Ricci curvature for biwarped product submanifolds in Kenmotsu space forms

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Abstract. The main objective of this paper is to achieve the Ricci curvature inequality for biwarped product submanifold isometrically immersed in a Kenmotsu space form in the expressions of the squared norm of mean curvature vector and warping functions. The equality cases are likewise discussed. In particular, we also derive Chen-Ricci inequality for CR-warped product submanifolds and semi slant warped product submanifolds.

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1 Introduction

The acknowledgment of warped product manifolds appeared after the methodology of Bishop and O'Neill [5] on the manifolds of negative curvature. Analyzing the way that a Riemannian product of manifolds can not have negative curvature, they build the model of warped product manifolds for the class of manifolds of negative (or non-positive) curvature which is characterized as follows:

Let $(\hat{S}_1, \hat{g}_1)$ and $(\hat{S}_2, \hat{g}_2)$ be two Riemannian manifolds with Riemannian metrics $\hat{g}_1$ and $\hat{g}_2$ respectively and ψ be a positive differentiable function on $\hat{S}_1$. If $a : \hat{S}_1 \times \hat{S}_2 \to S_1$ and $b : \hat{S}_1 \times \hat{S}_2 \to \hat{S}_2$ are the projection maps given by $a(x, y) = x$ and $b(x, y) = y$ for every $(x, y) \in \hat{S}_1 \times \hat{S}_2$, then the *warped product manifold* is the product manifold $\hat{M} = \hat{S}_1 \times \hat{S}_2$ endowed with the Riemannian structure such that

$$\hat{g}(\hat{X}, \hat{Y}) = \hat{g}_1(x_*\hat{X}, x_*\hat{Y}) + (\psi \circ x)^2 \hat{g}_2(y_*\hat{X}, y_*\hat{Y}),$$

for all $\hat{X}, \hat{Y} \in T\hat{M}$. The function ψ is called the *warping function* of the warped product manifold. If the warping function is constant, then the warped product is trivial i.e., simply Riemannian product. Further, if $\hat{X} \in TS_1$ and $\hat{Z} \in TS_2$, then from Lemma 7.3 of [5], we have the following well known result

$$\nabla_{\hat{X}}\hat{Z} = \nabla_{\hat{Z}}\hat{X} = (\frac{\hat{X}\psi}{\psi})\hat{Z}, \tag{1.1}$$

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where ∇ is the Levi-Civita connection on $\hat{M}$. In the light of the fact that warped product manifolds concede various applications in Physics and theory of relativity [4], this has been a subject of broad research. Warped products give numerous essential solutions for Einstein field equations [4]. The idea of displaying of space-time close to black holes admits a type of warped product manifold [16]. Schwartzschild space-time is a model of warped product $U \times_r S^2$, where the base $U = R \times R^+$ is a half plane $r > 0$ and the fibre S^2 is the sphere of unit radius. Under some specific circumstances, the Schwartzchild space-time turns into the black hole. A cosmological model to show the universe as a space-time recognized as Robertson-Walker model which is a warped product [20].

Some common properties of warped product manifolds were concentrated in [5]. B. Y. Chen ([8], [10]) played out an outward investigation of warped product submanifolds in a Kaehler manifold. From that point forward, numerous geometers have investigated warped product manifolds in various settings like almost complex and almost contact manifolds and different existence results have been researched (see the survey article [11]).

In 1999, Chen [9], discovered a relationship between Ricci curvature and squared mean curvature vector for an arbitrary Riemannian manifold. On the line of Chen a series of articles have been appeared to formulate the relationship between Ricci curvature and squared mean curvature in the setting of some important structures on Riemannian manifolds (see [2], [24], [15], [19], [25], [1], [23]). Recently, Meraj Ali Khan, Ibrahim Al-Dael [18] established a relationship between Ricci curvature and squared mean curvature for warped product submanifolds.

In this paper, our aim is to obtain a relationship between Ricci curvature and squared mean curvature for biwarped product submanifolds in the setting of Kenmotsu space forms.

2 Preliminaries

A $(2n+1)-$dimensional $C^\infty-$manifold $\hat{M}$ is said to have an *almost contact structure* if on $\hat{M}$ there exist a tensor field $\hat{\phi}$ of type $(1,1)$, a vector field $\hat{\xi}$ and a 1-form $\hat{\eta}$ satisfying [6]

$$\hat{\phi}^2 = -I + \hat{\eta} \otimes \hat{\xi}, \quad \hat{\phi}\hat{\xi} = 0, \quad \hat{\eta} \circ \hat{\phi} = 0, \quad \hat{\eta}(\hat{\xi}) = 1. \tag{2.1}$$

The manifold $\hat{M}$ with the structure $(\hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})$ is called *almost contact metric manifold*. There always exists a Riemannian metric $\hat{g}$ on an almost contact metric manifold $\hat{M}$, satisfying the following conditions

$$\hat{\eta}(\hat{X}) = \hat{g}(\hat{X}, \hat{\xi}), \quad \hat{g}(\hat{\phi}\hat{X}, \hat{\phi}\hat{Y}) = \hat{g}(\hat{X}, \hat{Y}) - \hat{\eta}(\hat{X})\hat{\eta}(\hat{Y}), \tag{2.2}$$

for all $\hat{X}, \hat{Y} \in T\hat{M}$ where $T\hat{M}$ is the tangent bundle of $\hat{M}$.

The connected almost contact metric manifolds whose automorphism groups possess maximum dimensions were classified by S. Tanno [22]. If the sectional curvature of a plane section containing $\hat{\xi}$ is a constant say $\hat{c}$, he divided such manifolds into three classes:

(a) Homogenous normal contact Riemannian manifolds with constant $\hat{\phi}-$ holomorphic sectional curvature $\hat{c} > 0$.

(b) Global Riemannian product of a line (or a circle) and Kaehlerian manifold with constant holomorphic sectional curvature $\hat{c} = 0$.

(c) A warped product space $R \times_{\psi} C^n$ if $\hat{c} < 0$.

The manifolds of class (a) are characterized by some tensorial equations and admits *Sasakian structure* and the manifolds of class (b) are *cosymplectic manifolds*. K. Kenmotsu [17] obtained some tensorial equations for the manifolds of class (c), these manifolds are called *Kenmotsu manifolds*. Genarally, these structures are not Sasakian.

An almost contact metric structure $(\hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})$ is said to be *Kenmotsu manifold* if it satisfies the following tensorial equation [17]

$$(\hat{\nabla}_{\hat{X}}\hat{\phi})\hat{Y} = \hat{g}(\hat{\phi}\hat{X}, \hat{Y})\hat{\xi} - \hat{\eta}(\hat{Y})\hat{\phi}\hat{X}, \tag{2.3}$$

for any $\hat{X}, \hat{Y} \in T\hat{M}$, where $\hat{\nabla}$ denotes the Riemannian connection of the metric $\hat{g}$. Moreover, for a Kenmotsu manifold

$$\hat{\nabla}_{\hat{X}}\hat{\xi} = \hat{X} - \hat{\eta}(\hat{X})\hat{\xi}. \tag{2.4}$$

A Kenmotsu manifold $\hat{M}$ is said to be a *Kenmotsu space form* [17] if it has constant $\hat{\phi}$-holomorphic sectional curvature $\hat{c}$ and is denoted by $\hat{M}(\hat{c})$. The curvature tensor $\hat{R}$ of Kenmotsu space form $\hat{M}(\hat{c})$ is given by

(2.5)
$$\begin{aligned}
\hat{R}(\hat{X}, \hat{Y}, \hat{Z}, \hat{W}) =& \frac{\hat{c}-3}{4}\{\hat{g}(\hat{Y}, \hat{Z})\hat{g}(\hat{X}, \hat{W}) - \hat{g}(\hat{X}, \hat{Z})\hat{g}(\hat{Y}, \hat{W})\} \\
&+ \frac{\hat{c}+1}{4}\{\hat{g}(\hat{X}, \hat{\phi}\hat{Z})\hat{g}(\hat{\phi}\hat{Y}, \hat{W}) - \hat{g}(\hat{Y}, \hat{\phi}\hat{Z})\hat{g}(\hat{\phi}\hat{X}, \hat{W}) \\
&+ 2\hat{g}(\hat{X}, \hat{\phi}\hat{Y})\hat{g}(\hat{\phi}\hat{Z}, \hat{W}) + \hat{\eta}(\hat{X})\hat{\eta}(\hat{Z})\hat{g}(\hat{Y}, \hat{W}) - \hat{\eta}(\hat{Y})\hat{\eta}(\hat{Z})\hat{g}(\hat{X}, \hat{W}) \\
&+ \hat{\eta}(\hat{Y})\hat{\eta}(\hat{W})\hat{g}(\hat{X}, \hat{Z}) - \hat{\eta}(\hat{X})\hat{\eta}(\hat{W})\hat{g}(\hat{Y}, \hat{Z})\},
\end{aligned}$$

for any vector fields $\hat{X}, \hat{Y}, \hat{Z}, \hat{W}$ on $\hat{M}$.

Let $\check{M}$ be an $n-$dimensional Riemannian manifold isometrically immersed in a $m-$dimensional Riemannian manifold $\hat{M}$. Then the Gauss and Weingarten formulas are $\hat{\nabla}_{\check{X}}\check{Y} = \check{\nabla}_{\check{X}}\check{Y} + h(\check{X}, \check{Y})$ and $\hat{\nabla}_{\check{X}}N = -A_N\check{X} + \nabla^{\perp}_{\check{X}}N$ respectively, for all $\check{X}, \check{Y} \in T\check{M}$ and $N \in T^{\perp}\check{M}$, where $\check{\nabla}$ is the induced Levi-Civita connection on $\check{M}$, N is a vector field normal to $\check{M}$, h is the second fundamental form of $\check{M}$, $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}\check{M}$ and A_N is the shape operator of the second fundamental form. The second fundamental form h and the shape operator are associated by the following formula

$$\hat{g}(h(\check{X}, \check{Y}), N) = \hat{g}(A_N\check{X}, \check{Y}). \tag{2.6}$$

The equation of Gauss is given by

$$\check{R}(\check{X}, \check{Y}, \check{Z}, \check{W}) = \hat{R}(\check{X}, \check{Y}, \check{Z}, \check{W}) + \hat{g}(h(\check{X}, \check{W}), h(\check{Y}, \check{Z})) - \hat{g}(h(\check{X}, \check{Z}), h(\check{Y}, \check{W})), \tag{2.7}$$

for all $\check{X}, \check{Y}, \check{Z}, \check{W} \in T\check{M}$, where $\hat{R}$ and $\check{R}$ are the curvature tensors of $\hat{M}$ and $\check{M}$ respectively.

For any $\check{X} \in T\check{M}$ and $N \in T^{\perp}\check{M}$, $\hat{\phi}\check{X}$ and $\hat{\phi}N$ can be decomposed as follows

$$\hat{\phi}\check{X} = P\check{X} + F\check{X} \tag{2.8}$$

and

$$\hat{\phi}N = tN + fN, \tag{2.9}$$

where $P\check{X}$ (resp. tN) is the tangential and $F\check{X}$ (resp. fN) is the normal component of $\hat{\phi}\check{X}$ (resp. ϕN).

Definition 2.1. [7] A submanifold $\check{M}$ of an almost contact metric manifold $\hat{M}$ is said to be *slant submanifold* if for any $x \in \check{M}$ and $\check{X} \in T_x\check{M} - \langle\check{\xi}\rangle$, the angle between $\check{X}$ and $\hat{\phi}\check{X}$ is constant. The constant angle $\theta \in [0, \pi/2]$ is then called *slant angle* of $\check{M}$ in $\hat{M}$. If $\theta = 0$, the submanifold is *invariant submanifold* and if $\theta = \pi/2$ then it is *anti-invariant submanifold.* If $\theta \neq 0, \pi/2$, it is *proper slant submanifold.*

For slant submanifolds of the contact metric manifolds J. L. Cabrerizo et al. [7] proved the following lemma.

Lemma 2.1. *Let $\check{M}$ be a submanifold of an almost contact metric manifold $\hat{M}$ such that $\check{\xi} \in T\check{M}$, then $\check{M}$ is a slant submanifold if and only if there exist a constant $\lambda \in [0, 1]$ such that*

$$P^2 = \lambda(I - \hat{\eta} \otimes \check{\xi}), \tag{2.10}$$

where $\lambda = -\cos^2\theta$.

Thus, one has the following consequences of the above formula.

$$\hat{g}(P\check{X}, P\check{Y}) = \cos^2\theta[\hat{g}(\check{X}, \check{Y}) - \hat{\eta}(\check{X})\hat{\eta}(\check{Y})], \tag{2.11}$$

$$\hat{g}(F\check{X}, F\check{Y}) = \sin^2\theta[\hat{g}(\check{X}, \check{Y}) - \hat{\eta}(\check{X})\hat{\eta}(\check{Y})] \tag{2.12}$$

for all $\check{X}, \check{Y} \in T\check{M}$.

For any orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space $T_x\check{M}$, the mean curvature vector $\check{H}(x)$ and its squared norm are defined as follows

$$\check{H}(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \quad \|\check{H}\|^2 = \frac{1}{n^2}\sum_{i,j=1}^{n} \hat{g}(h(e_i, e_i), h(e_j, e_j)), \tag{2.13}$$

where n is the dimension of $\check{M}$. If $h = 0$ then the submanifold is said to be totally geodesic and minimal if $\check{H} = 0$. If $h(\check{X}, \check{Y}) = \hat{g}(\check{X}, \check{Y})\check{H}$ for all $\check{X}, \check{Y} \in T\check{M}$, then $\check{M}$ is called totally umbilical.

The scalar curvature of $\hat{M}$ is denoted by $\hat{\tau}(\hat{M})$ and is defined as

$$\hat{\tau}(\hat{M}) = \sum_{1 \leq p < q \leq m} \hat{\kappa}_{pq}, \tag{2.14}$$

where $\hat{\kappa}_{pq} = \hat{\kappa}(e_p \wedge e_q)$ and m is the dimension of the Riemannian manifold $\hat{M}$. Throughout this study, we shall use the equivalent version of the above equation, which is given by

$$2\hat{\tau}(\hat{M}) = \sum_{1 \leq p < q \leq m} \hat{\kappa}_{pq}. \tag{2.15}$$

In a similar way, the scalar curvature $\hat{\tau}(L_x)$ of a $L-$plane is given by

$$\hat{\tau}(L_x) = \sum_{1 \leq p < q \leq m} \hat{\kappa}_{pq}. \tag{2.16}$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space $T_x\check{M}$ and if e_r belongs to the orthonormal basis $\{e_{n+1}, \ldots e_m\}$ of the normal space $T^{\perp}\check{M}$, then we have

$$h^r_{pq} = \hat{g}(h(e_p, e_q), e_r) \tag{2.17}$$

and

$$\|h\|^2 = \sum_{p,q=1}^{n} \hat{g}(h(e_p, e_q), h(e_p, e_q)). \tag{2.18}$$

Let $\check{\kappa}_{pq}$ and $\hat{\kappa}_{pq}$ be the sectional curvatures of the plane sections spanned by e_p and e_q at x in the submanifold $\check{M}^n$ and in the Riemannian space form $\hat{M}^m(\hat{c})$, respectively. Thus by Gauss equation, we have

$$\check{\kappa}_{pq} = \hat{\kappa}_{pq} + \sum_{r=n+1}^{m} (h^r_{pp}h^r_{qq} - (h^r_{pq})^2). \tag{2.19}$$

The global tensor field for orthonormal frame of vector field $\{e_1, \ldots, e_n\}$ on $\check{M}^n$ is defined as

$$\hat{S}(\check{X}, \check{Y}) = \sum_{i=1}^{n} \{\hat{g}(\hat{R}(e_i, \check{Y})\check{Y}, e_i)\}, \tag{2.20}$$

for all $\check{X}, \check{Y} \in T_x\check{M}^n$. The above tensor is called the Ricci tensor. If we fix a distinct vector e_u from $\{e_1, \ldots, e_n\}$ on $\check{M}^n$, which is governed by $\check{\chi}$. Then the Ricci curvature is defined by

$$\check{R}ic(\check{\chi}) = \sum_{\substack{p=1 \\ p \neq u}}^{n} \check{\kappa}(e_p \wedge e_u). \tag{2.21}$$

For a smooth function ψ on a Riemannian manifold $\hat{M}$ with Riemannian metric $\hat{g}$, the gradient of ψ is denoted by $\nabla\psi$ and is defined as

$$\hat{g}(\nabla\psi, \check{X}) = \check{X}\psi, \tag{2.22}$$

for all $\check{X} \in T\check{M}$.

Let the dimension of $\check{M}$ is n and $\{e_1, e_2, \ldots, e_n\}$ be a basis of $T\check{M}$. Then as a result of (2.22), we get

$$\|\nabla\psi\|^2 = \sum_{i=1}^{n}(e_i(\psi))^2. \tag{2.23}$$

The Laplacian of ψ is defined by

$$\Delta\psi = \sum_{i=1}^{n}\{(\nabla_{e_i}e_i)\psi - e_ie_i\psi\}. \tag{2.24}$$

3 Biwarped product submanifolds of a Kenmotsu manifold

B. Y. Chen and F. Dillen [13] generalize the definition of warped product submanifold to multiply warped product manifolds as follows.

Let $\{\hat{N}_i\}$, $i = 1, 2, \ldots, k$ be Riemannian manifolds with respective Riemannian metrics $\{\hat{g}_i\}_{i=1,2,\ldots,k}$ and $\{f_i\}_{i=2,3,\ldots,k}$ are positive valued functions on $\hat{N}_1$. Then the product manifold $\hat{M} = \hat{N}_1 \times \hat{N}_2 \times \cdots \times \hat{N}_k$ endowed with the Riemannian metric $\hat{g}$ given by

$$\hat{g} = \pi_1^*(\hat{g}_1) + \sum_{i=2}^{k}(f_i \circ \pi_1)^2\pi_i^*(\hat{g}_i)$$

is called multiply warped product manifold and denoted by $\hat{M} = \hat{N}_1 \times_{f_2} \hat{N}_2 \times \cdots \times_{f_k} \hat{N}_k$ where $\pi_i (i = 1, 2, \ldots, k)$ are the projection maps of $\hat{M}$ onto $\hat{N}_i$ respectively. The functions f_i are known as the warping functions [13]. If the warping functions are constants, the warped product is simply Riemannian product of manifolds. These warped product have been discussed by Andreea Olteanu in [21]. As a particular case of multiply warped product manifolds, we can define biwarped product manifolds for $i = 3$. For $i = 2$, multiply warped product manifold reduces to single warped product manifold. Consider the biwarped product manifold $\hat{M} = \hat{N}_0 \times_{f_1} \hat{N}_1 \times_{f_2} \hat{N}_2$ with the Levi-Civita connection of $\hat{N}_i$ for $i = 0, 1, 2$. Now, we have the following result for biwarped product submanifold.

Lemma 3.1. [3] *Let $\hat{M} = \hat{N}_0 \times_{f_1} \hat{N}_1 \times_{f_2} \hat{N}_2$ be a biwarped product manifold. Then we have*

$$\hat{\nabla}_{\hat{X}}\hat{Z} = \hat{\nabla}_{\hat{Z}}\hat{X} = \hat{X}(lnf_i)\hat{Z} \tag{3.1}$$

for $\hat{X} \in T\hat{N}_0$ and $\hat{Z} \in T\hat{N}_i$, for $i = 1, 2$.

In this section, we consider $n-$ dimensional biwarped product submanifold $\check{M}^n = \check{N}_T^{n_1} \times_{f_2} \check{N}_\perp^{n_2} \times_{f_3} \check{N}_\theta^{n_3}$ of a Kenmotsu manifold $(\hat{M}, \hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})$ with warping functions f_2 and f_3 such that $\check{N}_T^{n_1}$, $\check{N}_\perp^{n_2}$ and $\check{N}_\theta^{n_3}$ are the n_1- dimensional invariant, n_2- dimensional anti-invariant and n_3- dimensional proper slant submanifolds of $\hat{M}$ respectively. To answer the question that ξ would be parallel to which of the factor of biwarped product submanifold, we have the following:

Theorem 3.2. *Let $\hat{M}$ be a Kenmotsu manifold. If $\check{N}_T^{n_1} \times_{f_2} \check{N}_\perp^{n_2} \times_{f_3} \check{N}_\theta^{n_3}$ be a biwarped product submanifold of $\hat{M}$ such that $\check{N}_T^{n_1}$, $\check{N}_\perp^{n_2}$ and $\check{N}_\theta^{n_3}$ are the invariant, anti-invariant and proper slant submanifolds of $\hat{M}$ respectively. Then, we have*

(i) if $\check{\xi}$ is tangent to $\check{N}_\perp^{n_2}$, f_2 is constant,

(ii) if $\check{\xi}$ is tangent to $\check{N}_\theta^{n_3}$, f_3 is constant.

Proof. First we suppose $\check{\xi}$ is tangent to $\check{N}_\perp^{n_2}$. Making use of Gauss formula and the fact that $\hat{\eta}(\check{X}) = 0$ if $\check{\xi} \in T\check{N}_\perp^{n_2}$, we get

$$\check{\nabla}_{\check{X}}\check{\xi} + h(\check{X}, \check{\xi}) = \check{X}.$$

Now, using (3.1) and taking inner product with $\check{\xi}$, we derive $\check{X}lnf_2 = 0$, i.e. f_2 is constant.
Similarly, we can prove the part (ii) by taking $\check{\xi}$ tangent to $\check{N}_\theta^{n_3}$. □

In view of the above result, we remark that there do not exist non-trivial biwarped product submanifolds of the type $\check{N}_T^{n_1} \times_{f_2} \check{N}_\perp^{n_2} \times_{f_3} \check{N}_\theta^{n_3}$ of a Kenmotsu manifold if the structure vector field $\check{\xi}$ is tangential to $\check{N}_\perp^{n_2}$ or $\check{N}_\theta^{n_3}$.

Now let $\check{M}^n = \check{N}_T^{n_1} \times_{f_2} \check{N}_\perp^{n_2} \times_{f_3} \check{N}_\theta^{n_3}$ be a biwarped product submanifold of a Kenmotsu manifold $\hat{M}$ and we consider the vector field $\check{\xi}$ tangent to $\check{N}_T^{n_1}$. If $\check{D}$ is invariant distribution, $\check{D}^\perp$ is anti-invariant and $\check{D}^\theta$ is proper slant distribution with the slant angle θ, then the tangent bundle $T\check{M}$ of $\check{M}$ has the following decomposition

$$T\check{M}^n = \check{D} \oplus \check{D}^\perp \oplus \check{D}^\theta \oplus \langle\check{\xi}\rangle.$$

The normal bundle $T^\perp\check{M}$ is decompounded as

$$T^\perp\check{M}^n = \phi\check{D}^\perp \oplus F\check{D}^\theta \oplus \mu,$$

where μ is the orthogonal complementary distribution of $\phi\check{D}^\perp \oplus F\check{D}^\theta$ in $T^\perp\check{M}$. It is easy to see that μ is an invariant subbundle of $T^\perp\check{M}$ with respect to $\hat{\phi}$.

For a biwarped product submanifold $\check{M}^n = \check{N}_T^{n_1} \times_{f_2} \check{N}_\perp^{n_2} \times_{f_3} \check{N}_\theta^{n_3}$ of a Riemannian manifold from equation (3.5) of [13] one can conclude the following result

$$n_2\frac{\Delta f_2}{f_2} + n_3\frac{\Delta f_3}{f_3} = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\check{\kappa}(e_i, e_j) + \sum_{i=1}^{n_1}\sum_{k=1}^{n_3}\check{\kappa}(e_i, e_k). \tag{3.2}$$

Now, we have the following initial result.

Lemma 3.3. *Let $\check{M}^n = \check{N}_T^{n_1} \times_{f_2} \check{N}_\perp^{n_2} \times_{f_3} \check{N}_\theta^{n_3}$ be a biwarped product submanifold isometrically immersed in a Kenmotsu manifold $\hat{M}$. Then*

(i) $\hat{g}(h(\check{X}, \check{Y}), F\check{Z}) = 0$,

(ii) $\hat{g}(h(\check{X}, \check{Y}), \hat{\phi}\check{U}) = 0$,

(iii) $\hat{g}(h(\hat{\phi}\check{X}, \hat{\phi}\check{X}), N) = -\hat{g}(h(\check{X}, \check{X}), N)$,

for any $\check{X}, \check{Y} \in T\check{N}_T^{n_1}$, $\check{U} \in T\check{N}_\perp^{n_2}$, $\check{Z} \in T\check{N}_\theta^{n_3}$ and $N \in \mu$.

Proof. By using Gauss and Weingarten formulae in equation (2.3), we have

$$\check{\nabla}_{\check{X}} P\check{Z} + h(\check{X}, P\check{Z}) - A_{F\check{Z}}\check{X} + \nabla^{\perp}_{\check{X}} F\check{Z} + \phi\check{\nabla}_{\check{X}}\check{Z} + \check{\nabla}_{\check{Z}}\phi\check{X} + \phi h(\check{X}, \check{Z}) = 0,$$

taking inner product with $\check{Y}$ and using (3.1), we get the required result. In a similar way, we can prove the part (ii).
To prove (iii), for any $\check{X} \in T\check{N}_T$ we have

$$\hat{\nabla}_{\check{X}}\hat{\phi}\check{X} = -\hat{\eta}(\check{X})\hat{\phi}\check{X} + \hat{\phi}\hat{\nabla}_{\check{X}}\check{X},$$

using Gauss formula and (2.3), we get

$$\check{\nabla}_{\check{X}}\hat{\phi}\check{X} + h(\hat{\phi}\check{X}, \check{X}) = -\hat{\eta}(\check{X})\hat{\phi}\check{X} + \phi\check{\nabla}_{\check{X}}\check{X} + \hat{\phi}h(\check{X}, \check{X}),$$

and taking inner product with $\hat{\phi}N$, the above equation yields

$$\hat{g}(h(\hat{\phi}\check{X}, \check{X}), \hat{\phi}N) = \hat{g}(h(\check{X}, \check{X}), N). \tag{3.3}$$

By interchanging $\check{X}$ by $\hat{\phi}\check{X}$ the above equation gives

$$\hat{g}(h(\hat{\phi}\check{X}, \check{X}), \hat{\phi}N) = -\hat{g}(h(\hat{\phi}\check{X}, \hat{\phi}\check{X}), N). \tag{3.4}$$

From (3.3) and (3.4), we get the required result. □

Definition 3.1 The warped product $\check{N}_1 \times_{f_2} \check{N}_2 \times_{f_3} \check{N}_3$ isometrically immersed in a Riemannian manifold $\hat{M}$ is called $\check{N}_i$ totally geodesic if the partial second fundamental form h_i vanishes identically. It is called $\check{N}_i$-minimal if the partial mean curvature vector $\check{H}^i$ becomes zero for $i = 1, 2, 3$.

From Lemma 3.2 it is evident that the isometric immersion $\check{N}_T^{n_1} \times_{f_2} \check{N}_{\perp}^{n_2} \times_{f_3} \check{N}_{\theta}^{n_3}$ into a Kenmotsu manifold is $\check{D}-$ minimal. The $\check{D}-$ minimality property provides us a useful relationship between the biwarped product submanifold $\check{N}_T^{n_1} \times_{f_2} \check{N}_{\perp}^{n_2} \times_{f_3} \check{N}_{\theta}^{n_3}$ and the equation of Gauss.

Let $\{e_1, \ldots, e_p, e_{p+1} = \hat{\phi}e_1, \ldots, e_{n_1-1} = \hat{\phi}e_p, e_{n_1} = \check{\xi}, e_{n_1+1}, \ldots, e_{n_2}, e_{n_2+1} = e^1, \ldots, e_{n_2+q} = e^q, e_{n_2+q+1} = e^{q+1} = \sec\theta Pe^1, \ldots e_{(n_3=2q)} = e^{n_3} = \sec\theta Pe^q\}$ be a local orthonormal frame of vector fields on the biwarped product submanifold $\check{N}_T^{n_1} \times_{f_2} \check{N}_{\perp}^{n_2} \times_{f_3} \check{N}_{\theta}^{n_3}$ such that the set $\{e_1, \ldots, e_p, e_{p+1} = \hat{\phi}e_1, \ldots, e_{n_1-1} = \hat{\phi}e_p, e_{n_1} = \check{\xi}\}$ is tangent to $\check{N}_T$, the set $\{e_{n_1+1}, \ldots, e_{n_2}\}$ is tangent to $\check{N}_{\perp}^{n_2}$ and the set $\{e_{n_2+1}, \ldots, e_{n_2+q}, \ldots e^{n_3}\}$ is tangent to $\check{N}_{\theta}^{n_3}$. Moreover, $\{e_{n+1} = \hat{\phi}e_{n_1+1}, \ldots e_{n+n_2} = \hat{\phi}e_{n_2}, e_{n+n_2+1} = \csc\theta Fe^1, \ldots, e_{n+n_3} = \csc\theta Fe^q, e_{n+n_2+n_3+1} = \bar{e}^1, \ldots, e_m = \bar{e}^k\}$ is a basis for the normal bundle $T^{\perp}\check{M}$, such that the sets $\{e_{n+1} = \hat{\phi}e_{n_1+1}, \ldots e_{n+n_2} = \hat{\phi}e_{n_2}\}$ is tangent to $\hat{\phi}\check{D}^{\perp}$, $\{e_{n+1} = \csc\theta Fe^1, \ldots, e_{n+n_2} = \csc\theta Fe^q\}$ is tangent to $F\check{D}^{\theta}$ and $\{\bar{e}^1, \ldots, \bar{e}^k\}$ is tangent to the complementary invariant subbundle μ with even dimension k.

From Lemma 3.3, it is easy to conclude that

$$\sum_{r=n+1}^{m}\sum_{i,j=1}^{n_1} g(h(e_i, e_j), e_r) = 0. \tag{3.5}$$

Thus it follows that the trace of h due to $\check{N}_T^{n_1}$ becomes zero. Hence in view of the Definition 3.1, we obtain the following important result.

Theorem 3.4. *Let $\check{M}^n = \check{N}_T^{n_1} \times_{f_2} \check{N}_\perp^{n_2} \times_{f_3} \check{N}_\theta^{n_3}$ be a biwarped product submanifold isometrically immersed in a Kenmotsu manifold $\hat{M}$. Then $\check{M}^n$ is $\check{D}-$ minimal.*

So, it is easy to conclude the following

$$\|\check{H}\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{m} (h^r_{n_1+1 n_1+1} + \cdots + h^r_{n_2 n_2} + \cdots + h^r_{nn})^2, \tag{3.6}$$

where $\|\check{H}\|^2$ is the squared mean curvature.

4 Ricci curvature Inequalities for biwarped product submanifolds

In this section, we investigate Ricci curvature in terms of the squared norm of mean curvature and the warping functions as follows.

Theorem 4.1. *Let $\check{M}^n = \check{N}_T^{n_1} \times_{f_2} \check{N}_\perp^{n_2} \times_{f_3} \check{N}_\theta^{n_3}$ be a biwarped product submanifold isometrically immersed in a Kenmotsu space form $\hat{M}^m(\hat{c})$. Then for each orthogonal unit vector field $\check{\xi} \neq \check{\chi} \in T_x\check{M}$, either tangent to $\check{N}_T^{n_1}$, $\check{N}_\perp^{n_2}$ or $\check{N}_\theta^{n_3}$, we have*

(1) *The Ricci curvature satisfy the following inequalities*

(i) If $\check{\chi}$ is tangent to $\check{N}_T^{n_1}$, then

$$\begin{aligned} \frac{1}{4}n^2\|\check{H}\|^2 \geq & \check{R}ic(\check{\chi}) + \frac{n_2\Delta f_2}{f_2} + \frac{n_3\Delta f_3}{f_3} + \frac{\hat{c}+1}{4}(n - n_1 - \frac{1}{2}) \\ & - \frac{\hat{c}-3}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 1). \end{aligned} \tag{4.1}$$

(ii) If $\check{\chi}$ is tangent to $\check{N}_\perp^{n_2}$, then

$$\begin{aligned} \frac{1}{4}n^2\|\check{H}\|^2 \geq & \check{R}ic(\check{\chi}) + \frac{n_2\Delta f_2}{f_2} + \frac{n_3\Delta f_3}{f_3} + \frac{\hat{c}+1}{4}(n - n_1 + 1) \\ & - \frac{\hat{c}-3}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 1). \end{aligned} \tag{4.2}$$

(iii) $\check{\chi}$ is tangent to $\check{N}_\theta^{n_2}$, then

$$\begin{aligned} \frac{1}{4}n^2\|\check{H}\|^2 \geq & \check{R}ic(\check{\chi}) + \frac{n_2\Delta f_2}{f_2} + \frac{n_3\Delta f_3}{f_3} + \frac{\hat{c}+1}{4}(n - n_1 + 1 - \frac{3}{2}\cos^2\theta) \\ & - \frac{\hat{c}-3}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 1). \end{aligned} \tag{4.3}$$

(2) *If $\check{H}(x) = 0$ for each point $x \in \check{M}^n$, then there is a unit vector field $\check{\chi}$ which satisfies the equality case of (1) if and only if $\check{M}^n$ is mixed totally geodesic and $\check{\chi}$ lies in the relative null space $\mathcal{N}_x$ at x.*

(3) *For the equality case we have*

(*a*) *The equality case of* (4.1) *holds identically for all unit vector fields tangent to* $\check{N}_T^{n_1}$ *at each* $x \in \check{M}^n$ *if and only if* $\check{M}^n$ *is mixed totally geodesic and* $\check{D}-$*totally geodesic biwarped product submanifold in* $\hat{M}^m(\hat{c})$.

(*b*) *The equality case of* (4.2) *holds identically for all unit vector fields tangent to* $\check{N}_\perp^{n_2}$ *at each* $x \in \check{M}^n$ *if and only if* $\check{M}$ *is mixed totally geodesic and either* $\check{M}^n$ *is* $\check{D}^\perp$*- totally geodesic biwarped product or* $\check{M}^n$ *is a* $\check{D}^\perp$ *totally umbilical in* $\hat{M}^m(\hat{c})$ *with dim* $\check{D}^\perp = 2$.

(*c*) *The equality case of* (4.3) *holds identically for all unit vector fields tangent to* $\check{N}_\theta$ *at each* $x \in \check{M}^n$ *if and only if* $\check{M}$ *is mixed totally geodesic and either* $\check{M}^n$ *is* $\check{D}^\theta$*- totally geodesic biwarped product submanifold or* $\check{M}^n$ *is a* $\check{D}^\theta$ *totally umbilical in* $\hat{M}^m(\hat{c})$ *with dim* $\check{D}^\theta = 2$.

(*d*) *The equality case of* (1) *holds identically for all unit tangent vectors to* $\check{M}^n$ *at each* $x \in \check{M}^n$ *if and only if either* $\check{M}^n$ *is totally geodesic submanifold or* $\check{M}^n$ *is a mixed totally geodesic totally umbilical and* $\check{D}-$ *totally geodesic submanifold with dim* $\check{N}_\theta = 2$ *and dim* $\check{N}_\perp = 2$,

where n_1, n_2 *and* n_3 *are the dimensions of* $\check{N}_T^{n_1}$, $\check{N}_\perp^{n_2}$ *and* $\check{N}_\theta^{n_3}$ *respectively.*

Proof. Suppose that $\check{M}^n = \check{N}_T^{n_1} \times_{f_2} \check{N}_\perp^{n_2} \times_{f_3} \check{N}_\theta^{n_3}$ be a biwarped product submanifold of a Kenmotsu complex space form. From Gauss equation, we have

$$n^2\|\check{H}\|^2 = 2\tau(\check{M}^n) + \|h\|^2 - 2\hat{\tau}(\check{M}^n). \tag{4.4}$$

Let $\{e_1, \dots, e_{n_1}, e_{n_1+1}, \dots, e_{n_2}, \dots e_n\}$ be a local orthonormal frame of vector fields on $\check{M}^n$ such that $\{e_1, \dots, e_{n_1}\}$ are tangent to $\check{N}_T^{n_1}$, $\{e_{n_1+1}, \dots, e_{n_2}\}$ are tangent to $\check{N}_\perp^{n_2}$ and $\{e_{n_2+1}, \dots, e_n\}$ are tangent to $\check{N}_\theta^{n_3}$. So, the unit tangent vector $\check{\chi} = e_A \in \{e_1, \dots, e_n\}$ can be expanded (4.4) as follows

$$n^2\|\check{H}\|^2 = 2\check{\tau}(\check{M}^n) + \frac{1}{2}\sum_{r=n+1}^{m}\{(h_{11}^r + \dots h_{n_2n_2}^r + \dots + h_{nn}^r - h_{AA}^r)^2 + (h_{AA}^r)^2\}$$

$$-\sum_{r=n+1}^{m}\sum_{1\le i\ne j\le n} h_{ii}^r h_{jj}^r - 2\hat{\tau}(\check{M}^n). \tag{4.5}$$

The above expression can be written as follows

$$\begin{aligned} n^2\|\check{H}\|^2 = 2\check{\tau}(\check{M}^n) &+ \frac{1}{2}\sum_{r=n+1}^{m}\{(h_{11}^r + \dots h_{n_2n_2}^r + \dots + h_{nn}^r)^2 \\ &+ (2h_{AA}^r - (h_{11}^r + \dots + h_{nn}^r))^2\} + 2\sum_{r=n+1}^{m}\sum_{1\le i<j\le n}(h_{ij}^r)^2 \\ &- 2\sum_{r=n+1}^{m}\sum_{\substack{1\le i<j\le n\\ i,j\ne A}} h_{ii}^r h_{jj}^r - 2\hat{\tau}(\check{M}^n). \end{aligned}$$

In view of the Lemma 3.3, the preceding expression takes the form

$$
\begin{aligned}
n^2\|\check{H}\|^2 = {} & 2\check{\tau}(\check{M}^n) + \frac{1}{2}\sum_{r=n+1}^{m}\{(h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn})^2 \\
& + \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{AA} - (h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn}))^2 \\
& + 2\sum_{r=n+1}^{m}\sum_{1\le i<j\le n}(h^r_{ij})^2 - 2\sum_{r=n+1}^{m}\sum_{\substack{1\le i<j\le n\\ i,j\ne A}} h^r_{ii}h^r_{jj} - 2\hat{\tau}(\check{M}^n).
\end{aligned} \tag{4.6}
$$

Considering unit tangent vector $\check{\chi} = e_A$, we have three choices $\check{\chi}$ is either tangent to the base manifold $\check{N}^{n_1}_T$ or to the fibers $\check{N}^{n_2}_\perp$ and $\check{N}^{n_3}_\theta$.

Case 1: If $\check{\chi}$ is tangent to $\check{N}^{n_1}_T$, then we need to choose a unit vector field from $\{e_1, \dots, e_{n_1}\}$. Let $\check{\chi} = e_1$. Then from (2.21) and (3.5) we have

$$
\begin{aligned}
n^2\|\check{H}\|^2 \ge {} & \check{R}ic(\check{\chi}) + \frac{1}{2}\sum_{r=n+1}^{m}\{(h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn})^2 + \frac{n_2\Delta f_2}{f_2} \\
& + \frac{n_3\Delta f_3}{f_3} + \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{11} - (h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn}))^2 \\
& + \sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le n_1}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2) \\
& + \sum_{r=n+1}^{m}\sum_{n_1+1\le p<q\le n_2}(h^r_{pp}h^r_{qq} - (h^r_{pq})^2) \\
& + \sum_{r=n+1}^{m}\sum_{n_2+1\le s<t\le n}(h^r_{ss}h^r_{tt} - (h^r_{st})^2) \\
& + \sum_{r=n+1}^{m}\sum_{1\le i<j\le n}(h^r_{ij})^2 - \sum_{r=n+1}^{m}\sum_{2\le i<j\le n}(h^r_{ii}h^r_{jj}) \\
& - 2\hat{\tau}(\check{M}) + \sum_{2\le i<j\le n}\hat{\kappa}(e_i, e_j) + \hat{\tau}(\check{N}^{n_1}_T) + \hat{\tau}(\check{N}^{n_2}_\perp) + \hat{\tau}(\check{N}^{n_3}_\theta).
\end{aligned} \tag{4.7}
$$

Substituting $X, W = e_i, Y, Z = e_j$ in the formula (2.5), we have

$$
2\hat{\tau}(\check{M}) = \frac{\hat{c}-3}{4}(n(n-1)) - \frac{\hat{c}+1}{4}(2(n-1) - 3(n_1-1) - 3n_3\cos^2\theta) \tag{4.8}
$$

$$
\sum_{2\le i<j\le n}\hat{\kappa}(e_i, e_j) = \frac{\hat{c}-3}{8}((n-1)(n-2)) - \frac{\hat{c}+1}{8}(2(n-2) - 3(n_1-2) - 3n_3\cos^2\theta)
$$

$$
\hat{\tau}(\check{N}^{n_1}_T) = \frac{\hat{c}-3}{8}(n_1(n_1-1) + 3n_1) - \frac{\hat{c}+1}{8}(2(n_1-1) - 3(n_1-1))
$$

$$
\hat{\tau}(\check{N}^{n_2}_\perp) = \frac{\hat{c}-3}{8}(n_2(n_2-1))
$$

$$\hat{\tau}(\check{N}_{\theta}^{n_3}) = \frac{\hat{c}-3}{8}(n_3(n_3-1)) - \frac{\hat{c}+1}{8}(-3n_3\cos^2\theta).$$

Using these values in (4.7), we get

$$\begin{aligned}
n^2\|\check{H}\|^2 \geq & \check{R}ic(\check{\chi}) + \frac{1}{2}n^2\|\check{H}\|^2 + \frac{1}{2}\sum_{r=n+1}^{m}(2h_{11}^r - (h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r))^2 \\
& + \frac{n_2\Delta f_2}{f_2} + \frac{n_3\Delta f_3}{f_3} + \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_2}(h_{ij}^r)^2 \\
& + \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{k=n_2+1}^{n}(h_{ik}^r)^2 + \sum_{r=n+1}^{m}\sum_{\beta=2}^{n_1}h_{11}^r h_{\beta\beta}^r \\
& - \sum_{r=n+1}^{m}\sum_{i=2}^{n_1}\sum_{j=n_1+1}^{n_2}h_{ii}^r h_{jj}^r - \sum_{r=n+1}^{m}\sum_{i=2}^{n_1}\sum_{k=n_2+1}^{n}h_{ii}^r h_{kk}^r \\
& - \frac{\hat{c}-3}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 1) + \frac{\hat{c}+1}{4}(n - n_1 - \frac{1}{2}).
\end{aligned} \tag{4.9}$$

In view of Lemma 3.1

$$\begin{aligned}
\sum_{r=n+1}^{m}\sum_{\beta=2}^{n_1}h_{11}^r h_{\beta\beta}^r &= \sum_{r=n+1}^{m}(h_{11}^r)^2 - \sum_{r=n+1}^{m}\sum_{i=2}^{n_1}\Big[\sum_{j=n_1+1}^{n_2}h_{ii}^r h_{jj}^r + \sum_{k=n_2+1}^{n}h_{ii}^r h_{kk}^r\Big] \\
&= \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h_{11}^r h_{jj}^r.
\end{aligned}$$

Utilizing in (4.9), we have

$$\begin{aligned}
n^2\|\check{H}\|^2 \geq & \check{R}ic(\check{\chi}) + \frac{1}{2}n^2\|\check{H}\|^2 + \frac{1}{2}\sum_{r=n+1}^{m}(2h_{11}^r - (h_{n_1+1n_1+1}^r + \cdots + h_{nn}^r))^2 \\
& + \frac{n_2\Delta f_2}{f_2} + \frac{n_3\Delta f_3}{f_3} + \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_2}(h_{ij}^r)^2 \\
& + \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{k=n_2+1}^{n}(h_{ik}^r)^2 - \sum_{r=n+1}^{m}(h_{11}^r)^2 + \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}h_{ii}^r h_{jj}^r \\
& - \frac{\hat{c}-3}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 1) + \frac{\hat{c}+1}{4}(n - n_1 - \frac{1}{2}).
\end{aligned} \tag{4.10}$$

The third term on the right hand side can be written as

$$\begin{aligned}
& \frac{1}{2}\sum_{r=n+1}^{m}(2h_{11}^r - (h_{n_1+1n_1+1}^r + \cdots + h_{n_2n_2}^r + \cdots + h_{nn}^r))^2 \\
& \quad = 2\sum_{r=n+1}^{m}(h_{11}^r)^2 + \frac{1}{2}n^2\|\check{H}\|^2 - 2\sum_{r=n+1}^{m}\Big[\sum_{j=n_1+1}^{n_2}h_{11}^r h_{jj}^r + \sum_{k=n_2+1}^{n}h_{11}^r h_{kk}^r\Big].
\end{aligned} \tag{4.11}$$

Combining above two expressions, we have

$$\begin{aligned}\frac{1}{2}n^2\|\check{H}\|^2 \geq &\check{R}ic(\check{\chi}) + \sum_{r=n+1}^{m}(h^r_{11})^2 - \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n} h^r_{11}h^r_{jj}\\ &+\frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1}+\cdots+h^r_{n_2n_2}+\cdots+h^r_{nn})^2\\ &+\sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}(h^r_{ij})^2+\frac{n_2\Delta f_2}{f_2}+\frac{n_3\Delta f_3}{f_3}\\ &-\frac{\hat{c}-3}{4}(n+n_1n_2+n_2n_3+n_1n_3-1)+\frac{\hat{c}+1}{4}(n-n_1-\frac{1}{2}).\end{aligned} \tag{4.12}$$

Or equivalently

$$\begin{aligned}\frac{1}{4}n^2\|\check{H}\|^2 \geq &\check{R}ic(\check{\chi}) + \frac{1}{4}\sum_{r=n+1}^{m}(2h^r_{11}-(h^r_{n_1+1n_1+1}+\cdots+h^r_{n_2n_2}+\cdots+h^r_{nn}))^2\\ &+\sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}(h^r_{ij})^2+\frac{n_2\Delta f_2}{f_2}+\frac{n_3\Delta f_3}{f_3}\\ &-\frac{\check{c}-3}{4}(n+n_1n_2+n_2n_3+n_1n_3-1)+\frac{\check{c}+1}{4}(n-n_1-\frac{1}{2}),\end{aligned} \tag{4.13}$$

which gives the inequality (4.1).

Case 2. If $\check{\chi}$ is tangent to $\check{N}^{n_2}_{\perp}$, we chose the unit vector from $\{e_{n_1+1},\ldots,e_{n_2}\}$. Suppose $\check{\chi}=e_{n_2}$, then from (4.6), we deduce

$$\begin{aligned}n^2\|\check{H}\|^2 \geq &\check{R}ic(\check{\chi}) + \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1}+\ldots h^r_{n_2n_2}+\cdots+h^r_{nn})^2+\frac{n_2\Delta f_2}{f_2}\\ &+\frac{n_3\Delta f_3}{f_3}+\frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1}+\ldots h^r_{n_2n_2}+\cdots+h^r_{nn})-2h^r_{n_2n_2})^2\\ &+\sum_{r=n+1}^{m}\sum_{1\leq\alpha<\beta\leq n_1}(h^r_{\alpha\alpha}h^r_{\beta\beta}-(h^r_{\alpha\beta})^2)+\sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n_2}(h^r_{ss}h^r_{tt}-(h^r_{st})^2)\\ &+\sum_{r=n+1}^{m}\sum_{n_2+1\leq p<q\leq n}(h^r_{pp}h^r_{qq}-(h^r_{pq})^2)+\sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}(h^r_{ij})^2\\ &-\sum_{r=n+1}^{m}\sum_{\substack{1\leq i<j\leq n\\ i,j\neq n_2}}(h^r_{ii}h^r_{jj})-2\hat{\tau}(\check{M})+\sum_{\substack{1\leq i<j\leq n\\ i,j\neq n_2}}\bar{\kappa}(e_i,e_j)\\ &+\hat{\tau}(\check{N}^{n_1}_T)+\hat{\tau}(\check{N}^{n_2}_{\perp})+\hat{\tau}(\check{N}^{n_3}_{\theta}).\end{aligned} \tag{4.14}$$

From (2.5) by putting $X, W = e_i, Y, Z = e_j$, one can compute

$$\sum_{\substack{1\leq i<j\leq n\\ i,j\neq n_2}} \hat{\kappa}(e_i, e_j) = \tfrac{\hat{c}-3}{8}((n-1)(n-2)) - \tfrac{\hat{c}+1}{8}(2(n-2) - 3(n_1-1) - 3n_3\cos^2\theta)$$
$$\hat{\tau}(\check{N}_T^{n_1}) = \tfrac{\hat{c}-3}{8}(n_1(n_1-1)) - \tfrac{\hat{c}+1}{8}(2(n_1-1) - 3(n_1-1))$$
$$\hat{\tau}(\check{N}_\perp^{n_2}) = \tfrac{\hat{c}-3}{8}(n_2(n_2-1))$$
$$\hat{\tau}(\check{N}_\theta^{n_3}) = \tfrac{\hat{c}-3}{8}(n_3(n_3-1)) - \tfrac{\hat{c}+1}{8}(-3n_3\cos^2\theta).$$

Using these values together with (4.8) in (4.14) and applying similar techniques as in Case 1, we obtain

$$\begin{aligned}
n^2\|\check{H}\|^2 \geq & \check{R}ic(\check{\chi}) + \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn}) - 2h^r_{n_2n_2}))^2 \\
& + \frac{1}{2}n^2\|\check{H}\|^2 + \frac{n_2\Delta f_2}{f_2} + \frac{n_3\Delta f_3}{f_3} + \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}(h^r_{ij})^2 \\
& + \sum_{r=n+1}^{m}[\sum_{t=n_1+1}^{n_2-1} h^r_{n_2n_2}h^r_{tt} + \sum_{l=n_2+1}^{n} h^r_{n_2n_2}h^r_{ll}] \\
& \sum_{r=1}^{m}\sum_{i=1}^{n_1}[\sum_{j=n_1+1}^{n_2-1} h^r_{ii}h^r_{jj} + \sum_{k=n_2+1}^{n} h^r_{ii}h^r_{kk}] \\
& - \frac{\hat{c}-3}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 1) + \frac{\hat{c}+1}{4}(n - n_1 + 1)
\end{aligned} \tag{4.15}$$

By Lemma 3.3, one can conclude

$$\sum_{r=1}^{m}\sum_{i=1}^{n_1}[\sum_{j=n_1+1}^{n_2-1} h^r_{ii}h^r_{jj} + \sum_{k=n_2+1}^{n} h^r_{ii}h^r_{kk}] = 0.$$

The second and seventh terms on right hand side of (4.15) can be solved as follows.

(4.16)

$$\begin{aligned}
& \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \dots + h^r_{nn}) - 2h^r_{n_2n_2}))^2 + \sum_{r=n+1}^{m}[\sum_{t=n_1+1}^{n_2-1} h^r_{n_2n_2}h^r_{tt} + \sum_{l=n_2+1}^{n} h^r_{n_2n_2}h^r_{ll}] \\
&= \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \dots + h^r_{nn})^2 + 2\sum_{r=n+1}^{m}(h^r_{n_2n_2})^2 \\
&- 2\sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n} h^r_{n_2n_2}h^r_{jj} + \sum_{r=n+1}^{m}\sum_{t=n_1+1}^{n} h^r_{n_2n_2}h^r_{tt} - \sum_{r=n+1}^{m}(h^r_{n_2n_2})^2 \\
&= \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \dots + h^r_{nn})^2 + \sum_{r=n+1}^{m}(h^r_{n_2n_2})^2 \\
&- \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n} h^r_{nn}h^r_{jj}.
\end{aligned}$$

Utilizing these two values in (4.15), we arrive

(4.17)

$$\begin{aligned}\frac{1}{2}n^2\|\check{H}\|^2 \geq& \check{R}ic(\check{\chi}) + \sum_{r=n+1}^{m}(h^r_{n_2n_2})^2 - \sum_{r=n+1}^{m}\sum_{i=n_1+1}^{n} h^r_{nn}h^r_{jj}\\ &+ \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \cdots + h^r_n n)^2 + \frac{1}{2}n^2\|\check{H}\|^2 + \frac{n_2\Delta f_2}{f_2} + \frac{n_3\Delta f_3}{f_3}\\ &- \frac{\hat{c}-3}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 1) + \frac{\hat{c}+1}{4}(n - n_1 + 1)\end{aligned}$$

By using similar steps as in Case 1, the above inequality can be written as

$$\begin{aligned}\frac{1}{4}n^2\|\check{H}\|^2 \geq& \check{R}ic(\chi) + \frac{1}{4}\sum_{r=n+1}^{m}(2h^r_{n_2n_2} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2\\ &+ \frac{n_2\Delta f_2}{f_2} + \frac{n_3\Delta f_3}{f_3} + \frac{\hat{c}-3}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 1)\\ &+ \frac{\hat{c}+1}{4}(n - n_1 + 1),\end{aligned} \tag{4.18}$$

The last inequality leads to inequality (4.2).
Case 3. If $\check{\chi}$ is tangent to $\check{N}^{n_3}_\theta$, then we choose the unit vector field from $\{e_{n_2+1}, \ldots, e_n\}$. Suppose the vector $\check{\chi}$ is e_n. Then from (4.6)

(4.19)

$$\begin{aligned}n^2\|\check{H}\|^2 \geq& \check{R}ic(\check{\chi}) + \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \ldots h^r_{n_2n_2} + \cdots + h^r_{nn})^2 + \frac{n_2\Delta f_2}{f_2}\\ &+ \frac{n_3\Delta f_3}{f_3} + \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \ldots h^r_{n_2n_2} + \cdots + h^r_{nn}) - 2h^r_{nn})^2\\ &+ \sum_{r=n+1}^{m}\sum_{1\leq\alpha<\beta\leq n_1}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2) + \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n_2}(h^r_{ss}h^r_{tt} - (h^r_{st})^2)\\ &+ \sum_{r=n+1}^{m}\sum_{n_2+1\leq p<q\leq n}(h^r_{pp}h^r_{qq} - (h^r_{pq})^2) + \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}(h^r_{ij})^2\\ &- \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n-1}h^r_{ii}h^r_{jj} - 2\hat{\tau}(M) + \sum_{1\leq i<j\leq n-1}\hat{\kappa}(e_i, e_j)\\ &+ \hat{\tau}(\check{N}^{n_1}_T) + \hat{\tau}(\check{N}^{n_2}_\perp) + \hat{\tau}(\check{N}^{n_3}_\theta).\end{aligned}$$

From (2.5), one can compute

$$\sum_{1\leq i<j\leq n-1}\hat{\kappa}(e_i, e_j) = \frac{\hat{c}-3}{8}((n-1)(n-2)) - \frac{\hat{c}+1}{8}(2(n-2) - 3(n_1-1) - 3(n_3-1)\cos^2\theta)$$

$$\hat{\tau}(\check{N}^{n_1}_T) = \frac{\hat{c}-3}{8}(n_1(n_1-1)) - \frac{\hat{c}+1}{8}(2(n_1-1) - 3(n_1-1))$$

$$\hat{\tau}(\check{N}_{\perp}^{n_2}) = \frac{\hat{c}-3}{8}(n_2(n_2-1))$$

$$\hat{\tau}(\check{N}_{\theta}^{n_3}) = \frac{\hat{c}-3}{8}(n_3(n_3-1)) - \frac{\hat{c}+1}{8}(-3n_3\cos^2\theta).$$

By usage of these values together with (4.8) in (4.19) and analogous to case 1 and case 2, we obtain

(4.20)

$$\begin{aligned}
n^2\|\check{H}\|^2 \geq & \check{R}ic(\check{\chi}) + \frac{1}{2}n^2\|\check{H}\|^2 + \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn}) - 2h^r_{nn})^2 \\
& + \frac{n_2\Delta f_2}{f_2} + \frac{n_3\Delta f_3}{f_3} + \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}(h^r_{ij})^2 \\
& + \sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n-1} h^r_{nn}h^r_{qq} - \sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n-1} h^r_{ii}h^r_{jj} \\
& - \frac{\hat{c}-3}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 1) + \frac{\hat{c}+1}{4}(n - n_1 + 1 - \frac{3}{2}\cos^2\theta)
\end{aligned}$$

On applying the Lemma 3.3, it is easy to verify

$$\sum_{r=n+1}^{m}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n-1} h^r_{ii}h^r_{jj} = 0. \tag{4.21}$$

Using in (4.20), we obtain

(4.22)

$$\begin{aligned}
n^2\|\check{H}\|^2 \geq & \check{R}ic(\chi) + \frac{1}{2}n^2\|H\|^2 + \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn}) - 2h^r_{nn})^2 \\
& + \frac{n_2\Delta f_2}{f_2} + \frac{n_3\Delta f_3}{f_3} + \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}(h^r_{ij})^2 + \sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n-1} h^r_{nn}h^r_{qq} \\
& - \frac{\hat{c}-3}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 1) + \frac{\hat{c}+1}{4}(n - n_1 + 1 - \frac{3}{2}\cos^2\theta)
\end{aligned}$$

The third and seventh terms on the right hand side of (4.22) in a similar way as in case 1 and case 2 can be simplified as

$$\begin{aligned}
& \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn}) - 2h^r_{nn})^2 + \sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n-1} h^r_{nn}h^r_{qq} \\
& = \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \dots h^r_{n_2n_2} + \dots + h^r_{nn})^2 + \sum_{r=n+1}^{m}(h^r_{nn})^2 \\
& - \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n} h^r_{nn}h^r_{jj}.
\end{aligned} \tag{4.23}$$

By combining (4.22) and (4.23) and using similar techniques as used in case 1 and case 2, we can derive

$$\begin{aligned}\frac{1}{4}n^2\|\check{H}\|^2 \geq \check{R}ic(\check{\chi}) + \frac{1}{4}\sum_{r=n+1}^{m}(2h^r_{nn} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2 \\ + \frac{n_2\Delta f_2}{f_2} + \frac{n_3\Delta f_3}{f_3} - \frac{c-3}{4}(n + n_1n_2 + n_2n_3 + n_1n_3 - 1) \\ + \frac{\hat{c}+1}{4}(n - n_1 + 1 - \frac{3}{2}\cos^2\theta)\end{aligned} \tag{4.24}$$

The last inequality leads to inequality (iii) in (1).

Next, we explore the equality cases of (1). First, we redefine the notion of the relative null space $\mathcal{N}_x$ of the submanifold M^n in the complex space form $\hat{M}^m(\hat{c})$ at any point $x \in \check{M}^n$, the relative null space was defined in [9], as follows

$$\mathcal{N}_x = \{\check{X} \in T_x\check{M}^n : h(\check{X}, \check{Y}) = 0, \forall \check{Y} \in T_x\check{M}^n\}.$$

For $A \in \{1, \ldots, n\}$ a unit vector field e_A tangent to $\check{M}^n$ at x satisfies the equality sign of (4.1) identically if and only if

$$(i)\ \sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0\ (ii)\ \sum_{b=1}^{n}\sum_{\substack{A=1\\ b\neq A}}^{n} h^r_{bA} = 0\ (iii)\ 2h^r_{AA} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{4.25}$$

such that $r \in \{n+1, \ldots m\}$ the condition (i) implies that M^n is mixed totally geodesic biwarped product submanifold. Combining statements (ii) and (iii) with the fact that $\check{M}^n$ is biwarped product submanifold, we get that the unit vector field $\check{\chi} = e_A$ belongs to the relative null space $\mathcal{N}_x$. The converse is trivial, this proves statement (2).

For a biwarped product submanifold, the equality sign of (4.1) holds identically for all unit tangent vector belong to $\check{N}_T$ at x if and only if

$$(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0\quad (ii)\sum_{b=1}^{n}\sum_{\substack{A=1\\ b\neq A}}^{n_1} h^r_{bA} = 0\quad (iii)\ 2h^r_{pp} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{4.26}$$

where $p \in \{1, \ldots, n_1\}$ and $r \in \{n+1, \ldots, m\}$. Since $\check{M}^n$ is biwarped product submanifold, the third condition implies that $h^r_{pp} = 0, \quad p \in \{1, \ldots, n_1\}$. Using this in the condition (ii), we conclude that $\check{M}^n$ is $\check{D}-$totally geodesic biwarped product submanifold in $\hat{M}^m(\hat{c})$ and mixed totally geodesicness follows from the condition (i), which proves (a) in the statement (3).

For a biwarped product submanifold, the equality sign of (4.2) holds identically for all unit tangent vector fields tangent to $\check{N}_\perp$ at x if and only if

$$(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0\ (ii)\sum_{b=1}^{n}\sum_{\substack{A=n_1+1\\ b\neq A}}^{n_2} h^r_{bA} = 0\ (iii)\ 2h^r_{KK} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{4.27}$$

such that $K \in \{n_1+1, \ldots, n_2\}$ and $r \in \{n+1, \ldots, m\}$. From the condition (iii) two cases emerge, that is

$$h^r_{KK} = 0, \quad \forall K \in \{n_1+1, \ldots, n_2\} \ \text{ and } \ r \in \{n+1, \ldots, m\} \ \text{ or } \ dim\ N_\perp = 2. \tag{4.28}$$

If the first case of (4.27) satisfies, then by virtue of condition (ii), it is easy to conclude that $\check{M}^n$ is a $\check{D}^{\perp}-$ totally geodesic biwarped product submanifold in $\hat{M}^m(c)$. This is the first case of part (b) of statement (3).

For a biwarped product submanifold, the equality sign of (4.3) holds identically for all unit tangent vector fields tangent to $\check{N}_{\theta}^{n_3}$ at x if and only if

$$(4.29)\qquad (i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}h^r_{pq}=0\ (ii)\sum_{b=1}^{n}\sum_{\substack{A=n_2+1\\ b\neq A}}^{n_3}h^r_{bA}=0\ (iii)\ 2h^r_{LL}=\sum_{q=n_1+1}^{n}h^r_{qq},$$

such that $L\in\{n_2+1,\dots,n\}$ and $r\in\{n+1,\dots,m\}$. From the condition (iii) two cases arise, that is

$$(4.30)\quad h^r_{LL}=0,\ \ \forall L\in\{n_2+1,\dots,n\}\ \text{ and }\ r\in\{n+1,\dots,m\}\ \text{ or }\ dim\ N_{\theta}=2.$$

If the first case of (4.29) satisfies, then by virtue of condition (ii), it is easy to conclude that $\check{M}^n$ is a $\check{D}^{\theta}-$ totally geodesic biwarped product submanifold in $\hat{M}^m(\hat{c})$. This is the first case of part (c) of statement (3).

For the other case, assume that $\check{M}^n$ is not $\check{D}^{\theta}-$totally geodesic biwarped product submanifold and dim $\check{N}_{\theta}=2$. Then condition (ii) of (4.29) implies that $\check{M}^n$ is $\check{D}^{\theta}-$ totally umbilical biwarped product submanifold in $\hat{M}(\hat{c})$, which is second case of this part. This verifies part (c) of (3).

To prove (d) using parts (a), (b) and (c) of (3), we combine (4.26),(4.27) and (4.29). For the first case of this part, assume that $dim\check{N}_{\perp}\neq 2$ and $dim\check{N}_{\theta}\neq 2$. Since from parts (a), (b) and (c) of statement (3) we conclude that $\check{M}^n$ is $\check{D}-$totally geodesic, $\check{D}^{\perp}-$ totally geodesic and $\check{D}^{\theta}-$ totally geodesic submanifolds in $\hat{M}^m(c)$. Hence $\check{M}^n$ is a totally geodesic submanifold in $\hat{M}^m(\hat{c})$.

On the other hand, suppose that first case does not satisfy. Then parts (a), (b) and (c) provide that $\check{M}^n$ is mixed totally geodesic and $\check{D}-$ totally geodesic submanifold of $\hat{M}^m(c)$ with $dim\check{N}_{\perp}=2$ and $dim\check{N}_{\theta}=2$. From the conditions (b) and (c) it follows that $\check{M}^n$ is $\check{D}^{\perp}-$ and $\check{D}^{\theta}-$totally umbilical biwarped product submanifolds and from (a) it is $\check{D}-$totally geodesic, which is part (d). This proves the theorem. □

If $N_{\perp}^{n_2}=\{0\}$, then the biwarped product submanifold becomes the semi-slant warped product submanifold that is $\check{M}^n=\check{N}_T^{n_1}\times_{f_2}\check{N}_{\theta}^{n_3}$. Now, we have the following corollary which can be deduced from the Theorem 4.1.

Corollary 4.2. *Let $\check{M}^n=\check{N}_T^{n_1}\times_{f_3}\check{N}_{\theta}^{n_3}$ be a semi-slant warped product submanifold isometrically immersed in a Kenmotsu space form $\hat{M}(\hat{c})$. Then for each orthogonal unit vector field $\check{\xi}\neq\check{\chi}\in T_x\check{M}$, either tangent to $\check{N}_T^{n_1}$ or $\check{N}_{\theta}^{n_3}$, we have*

(1) *The Ricci curvature satisfy the following inequalities*

(i) If $\check{\chi}$ is tangent to $\check{N}_T^{n_1}$, then

$$(4.31)\quad \frac{1}{4}n^2\|\check{H}\|^2\geq \check{R}ic(\check{\chi})+\frac{n_3\Delta f_3}{f_3}-\frac{\hat{c}-3}{4}(n+n_1n_3-1)+\frac{\hat{c}+1}{4}(n_3-\frac{1}{2}).$$

(ii) $\check{\chi}$ *is tangent to* $\check{N}_\theta^{n_2}$, *then*

$$\frac{1}{4}n^2\|\check{H}\|^2 \geq \check{R}ic(\chi) + \frac{n_3\Delta f_3}{f_3} - \frac{\hat{c}-3}{4}(n + n_1 n_3 - 1) + \frac{\hat{c}+1}{4}(n_3 + 1 - \frac{3}{2}\cos^2\theta). \tag{4.32}$$

(2) *If* $\check{H}(x) = 0$, *then each point* $x \in \check{M}^n$ *there is a unit vector field* $\check{\chi}$ *which satisfies the equality case of* (1) *if and only if* $\check{M}^n$ *is mixed totally geodesic and* $\check{\chi}$ *lies in the relative null space* $\mathcal{N}_x$ *at* x.

(3) *For the equality case we have*

(a) The equality case of (4.31) *holds identically for all unit vector fields tangent to* $\check{N}_T$ *at each* $x \in \check{M}^n$ *if and only if* $\check{M}^n$ *is mixed totally geodesic and* $\check{D}-$*totally geodesic point wise semi slant warped product submanifold in* $\hat{M}^m(c)$.

(b) The equality case of (4.32) *holds identically for all unit vector fields tangent to* $\check{N}_\theta^{n_3}$ *at each* $x \in \check{M}^n$ *if and only if* $\check{M}$ *is mixed totally geodesic and either* $\check{M}^n$ *is* $\check{D}^\theta$*- totally geodesic point wise semi slant warped product submanifold or* $\check{M}^n$ *is a* $\check{D}^\theta$ *totally umbilical in* $\hat{M}^m(\hat{c})$ *with dim* $\check{D}^\theta = 2$.

(c) The equality case of (1) *holds identically for all unit tangent vectors to* $\check{M}^n$ *at each* $x \in \check{M}^n$ *if and only if either* $\check{M}^n$ *is totally geodesic submanifold or* $\check{M}^n$ *is a mixed totally geodesic totally umbilical and* $\check{D}-$ *totally geodesic submanifold with dim* $\check{N}_\theta = 2$

where n_1 *and* n_3 *are the dimensions of* $\check{N}_T^{n_1}$ *and* $\check{N}_\theta^{n_3}$ *respectively.*

Now, we have another case that is if $\check{N}_\theta^{n_3} = \{0\}$ then the biwarped product submanifold becomes the contact CR-warped product submanifold. In this case we have the following corollary.

Corollary 4.3. *Let* $\check{M}^n = \check{N}_T^{n_1} \times_{f_2} \check{N}_\perp^{n_2}$ *be a contact CR-warped product submanifold isometrically immersed in a Kenmotsu space form* $\hat{M}^m(\hat{c})$. *Then for each orthogonal unit vector field* $\xi \neq \check{\chi} \in T_x\check{M}$, *either tangent to* $\check{N}_T^{n_1}$ *or* $\check{N}_\perp^{n_2}$, *we have*

(1) *The Ricci curvature satisfy the following inequalities*

(i) If $\check{\chi}$ *is tangent to* $\check{N}_T^{n_1}$, *then*

$$\frac{1}{4}n^2\|\check{H}\|^2 \geq \check{R}ic(\check{\chi}) + \frac{n_2\Delta f_2}{f_2} - \frac{\hat{c}-3}{4}(n + n_1 n_2 - 1) + \frac{\hat{c}+1}{4}(n_2 - \frac{1}{2}). \tag{4.33}$$

(ii) If $\check{\chi}$ *is tangent to* $\check{N}_\perp^{n_2}$, *then*

$$\frac{1}{4}n^2\|H\|^2 \geq \check{R}ic(\chi) + \frac{n_2\Delta f_2}{f_2} - \frac{\hat{c}-3}{4}(n + n_1 n_2 - 1) + \frac{\hat{c}+1}{4}(n_2 + 1). \tag{4.34}$$

(2) *If $\check{H}(x) = 0$, then each point $x \in \check{M}^n$ there is a unit vector field $\check{\chi}$ which satisfies the equality case of* (1) *if and only if $\check{M}^n$ is mixed totally geodesic and $\check{\chi}$ lies in the relative null space $\mathcal{N}_x$ at x.*

(3) *For the equality case we have*

   (a) *The equality case of* (4.33) *holds identically for all unit vector fields tangent to $\check{N}_T$ at each $x \in \check{M}^n$ if and only if $\check{M}^n$ is mixed totally geodesic and $\check{D}-$totally geodesic CR-warped product submanifold in $\hat{M}^m(\hat{c})$.*

   (b) *The equality case of* (4.34) *holds identically for all unit vector fields tangent to $\check{N}_\perp^{n_2}$ at each $x \in \check{M}^n$ if and only if $\check{M}$ is mixed totally geodesic and either $\check{M}^n$ is $\check{D}^\perp$- totally geodesic biwarped product or $\check{M}^n$ is a $\check{D}^\perp$ totally umbilical in $\hat{M}^m(c)$ with dim $\check{D}^\perp = 2$.*

   (c) *The equality case of* (1) *holds identically for all unit tangent vectors to $\check{M}^n$ at each $x \in \check{M}^n$ if and only if either $\check{M}^n$ is totally geodesic submanifold or $\check{M}^n$ is a mixed totally geodesic totally umbilical and $\check{D}-$ totally geodesic submanifold with dim $\check{N}_\perp = 2$*

where n_1 and n_2 are the dimensions of $\check{N}_T^{n_1}$ and $\check{N}_\perp^{n_2}$ respectively.

5 Conclusions

In this paper, Chen-Ricci inequalities are obtained for biwarped product submanifolds in Kenmotsu space forms. In particular, we also investigate Chen-Ricci inequalities for semi-slant warped product submanifolds and contact CR-warped product submanifolds. Moreover, the results acquired in this paper can also be extended for generalized Sasakian space forms.

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