_v(t)\big)\big] = \operatorname{tr}\big(J_v'(t)J_v^{-1}(t)\big) = u\big(\varphi_t(v)\big).$$

Thus

$$\det\big(J_v(t)\big) = \exp\Big[\int_0^t u\big(\varphi_s(v)\big)ds\Big]. \tag{3.3}$$

Our proof of the main theorem is based on the following result. For the sake of completeness we sketch the proof.

Lemma 3.2. *([11, Lemma 1]) Let $A_v(t), J_v(t)$, and u be as above. Then for all $v \in SM$*

$$\big[\det\big(A_v(r)\big)\big]^{\frac{1}{n-1}} \geq \int_0^r \exp\Big\{\frac{1}{n-1}\Big[\int_0^r u\big(\varphi_t(v)\big)dt - 2\int_0^s u\big(\varphi_t(v)\big)dt\Big]\Big\}ds$$

with equality if and only if $J_v(t) = \sqrt{\lambda(t)}\,\mathrm{Id}$ for a function λ.

Proof. From [3, appendix D.14] we have that

$$\det\Big[\int_0^t J_v^{-1}(s)J_v^{-1^*}(s)ds\Big] \geq \Big\{\int_0^t \big[\det\big(J_v(s)\big)\big]^{\frac{-2}{n-1}}ds\Big\}^{n-1} \tag{3.4}$$

with equality if and only if $J_v^*(t)J_v(t) = \lambda(t)\,\mathrm{Id}$. Applying this to equations (3.1) and (3.3) yields

$$\begin{aligned}
\det\big(J_v(t)\big) &\geq \exp\Big[\int_0^r u\big(\varphi_t(v)\big)dt\Big]\Big[\int_0^r \exp\Big\{\frac{-2}{n-1}\int_0^s u\big(\varphi_t(v)\big)dt\Big\}ds\Big]^{n-1}\\
&= \Big(\int_0^r \exp\Big\{\frac{1}{n-1}\int_0^r u\big(\varphi_t(v)\big)dt\Big\}\exp\Big\{\frac{-2}{n-1}\int_0^s u\big(\varphi_t(v)\big)dt\Big\}ds\Big)^{n-1}\\
&= \Big(\int_0^r \exp\Big\{\frac{1}{n-1}\Big[\int_0^r u\big(\varphi_t(v)\big)dt - 2\int_0^s u\big(\varphi_t(v)\big)dt\Big]\Big\}ds\Big)^{n-1}.
\end{aligned}$$

If we have equality in (3.4) then $J_v^*(t)J_v(t) = \lambda(t)\,\mathrm{Id}$ and hence $J_v(t) = \sqrt{\lambda(t)}\,\mathrm{Id}$. $\square$

Now we are ready to prove main theorem using Theorem 3.1 and Lemma 3.2.

Theorem 3.3. *Let M be an n-dimensional compact Finsler manifold without conjugate points and let $\delta := \delta(h)$, then for every $r > 0$,*

$$\text{ave}\left[\mu\big(\widetilde{B}(x,r)\big)\right] \geq e^{-|S_{max}|} \cdot \mu\big(B_\delta(r)\big), \tag{3.5}$$

with equality holding if and only if M has constant flag curvature δ.

Proof. We use the scheme of [11, section 1], which is a modified version for Finsler manifolds. Let $\widetilde{S}(x,r)$ be the a metric sphere of radius r in the universal covering $\widetilde{M}$ centered at a point $\widetilde{x}$ that projects to x and $S_\delta(r)$ a metric sphere of r in M_δ^n. By definition (2.1) and Lemma 3.2, we have

$$\text{ave}\left[\mu\big(\widetilde{S}(x,r)\big)\right] = \frac{1}{\mu(M)} \int_{SM} e^{-\tau(v)} \det\big(A_v(r)\big) dV$$

$$\geq \mathfrak{c}_{n-1}\left[\int_{SM} e^{-\frac{\tau(v)}{n-1}} \int_0^r \exp\left\{\frac{1}{n-1}\left[\int_0^r u\big(\varphi_t(v)\big)dt - 2\int_0^s u\big(\varphi_t(v)\big)dt\right]\right\} ds dV\right]^{n-1}.$$

Hence by changing order of integration and the applying Jensen's inequality to the convex function exp, we obtain

$$\begin{aligned}
\left(\frac{\text{ave}\left[\mu\big(\widetilde{S}(x,r)\big)\right]}{\mathfrak{c}_{n-1}\cdot e^{-|S_{max}|}}\right)^{\frac{1}{n-1}} &\geq \int_0^r \left(\int_{SM} \exp\left\{\frac{1}{n-1}\left[\int_0^r u\big(\varphi_t(v)\big)dt - 2\int_0^s u\big(\varphi_t(v)\big)dt\right]\right\}dV\right)ds \\
&\geq \int_0^r \exp\left\{\int_{SM} \frac{1}{n-1}\left[\int_0^r u\big(\varphi_t(v)\big)dt - 2\int_0^s u\big(\varphi_t(v)\big)dt\right]dV\right\}ds. \qquad (3.6)
\end{aligned}$$

Now the fact that dV is invariant under the geodesic flow φ_t yields

$$\begin{aligned}
\int_{SM}\left[\int_0^r u\big(\varphi_t(v)\big)dt\right]dV &= \int_0^r \left[\int_{SM} u\big(\varphi_t(v)\big)dV\right]dt \\
&= \int_0^r \left[\int_{SM} u(v)dV\right]dt = r\cdot h.
\end{aligned}$$

Applying this to above inequality yields

$$\begin{aligned}
\left(\frac{\text{ave}\left[\mu\big(\widetilde{S}(x,r)\big)\right]}{\mathfrak{c}_{n-1}\cdot e^{-|S_{max}|}}\right)^{\frac{1}{n-1}} &\geq \int_0^r \exp\left\{\frac{1}{n-1}(r-2s)h\right\}ds \\
&= \int_0^r e^{(r-2s)\sqrt{-\delta}}ds \\
&= \begin{cases} r, & \text{if } \delta = 0; \\ \frac{1}{\sqrt{-\delta}\sinh(\sqrt{-\delta}r)}, & \text{if } \delta < 0. \end{cases}
\end{aligned}$$

Thus we see that $\text{ave}\left[\mu\big(\widetilde{S}(x,r)\big)\right] \geq e^{-|S_{max}|}\cdot\mu\big(S_\delta(r)\big)$ and get main theorem by integration.

If we have equality in (3.5) then for every $v \in SM$ and

$$0 \leq t \leq r, \tau\big(\varphi_t(v)\big) = \tau(v) + t|S_{max}|.$$

By the definition of S_{max}, $\tau\big(\varphi_t(v)\big) \leq \tau(v) + t|S_{max}|$ which implies that $S_{max} = 0$. If equality holds in (3.6) then we have equality in the Jensen's inequality for a.e. s and in Lemma 3.2 for a.e. v. Hence for a.e. s and a.e. v

$$(r-2s)(n-1)\sqrt{-\delta} = \int_0^r u\big(\varphi_t(v)\big)dt - 2\int_0^s u\big(\varphi_t(v)\big)dt$$

so $u(v) = (n-1)\sqrt{-\delta}$ for a.e. v. Equality in Lemma 3.2 says $J_v(t) = \sqrt{\lambda(t)}\,\mathrm{Id}$ for a function λ. By (3.3) and the above we see $\lambda(t) = e^{2t\sqrt{-\delta}}\,\mathrm{Id}$. Hence $J_v(t) = e^{t\sqrt{-\delta}}\,\mathrm{Id}$ for a.e. v and by the Riccati equation (3.2) we see that M has constant flag curvature δ. □

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