Semi-invariant lightlike submanifolds of a metallic semi-Riemannian manifold

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Abstract. In this paper, we study the geometry of screen semi-invariant lightlike submanifolds of a metallic semi-Riemannian manifold. We find some conditions for integrability of distributions. Furthermore, we investigate totally geodesic and mixed geodesic distributions of such a submanifold.

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1 Introduction

Lightlike submanifolds are one of the most interesting topics in differential geometry. It is well known that a submanifold of a Riemannian manifold is always a Riemannian one. Contrary to that case, in semi-Riemannian manifolds, the induced metric by the semi-Riemannian metric on the ambient manifold is not necessarily non-degenerate. Since the induced metric is degenerate on lightlike submanifolds, the tools which are used to investigate the geometry of submanifolds in Riemannian case are not favorable in semi-Riemannian case and so the classical theory can not be used to define an induced object on a lightlike submanifold. The main difficulties arise from the fact that intersection of the normal bundle and the tangent bundle of a lightlike submanifold is non-zero. The geometry of lightlike submanifolds of semi-Riemannian manifolds was studied by Duggal and Bejancu [5].

Different kinds of geometric structures (such as almost product, almost contact, almost para-contact etc.) allow to get rich results while studying the geometry of submanifolds. Recently, Riemannian manifolds with metallic structures is one of the most studied topics in differential geometry. In 2002, as a generalization of the Golden mean, metallic means family was itnroduced by De Spinadel [4], which contains the Silver mean, Bronze mean, Copper mean and Nickel mean etc.

The metallic means family play an important role to establish a relationship between mathematics and architecture. For example, Golden mean and Silver mean can

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be seen in the sacred art of Egypt, Turkey, India, China and other ancient civilizations [3]. Polynomial structures on manifolds were introduced by Goldberg and Petridis [10], Goldberg and Yano [11]. Crasmareanu and Hretcanu [2] defined Golden structure as a particular case of polynomial structure and some generalizations of this, called metallic structure. Being inspired by the metallic mean, they defined the notion of metallic manifold $\tilde{N}$ by a $(1,1)$-tensor field $\tilde{J}$ on $\tilde{N}$, which satisfies $\tilde{J}^2 = p\tilde{J} + qI$, where I is the identity operator on $\Gamma(T\tilde{N})$ and p, q are fixed positive integers. Moreover, if $(\tilde{N}, \tilde{g})$ is a Riemannian manifold endowed with metallic structure $\tilde{J}$ such that the Riemannian metric $\tilde{g}$ is $\tilde{J}$- compatible, i.e., $\tilde{g}(\tilde{J}V, W) = \tilde{g}(V, \tilde{J}W)$ for any V, W $\in \Gamma(T\tilde{N})$, then $(\tilde{g}, \tilde{J})$ is called metallic Riemannian structure and $(\tilde{N}, \tilde{g}, \tilde{J})$ a metallic Riemannian manifold.

Metallic structure on the ambient Riemannian manifold provides important geometrical results on submanifolds since it is an important tool to investigate the geometry of submanifolds. Invariant, anti-invariant, semi-invariant, slant, semi-slant and hemi-slant submanifolds of metallic Riemannian manifold were studied in [12, 13, 14].

Recently, many authors have studied Golden Riemannian manifolds and their submanifolds. Poyraz and Yaar [16] initiated the study of lightlike geometry in Golden semi-Riemannian manifolds. Acet [1] has worked on lightlike hypersurfaces of a metallic semi-Riemannian manifold .

2 Preliminaries

A submanifold N'^m immersed in a semi-Riemannian manifold $(\tilde{N}^{m+n}, \tilde{g})$ is called a lightlike submanifold [5] if it admits a degenerate metric g induced from $\tilde{g}$ on N'. If $\tilde{g}$ is degenerate on the tangent bundle TN' of N', then N' is called lightlike submanifold. For a degenerate metric g on N', $TN'^{\perp}$ is a degenerate n-dimensional subspace of $T_x\tilde{N}$. Thus both T_xN' and $T_xN'^{\perp}$ are degenerate orthogonal subspaces but not complementary to each other. Therefore there exists a subspace $Rad(TN') = T_xN' \cap T_xN'^{\perp}$, known as Radical subspace. If the mapping $Rad(TN') : N' \longrightarrow TN'$, such that $x \in N' \mapsto Rad(T_xN')$, defines a smooth distribution of rank $r > 0$ on N', then N' is said to be an r-lightlike submanifold and the distribution $Rad(TN')$ is said to be radical distribution on N'. The non-degenerate complementary subbundles $S(TN')$ and $S(TN'^{\perp})$ of $Rad(TN')$ are known as screen distribution in TN' and screen transversal distribution in $TN'^{\perp}$ respectively, i.e.,

$$TN' = Rad(TN') \perp S(TN') \ \ \& \ \ TN'^{\perp} = Rad(TN') \perp S(TN'^{\perp}). \tag{2.1}$$

Let $ltr(TN')$(lightlike transversal bundle) and $tr(TN')$(transversal bundle) be complementary but not orthogonal vector bundles to $Rad(TN')$ in $S(TN'^{\perp})^{\perp}$ and TN' in $T\tilde{N}'|_{N'}$ respectively.
Then, the transversal vector bundle $tr(TN')$ is given by[6]

$$tr(TN') = ltr(TN') \perp S(TN'^{\perp}). \tag{2.2}$$

From (2.1) and (2.2), we get

$$T\tilde{N}|_{N'} = TN' \oplus tr(TN') = (Rad(TN') \oplus ltr(TN')) \perp S(TN') \perp S(TN'^{\perp}). \tag{2.3}$$

Theorem 2.1. *[5] Let $(N', g, S(TN'), S(TN'^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\tilde{N}, \tilde{g})$. Then there exists a complementary vector bundle $ltr(TN')$ of $Rad(TN')$ in $S(TN'^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TN')|u)$ consisting of a smooth section $\{N_i\}$ of $S(TN'^{\perp})^{\perp}|_u$, where u is a coordinate neighbourhood of N' such that*

$$\tilde{g}_{ij}(N_i, \xi_j) = \delta_{ij}, \quad \tilde{g}_{ij}(N_i, N_j) = 0, \tag{2.4}$$

for any i, $j \in \{1, 2, ..., r\}$.

A submanifold $(N', g, S(TN'), S(TN'^{\perp}))$ of $\tilde{N}$ is said to be
(i) r-lightlike if $r < min\{m, n\}$;
(ii) coisotropic if $r = n < m, S(TN'^{\perp}) = 0$;
(iii) isotropic if $r = m = n, S(TN') = 0$;
(iv) totally lightlike if $r = m = n, S(TN') = 0 = S(TN'^{\perp})$.
Let $\overline{\nabla}, \nabla$ and ∇^t denote the linear connections on $\tilde{N}, N'$ and vector bundle $tr(TN')$, respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall\ X, Y \in \Gamma(TN'), \tag{2.5}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U, \forall\ X \in \Gamma((TN')),\ U \in \Gamma(tr(TN')) \tag{2.6}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TN')$ and $\Gamma(tr(TN'))$ respectively, the linear connections ∇ and ∇^t are on N' and on the vector bundle $tr(TN')$ respectively, the second fundamental form h is a symmetric $F(N)$-bilinear form on $\Gamma(TN')$ with values in $\Gamma(tr(TN'))$ and the shape operator A_N is a linear endomorphism of $\Gamma(TN')$.
From 2.5 and 2.6 , for any $X, Y \in \Gamma(tr(TN'))$, $N \in \Gamma(ltr(TN'))$ and $L \in \Gamma(S(TN'^{\perp}))$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_W(N) + D^s(W, N), \tag{2.8}$$

$$\overline{\nabla}_X L = -A_L X + \nabla^s_X(L) + D^l(X, L), \tag{2.9}$$

where $D^l(X, L), D^s(X, N)$ are the projections of ∇^t on $\Gamma(ltr(TN'))$ and $\Gamma(S(TN'^{\perp}))$ respectively, ∇^l, ∇^s are linear connections on $\Gamma(ltr(TN'))$ and $\Gamma(S(TN'^{\perp}))$, respectively and A_N, A_L are shape operators on N' with respect to N and L, respectively. Using (2.5) and (2.7) -(2.9) , we obtain

$$\tilde{g}(h^s(X, Y), L) + \tilde{g}(Y, D^l(X, L)) = g(A_L X, Y), \tag{2.10}$$

$$\tilde{g}(D^s(X, N), L) = \tilde{g}(N, A_L X). \tag{2.11}$$

for $X, Y \in \Gamma(TN')$, $L \in \Gamma(S(TN'^{\perp})$ and $N \in \Gamma(ltr(TN'))$.
Let J denote the projection of TN' on $S(TN')$ and let ∇^*, ∇^{*t} denote the linear

connections on $S(TN')$ and $Rad(TN')$, respectively. Then from the decomposition of tangent bundle of lightlike submanifold, we have

$$\nabla_X JY = \nabla^*_X JY + h^*(X, JY), \tag{2.12}$$

$$\nabla_X E = -A^*_E X + \nabla^{*t}_X(E), \tag{2.13}$$

for $X, Y \in \Gamma(TN')$ and $E \in \Gamma(RadTN')$, where h^*, A^* are the second fundamental form and shape operator of distributions $S(TN')$ and $Rad(TN')$, respectively.
by using above equations, we get

$$\tilde{g}(h^l(X, JY), E) = g(A^*_E X, JY), \tag{2.14}$$

$$\tilde{g}(h^*(X, JY), N) = g(A_N X, JY), \tag{2.15}$$

$$\tilde{g}(h^i(X, E), E) = 0, \;\; A^*_E E = 0. \tag{2.16}$$

Let $(\tilde{N}, \tilde{g})$ be a smooth manifold, then $\tilde{N}$ is called golden semi-Riemannian manifold if $\exists$ an $(1,1)$ tensor field $\tilde{J}$ on $\tilde{N}$ such that

$$\tilde{J}^2 = \tilde{J} + I, \tag{2.17}$$

where I is the identity map on $\tilde{N}$. Also

$$\tilde{g}(\tilde{J}X, Y) = \tilde{g}(X, \tilde{J}Y). \tag{2.18}$$

Let (N', g', P') is called golden semi-Riemannian manifold. Also, we have

$$\triangledown'_X \tilde{J}Y = \tilde{J}\triangledown'_X Y, \tag{2.19}$$

if $\tilde{J}$ is a metallic structure, then 2.19 is equivalent to

$$\tilde{g}(\tilde{J}X, \tilde{J}Y) = \tilde{g}(\tilde{J}X, Y) + \tilde{g}(X, Y), \tag{2.20}$$

for any $X, Y \in \Gamma(TN')$.

3 Semi-Invariant Lightlike Submanifolds of Metallic semi-Riemannian Manifold

Let $(N', g, S(TN'), S(TN'^{\perp}))$ be a lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then, for any $X \in \Gamma(TN')$, we have

$$\tilde{J}X = JX + WX, \tag{3.1}$$

where JX and WX are the tangential and transversal components of $\tilde{J}X$, respectively. Similarly, for any $U \in tr(TN')$, we have

$$\tilde{J}U = BU + CU, \tag{3.2}$$

where BU and CU are the tangential and transversal components of $\tilde{J}X$, respectively.

Definition 3.1. Let N' be a lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then, N' is a semi-invariant lightlike submanifold, if the following conditions are satisfied:

$$\tilde{J}(Rad(TN')) \subseteq S(TN'), \tag{3.3}$$

$$\tilde{J}(ltr(TN')) \subseteq S(TN'), \tag{3.4}$$

$$\tilde{J}(S(TN'^{\perp})) \subseteq S(TN'). \tag{3.5}$$

If we set $D_1 = \tilde{J}(Rad(TN')), D_2 = \tilde{J}(ltr(TN'))$ and $D_3 = \tilde{J}(S(TN'^{\perp}))$ then we have

$$S(TN') = D_0 \perp \{D_1 \oplus D_2\} \perp D_3. \tag{3.6}$$

Thus we drive

$$TN' = D_0 \perp \{D_1 \oplus D_2\} \perp D_3 \perp Rad(TN'), \tag{3.7}$$

$$TN' = D_0 \perp \{D_1 \oplus D_2\} \perp D_3 \perp S(TN'^{\perp}) \perp \{Rad(TN') \oplus ltr(TN')\}. \tag{3.8}$$

From above definition, we can write

$$D = D_0 \perp D_1 \perp Rad(TN') \tag{3.9}$$

and

$$D^{\perp} = D_2 \perp D_3. \tag{3.10}$$

Thus we have

$$TN' = D \oplus D^{\perp}. \tag{3.11}$$

For case (2), we know that $S(TN') = \{0\}$. Then we drive

$$S(TN') = \{D_1 \oplus D_2\} \perp D_0, \tag{3.12}$$

$$TN' = \{D_1 \oplus D_2\} \perp D_0 \perp Rad(TN'), \tag{3.13}$$

$$T\tilde{N} = \{D_1 \oplus D_2\} \perp D_0 \perp \{Rad(TN') \oplus ltr(TN')\}, \tag{3.14}$$

$$TN' = D \oplus D_2. \tag{3.15}$$

Remark 3.2. The distribution D_0 and D are invariant distributions with respect to $\tilde{J}$.

Remark 3.3. Let N' be a semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$ with $\overline{\nabla}\tilde{J} = 0$. Then, we have

$$(\nabla_X P)Y = A_{wY}X + Bh(X, Y), \tag{3.16}$$

$$w\nabla_X Y = \nabla^t_X wY + h(X, JY), \tag{3.17}$$

$$\nabla_X BU = -PA_U X + B\nabla^t_X U, \tag{3.18}$$

$$h(X, BU) = -wA_U X, \tag{3.19}$$

for any $X, Y \in \Gamma(TN')$ and $U \in \Gamma(tr(TN'))$.
Throughout this paper, we assume $\overline{\nabla}\tilde{J} = 0$.

Remark 3.4. Let N' be a semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then, we have

$$J^2X = JX + X - BwX, \tag{3.20}$$

$$wJX = wX, JBU = BU, wBU = U, \tag{3.21}$$

$$g(JX, Y) - g(X, PY) = g(X, wY) - g(wX, Y), \tag{3.22}$$

(3.23)
$$g(JX, JY) = g(JX, Y) + g(X, Y) + g(wX, Y) - g(JX, wY) - g(wX, JY) - g(wX, wY),$$

for any $X, Y \in \Gamma(TN')$ and $U \in \Gamma(tr(TN'))$.

Remark 3.5. Let N' be a semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then J is metallic structure on N' iff $wX = 0$.

Theorem 3.1. *Let N' be a semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. For any $X, Y \in \Gamma(D)$, then D is integrable iff*

$$h^l(\tilde{J}X, \tilde{J}Y) = ph^l(\tilde{J}X, Y) + qh^l(X, Y), \tag{3.24}$$

$$h^s(\tilde{J}X, \tilde{J}Y) = ph^s(\tilde{J}X, Y) + qh^s(X, Y). \tag{3.25}$$

Proof. For any $X, Y \in \Gamma(D)$, $E \in \Gamma(Rad(TN'))$, $L \in \Gamma(S(TN'^{\perp}))$ and $N \in \Gamma(ltr(TN'))$. The distribution D is integrable iff

$$(\tilde{g}[\tilde{J}X, Y], \tilde{J}E) = (\tilde{g}[\tilde{J}X, Y], \tilde{J}L) = 0. \tag{3.26}$$

Then from (2.7) and (2.17), we obtain

$$(\tilde{g}[\tilde{J}X, Y], \tilde{J}E) = \tilde{g}(\overline{\nabla}_{\tilde{J}X}Y - \overline{\nabla}_Y\tilde{J}X, \tilde{J}E),$$
$$(\tilde{g}[\tilde{J}X, Y], \tilde{J}E) = (h^l(\tilde{J}X, \tilde{J}Y) - ph^l(\tilde{J}X, Y) - qh^l(X, Y), E).$$

Using (3.26), we get

$$h^l(\tilde{J}X, \tilde{J}Y) = ph^l(\tilde{J}X, Y) + qh^l(X, Y). \tag{3.27}$$

Further,

$$(\tilde{g}[\tilde{J}X, Y], \tilde{J}L) = \tilde{g}(\overline{\nabla}_{\tilde{J}X}Y - \overline{\nabla}_Y\tilde{J}X, \tilde{J}L), \tag{3.28}$$

$$(\tilde{g}[\tilde{J}X, Y], \tilde{J}L) = (h^s(\tilde{J}X, \tilde{J}Y) - ph^l(\tilde{J}X, Y) - qh^s(X, Y), L), \tag{3.29}$$

Using (3.26), we have

$$h^s(\tilde{J}X, \tilde{J}Y) = ph^s(\tilde{J}X, Y) + qh^l(X, Y). \tag{3.30}$$

Thus, proof is completed. □

Theorem 3.2. *Let N' be a semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. If D is integrable, then the leaves of D have a metallic structure.*

Proof. Let N' be a semi-invariant lightlike submanifold and M' be leaf of D. Since D is integrable, then for any $j \in M'$, we have $T_jM' = (D)_j$.
Letting $J' = J|D$, we say that J' defines $(1,1)$-tensor field on M', because D is $\tilde{J}$-invariant.
If $X \in \Gamma(D)$, then $wX = 0$, we drive

$$J'^2X = J^2X = \tilde{J}^2X = \tilde{J}pX + qX = JpX + qX = J'pX + qX,$$

which proves the theorem. □

Theorem 3.3. *Let N' be a semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then, for any $X, Y \in \Gamma(D^\perp)$, $Z \in \Gamma(D_0)$, $N \in \Gamma(ltr(TN'))$ and $U_1, U_2 \in \Gamma(ltr(TN'))$, the distribution $D^\perp$ is integrable iff*

(i) $p\tilde{g}(h^*(X, Y) - h^*(Y, X), N) = q\tilde{g}(A_{U_2}X - A_{U_1}Y, N)$,

(ii) $\tilde{g}A_{U_1}Y, \tilde{J}N) = \tilde{g}(A_{U_2}X, \tilde{J}N)$,

(iii) $\tilde{g}A_{U_1}Y, \tilde{J}Z) = \tilde{g}(A_{U_2}X, \tilde{J}Z)$.

Proof. For any $X, Y \in \Gamma(D^\perp)$, $Z \in \Gamma(D_0)$, $N \in \Gamma(ltr(TN'))$ and $U_1, U_2 \in \Gamma(ltr(TN'))$, the distribution $D^\perp$ is integrable iff

$$\tilde{g}([X, Y], \tilde{J}N) = \tilde{g}([X, Y], N) = \tilde{g}([X, Y], Z) = 0. \tag{3.31}$$

Choosing $X, Y \in \Gamma(D^\perp)$, there is a vector field $U_1, U_2 \in \Gamma(tr(TN'))$, such that $X = \tilde{J}U_1$ and $Y = \tilde{J}U_2$,
Using (2.6), (2.7) and (2.12), we obtain

$$\tilde{g}([X, Y], \tilde{J}N) = \tilde{g}(\overline{\nabla}_XY - \overline{\nabla}_YX, \tilde{J}N). \tag{3.32}$$

Using (2.17), (2.18), $X = \tilde{J}U_1$ and $Y = \tilde{J}U_2$, in (2.1), we get

$$\tilde{g}([X,Y],\tilde{J}N) = p\tilde{g}(\nabla_X \tilde{J}U_2 - \nabla_Y \tilde{J}U_1, N) + q\tilde{g}(\nabla_X U_2 - \nabla_Y U_1, N). \tag{3.33}$$

Using (2.6) and (2.12) in (3.33), we get

$$\tilde{g}([X,Y],\tilde{J}N) = p\tilde{g}(h^*(X,\tilde{J}U_2) - h^*(Y,\tilde{J}U_1), N) - q\tilde{g}(A_{U_2}X - A_{U_1}Y, N). \tag{3.34}$$

Now

$$\tilde{g}([X,Y],N) = \tilde{g}(\overline{\nabla}_X Y - \overline{\nabla}_Y X, N),$$

Using (2.6) ,(2.18), $X = \tilde{J}U_1$ and $Y = \tilde{J}U_2$, in above equation, we get

$$= \tilde{g}(A_{U_1}Y - A_{U_2}X, \tilde{J}N). \tag{3.35}$$

For any $Z \in \Gamma(D_0)$, we have

$$\tilde{g}([X,Y],Z) = \tilde{g}(\overline{\nabla}_X Y - \overline{\nabla}_Y X, Z),$$

Using (2.6) ,(2.18), $X = \tilde{J}U_1$ and $Y = \tilde{J}U_2$, in above equation, we get

$$= \tilde{g}(A_{U_1}Y - A_{U_2}X, \tilde{J}Z). \tag{3.36}$$

Using conditions of (3.31) in (3.34), (3.35) and (3.36), we obtain required results. This completes the proof. □

Theorem 3.4. *Let N' be a semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N},\tilde{g},\tilde{J})$. Then, for any $X \in \Gamma(D_0)$, $E, E', E_1 \in \Gamma(Rad(TN'))$, $N \in \Gamma(ltr(TN'))$ and $L \in \Gamma(S(TN'^{\perp}))$, $Rad(TN')$ is integrable iff*

(i) $g(h^*(E,\tilde{J}E'),N) = \tilde{g}(h^*(E',\tilde{J}E),N)$,

(ii) $\tilde{g}(h^l(E,\tilde{J}E'),E_1) = \tilde{g}(h^l(E',\tilde{J}E),E_1)$,

(iii) $\tilde{g}(h^s(E,\tilde{J}E'),L) = \tilde{g}(h^s(E',\tilde{J}E),L)$,

(iv) $g(A^*_E E',X) = g(A^*_{E'}E,X)$.

Proof. $Rad(TN')$ is integrable iff

$$\tilde{g}([E,E'],\tilde{J}N) = \tilde{g}([E,E'],\tilde{J}E_1) = \tilde{g}([E,E'],\tilde{J}L) = \tilde{g}([E,E'],X) = 0, \tag{3.37}$$

for any $X \in \Gamma(D_0)$, $E, E', E_1 \in \Gamma(Rad(TN'))$, $N \in \Gamma(ltr(TN'))$ and $L \in \Gamma(S(TN'^{\perp}))$. By using (2.6) and (2.7), we get

$$\tilde{g}([E,E'],\tilde{J}N) = \tilde{g}(\nabla_E E' - \nabla_{E'}E, \tilde{J}N),$$

$$\tilde{g}([E,E'],\tilde{J}N) = \tilde{g}(\nabla_E \tilde{J}E' - \nabla_{E'}\tilde{J}E, N),$$

using (2.12) in (3.37), we obtain

$$= \tilde{g}(h^*(E,\tilde{J}E)' - h^*(E',\tilde{J}E), N) \tag{3.38}$$

Now

$$\tilde{g}([E,E'],\tilde{J}E_1) = \tilde{g}(\overline{\nabla}_E E' - \overline{\nabla}_{E'}E, \tilde{J}E_1),$$

using (2.7) and (2.12) in (3.37), we get

$$\tilde{g}([E,E'],\tilde{J}E_1)=\tilde{g}(h^l(E,\tilde{J}E')-h^l(E',\tilde{J}E),E_1) \tag{3.39}$$

$$\tilde{g}([E,E'],\tilde{J}L)=\tilde{g}(\overline{\nabla}_E E'-\overline{\nabla}_{E'}E,\tilde{J}L),$$

from (2.7) and (2.12) in (3.37), we have

$$\tilde{g}([E,E'],\tilde{J}L)=\tilde{g}(h^sE,\tilde{J}E'-h^sE',\tilde{J}E),L). \tag{3.40}$$

Now

$$\tilde{g}([E,E'],X)=\tilde{g}(\overline{\nabla}_E E'-\overline{\nabla}_{E'}E,X),$$

using (2.13) in (3.37), we obtain

$$\tilde{g}([E,E'],X)=g(A^*_E E'-A^*_{E'}E,X). \tag{3.41}$$

This completes the proof. □

Theorem 3.5. *Let N' be a semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N},\tilde{g},\tilde{J})$. Then, for any $X\in\Gamma(D_0)$, $E,E',E_1\in\Gamma(Rad(TN'))$, $N\in\Gamma(ltr(TN'))$ and $L\in\Gamma(S(TN'^{\perp}))$, $\tilde{J}Rad(TN')$ is integrable iff*

(i) $\tilde{g}(h^l(\tilde{J}E_1,E_2),E)=\tilde{g}(h^l(\tilde{J}E_2,E_1),E)$,

(ii) $\tilde{g}(h^s(\tilde{J}E_1,E_2),L)=\tilde{g}(h^s(\tilde{J}E_2,E_1),L)$,

(iii) $g(A_N\tilde{J}E_1,\tilde{J}E_2)=g(A_N\tilde{J}E_2,\tilde{J}E_1)$,

(iv) $g(A^*_{E_1}\tilde{J}E_2,\tilde{J}X)=g(A^*_{E_2}\tilde{J}E_1,\tilde{J}X)$.

Proof. $Rad(TN')$ is integrable iff

$$\tilde{g}([\tilde{J}E_1,\tilde{J}E_2],\tilde{J}E)=\tilde{g}([\tilde{J}E_1,\tilde{J}E_2],\tilde{J}L)=\tilde{g}([\tilde{J}E_1,\tilde{J}E_2],N)=\tilde{g}([\tilde{J}E_1,\tilde{J}E_2],X)=0, \tag{3.42}$$

for any $X\in\Gamma(D_0)$, $E,E',E_1\in\Gamma(Rad(TN'))$, $N\in\Gamma(ltr(TN'))$ and $L\in\Gamma(S(TN'^{\perp}))$. Since $\overline{\nabla}$ is a metric connection, by using (2.7) and (2.8), we obtain

$$\tilde{g}([\tilde{J}E_1,\tilde{J}E_2],\tilde{J}E)=\tilde{g}(\overline{\nabla}_{\tilde{J}E_1}\tilde{J}E_2-\overline{\nabla}_{\tilde{J}E_2}\tilde{J}E_1,\tilde{J}E),$$

using (2.13) and (2.17), we have

$$\tilde{g}([\tilde{J}E_1,\tilde{J}E_2],\tilde{J}E)=\tilde{g}p(\overline{\nabla}_{\tilde{J}E_1}\tilde{J}E_2,E)+\tilde{g}q(\overline{\nabla}_{\tilde{J}E_1}E_2,E)-\tilde{g}p(\overline{\nabla}_{\tilde{J}E_2}\tilde{J}E_1,E)+\tilde{g}q(\overline{\nabla}_{\tilde{J}E_2}E_1,E), \tag{3.43}$$

$$=p\tilde{g}(h^l(\tilde{J}E_1,\tilde{J}E_2),E)+q\tilde{g}(h^l(\tilde{J}E_1,E_2),E)-p\tilde{g}(h^l(\tilde{J}E_2,\tilde{J}E_1),E)-q\tilde{g}(h^l(\tilde{J}E_2,E_1),E),$$

from (3.42), we get

$$=q\tilde{g}(h^l(\tilde{J}E_1,E_2),E)-q\tilde{g}(h^l(\tilde{J}E_2,E_1),E).$$

Now, we have

$$\tilde{g}([\tilde{J}E_1,\tilde{J}E_2],\tilde{J}L)=\tilde{g}(\overline{\nabla}_{\tilde{J}E_1}\tilde{J}E_2-\overline{\nabla}_{\tilde{J}E_2}\tilde{J}E_1,\tilde{J}L),$$

using (2.13) and (2.17), we have
(3.44)
$\tilde{g}([\tilde{J}E_1,\tilde{J}E_2],\tilde{J}L) = p\tilde{g}(\overline{\nabla}_{\tilde{J}E_1}\tilde{J}E_2,L)+q\tilde{g}(\overline{\nabla}_{\tilde{J}E_1}E_2,L)-p\tilde{g}(\overline{\nabla}_{\tilde{J}E_2}\tilde{J}E_1,L)+q\tilde{g}(\overline{\nabla}_{\tilde{J}E_2}E_1,L),$

$= p\tilde{g}(h^s(\tilde{J}E_1,\tilde{J}E_2),L)+q\tilde{g}(h^s(\tilde{J}E_1,E_2),L)-p\tilde{g}(h^s(\tilde{J}E_2,\tilde{J}E_1),L)-q\tilde{g}(h^s(\tilde{J}E_2,E_1),L),$

now using (3.42), we have

$$= \tilde{g}q(h^s(\tilde{J}E_1,E_2),L) - \tilde{g}q(h^s(\tilde{J}E_2,E_1),L).$$

Using (2.7) and (2.13), we obtain

$$\tilde{g}([\tilde{J}E_1,\tilde{J}E_2],N) = \tilde{g}(\overline{\nabla}_{\tilde{J}E_1}\tilde{J}E_2 - \overline{\nabla}_{\tilde{J}E_2}\tilde{J}E_1,N),$$

$$= -\tilde{g}(\tilde{J}E_2,\overline{\nabla}_{\tilde{J}E_1}N) + \tilde{g}(\tilde{J}E_1,\overline{\nabla}_{\tilde{J}E_2}N),$$

$$= g(A_N\tilde{J}E_1,\tilde{J}E_2) - g(A_N\tilde{J}E_2,\tilde{J}E_1).$$

Now

$$\tilde{g}([\tilde{J}E_1,\tilde{J}E_2],X) = \tilde{g}(\overline{\nabla}_{\tilde{J}E_1}\tilde{J}E_2 - \overline{\nabla}_{\tilde{J}E_2}\tilde{J}E_1,X),$$

using (2.13), we get

$$= g(A^*_{E_1}\tilde{J}E_2 - A^*_{E_2}\tilde{J}E_1,\tilde{J}X).$$

Thus the proof is completed. □

Theorem 3.6. *Let N' be a semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N},\tilde{g},\tilde{J})$. Then, for any $E,E_1,E_2 \in \Gamma(Rad(TN'))$, $N \in \Gamma(ltr(TN'))$ and $L \in \Gamma(S(TN'^{\perp}))$, each leaf of radical distribution is totally geodesic on N' iff*

(i) $A^*_{E_2}E_1 \in \Gamma(D_1 \perp D_3)$,

(ii) $\tilde{g}(h^*(E_1,\tilde{J}E_2),N) = 0$,

(iii) $\tilde{g}(h^s(E_1,\tilde{J}E_2),L) = 0$.

Proof. Radical distribution is totally geodesic iff

$$g(\nabla_{E_1}E_2,\tilde{J}E) = g(\nabla_{E_1}E_2,\tilde{J}L) = g(\nabla_{E_1}E_2,\tilde{J}N) = g(\nabla_{E_1}E_2,X) = 0, \tag{3.45}$$

for any $X \in \Gamma(D_0)$, $E,E_1,E_2 \in \Gamma(Rad(TN'))$, $N \in \Gamma(ltr(TN'))$ and $L \in \Gamma(S(TN'^{\perp}))$.
By using (2.7), (2.12) and (2.13), we have

$$g(\nabla_{E_1}E_2,\tilde{J}E) = \tilde{g}(\overline{\nabla}_{E_1}E_2,\tilde{J}E),$$

$$= -g(A^*_{E_2}E_1,\tilde{J}E). \tag{3.46}$$

$$g(\nabla_{E_1}E_2,X) = \tilde{g}(\overline{\nabla}_{E_1}E_2,X),$$

$$= -g(A^*_{E_2}E_1,X). \tag{3.47}$$

$$g(\nabla_{E_1}E_2,\tilde{J}N)=\tilde{g}(\overline{\nabla}_{E_1}E_2,\tilde{J}N),$$

$$=\tilde{g}(h^*(E_1,\tilde{J}E_2),N). \tag{3.48}$$

$$g(\nabla_{E_1}E_2,\tilde{J}L)=\tilde{g}(\overline{\nabla}_{E_1}E_2,\tilde{J}L),$$

$$=\tilde{g}(h^s(E_1,\tilde{J}E_2),L). \tag{3.49}$$

Hence theorem proved. □

Definition 3.6. Let N' be a proper semi-invariant r-lightlike submanifold of a metallic semi-Riemannian manifold $\tilde{N}$. If

$$h(X,Y)=0,\ \forall X\in\Gamma(D),X\in\Gamma(D^{\perp}),$$

then, N' is called as mixed-geodesic submanifold.

Theorem 3.7. *Let N' be a semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N},\tilde{g},\tilde{J})$. Then, for any $X\in\Gamma(D)$, $Y\in\Gamma(D^{\perp})$, $E\in\Gamma(Rad(TN'))$, $U\in\Gamma(tr(TN'))$ and $L\in\Gamma(S(TN'^{\perp}))$, the following are equivalent*

(i) N' is mixed geodesic,

(ii) A_UX has only component in $\Gamma(D)$,

(iii) A^*_EX and A_LX have no components in D_1 and D_3,

(iv) $\nabla^*_Y\tilde{J}E,\nabla^*_Y\tilde{J}L\in\Gamma(D_0\perp D_2)$.

Proof. N' is mixed-geodesic iff

$$\tilde{g}(h(X,Y),E)=0$$

and

$$\tilde{g}(h(X,Y),L)=0,$$

for any $X\in\Gamma(D)$, $Y\in\Gamma(D^{\perp})$, $E\in\Gamma(Rad(TN'))$ and $L\in\Gamma(S(TN'^{\perp}))$.
Choosing $Y\in\Gamma D^{\perp}$, there is a vector field $U\in\Gamma(tr(TM))$ such that $Y=\tilde{J}U$, by using (2.5) and (2.6), we have

$$\tilde{g}(h(X,Y),E)=\tilde{g}(\overline{\nabla}_XY,E)=\tilde{g}(\overline{\nabla}_X\tilde{J}U,E),$$

$$\tilde{g}(\overline{\nabla}_XU,\tilde{J}E)=-\tilde{g}(A_UX,\tilde{J}E), \tag{3.50}$$

$$\tilde{g}(h(X,Y),L)=\tilde{g}(\overline{\nabla}_XY,L)=\tilde{g}(\overline{\nabla}_X\tilde{J}U,L),$$

$$\tilde{g}(\overline{\nabla}_XU,\tilde{J}L)=-\tilde{g}(A_UX,\tilde{J}L), \tag{3.51}$$

thus we drive (i) $\iff$ (ii), from (2.5), (2.9) and (2.13), we get

$$\tilde{g}(h(X,Y),E)=g(Y,A^*_EX), \tag{3.52}$$

$$\tilde{g}(h(X,Y),L) = \tilde{g}(Y, A_L X - D^l(X,L)). \tag{3.53}$$

Hence we obtain (i) $\iff$ (iii). Since $\overline{\nabla}$ is a metric connection, from (2.5) we drive

$$\begin{aligned}
\tilde{g}(h(\tilde{J}X,Y),E) &= \tilde{g}(h(Y,\tilde{J}X),E),\\
&= \tilde{J}(\overline{\nabla}_Y \tilde{J}X, E),\\
&= -\tilde{g}(\tilde{J}X, \overline{\nabla}_Y E),\\
&= -\tilde{g}(X, \overline{\nabla}_Y \tilde{J}E),\\
&= -\tilde{g}(X, \overline{\nabla}^*_Y \tilde{J}E).\\
\tilde{g}(h(\tilde{J}X,Y),L) &= \tilde{g}(h(Y,\tilde{J}X),L),\\
&= \tilde{J}(\overline{\nabla}_Y \tilde{J}X, L),\\
&= -\tilde{g}(\tilde{J}X, \overline{\nabla}_Y L),\\
&= -\tilde{g}(X, \overline{\nabla}_Y \tilde{J}L),\\
&= -\tilde{g}(X, \overline{\nabla}^*_Y \tilde{J}L),
\end{aligned}$$

thus we drive (i) $\iff$ (iv). □

Definition 3.7. A semi-invariant submanifold N' of a metallic semi-Riemannian manifold N' is named as D-totally geodesic (resp. $D^{\perp}$-totally geodesic) if its the second fundamental form h satisfies $h(X,Y) = 0$ (resp. $h(Z,L) = 0$), for any $X, Y \in \Gamma(D)$, $(Z, L \in \Gamma(D^{\perp}))$.

Theorem 3.8. *Let N' be a semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then, for any $X, Y \in \Gamma(D)$, $E \in \Gamma(Rad(TN'))$ and $L \in \Gamma(S(TN'^{\perp}))$, the following are equivalent*

(i) N' is D-geodesic,

(ii) $A^*_E X \in \Gamma(D_1 \perp D_3)$ and $g(Y, A_L X) = g(Y, D^l(X,L))$,

(iii) $\nabla^*_X Y$ has no components in D_2 and D_3,

(iv) $\nabla^*_X \tilde{J}E, \nabla^*_X \tilde{J}L \in \Gamma(D_1 \perp D_3)$.

Proof. For any $X, Y \in \Gamma(D)$, $E \in \Gamma(Rad(TN'))$ and $L \in \Gamma(S(TN'^{\perp}))$,
by using (2.5), (2.9) and (2.13), we obtain

$$\tilde{g}(h(X,Y),E) = g(Y, A^*_E X), \tag{3.54}$$

$$\tilde{g}(h(X,Y),L) = \tilde{g}(Y, A_L X - D^l(X,L)). \tag{3.55}$$

Hence we drive (i) $\iff$ (ii). Since $\overline{\nabla}$ is a metric connection, from (2.5), (2.9) and (2.13), we get

$$\begin{aligned}
g(\nabla^*_X Y, \tilde{J}E) &= \tilde{g}(\overline{\nabla}_X Y, \tilde{J}E),\\
&= \tilde{g}(\overline{\nabla}_X \tilde{J}Y, E),
\end{aligned}$$

$$= \tilde{g}(\tilde{J}Y, \overline{\nabla}_X E),$$

$$= \tilde{g}(\tilde{J}Y, A^*_E X), \tag{3.56}$$

$$g(\nabla^*_X Y, \tilde{J}L) = \tilde{g}(\overline{\nabla}_X Y, \tilde{J}L),$$
$$= \tilde{g}(\overline{\nabla}_X \tilde{J}Y, L),$$
$$= \tilde{g}(\tilde{J}Y, \overline{\nabla}_X L),$$

$$= \tilde{g}(\tilde{J}Y, A_L X - D^l(X, L)). \tag{3.57}$$

This is (ii) $\iff$ (iii). Similarly, since $\overline{\nabla}$ is a metric connection, from (2.5) and (2.12) we have

$$g(\nabla^*_X \tilde{J}E, Y) = \tilde{g}(\overline{\nabla} \tilde{J}E, Y),$$
$$= -\tilde{g}(\tilde{J}E, \overline{\nabla}_X Y),$$

$$= -\tilde{g}(\tilde{J}E, \nabla^*_X Y). \tag{3.58}$$

$$g(\nabla^*_X \tilde{J}L, Y) = \tilde{g}(\overline{\nabla} \tilde{J}L, Y),$$
$$= -\tilde{g}(\tilde{J}L, \overline{\nabla}_X Y),$$

$$= -\tilde{g}(\tilde{J}L, \nabla^*_X Y). \tag{3.59}$$

Hence we get (iii) $\iff$ (iv). Since $\overline{\nabla}$ is a metric connection, using (2.5), (2.12) and (2.17) , we get

$$\tilde{g}(h(X, Y), E) = \tilde{g}(\overline{\nabla}_X Y, E),$$
$$= \tilde{g}(\overline{\nabla}_X \tilde{J}Y, \tilde{J}E) - \tilde{g}(\overline{\nabla}_X Y, \tilde{J}E),$$
$$= -\tilde{g}(\tilde{J}Y, \overline{\nabla}_X \tilde{J}E) + \tilde{g}(Y, \overline{\nabla}_X \tilde{J}E),$$

$$= -\tilde{g}(\tilde{J}Y, \nabla^*_X \tilde{J}E) + \tilde{g}(Y, \nabla^*_X \tilde{J}E). \tag{3.60}$$

$$\tilde{g}(h(X, Y), L) = \tilde{g}(\overline{\nabla}_X Y, L),$$
$$= \tilde{g}(\overline{\nabla}_X \tilde{J}Y, \tilde{J}L) - \tilde{g}(\overline{\nabla}_X Y, \tilde{J}L),$$
$$= -\tilde{g}(\tilde{J}Y, \overline{\nabla}_X \tilde{J}L) + \tilde{g}(Y, \overline{\nabla}_X \tilde{J}L),$$

$$= -\tilde{g}(\tilde{J}Y, \nabla^*_X \tilde{J}L) + \tilde{g}(Y, \nabla^*_X \tilde{J}L). \tag{3.61}$$

Thus we get (iv) $\iff$ (i). □

Theorem 3.9. *Let N' be a semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{J})$. Then, for any $X, Y \in \Gamma(D^\perp)$, $E \in \Gamma(Rad(TN'))$, $U \in \Gamma(tr(TN'))$ and $L \in \Gamma(S(TN'^\perp))$, the following are equivalent*

(i) N' is $D^\perp$-geodesic,

(ii) $A^*_E X$ and $A_L X$ have no components in D_2 and D_3, and $g(Y, A_L X) = g(Y, D^l(X, L))$,

(iii) $A_U X$ *has no components in* D_2 *and* D_3.

Proof. N' is $D^{\perp}$- geodesic iff

$$\tilde{g}(h(X,Y),E)=0$$

and

$$\tilde{g}(h(X,Y),L)=0,$$

for any $X,Y\in\Gamma(D^{\perp})$, $E\in\Gamma(Rad(TN'))$, $U\in\Gamma(tr(TN'))$ and $L\in\Gamma(S(TN'^{\perp}))$, from (2.5), (2.9) and (2.13), we get

$$\tilde{g}(h(X,Y),E)=g(Y,A_E^*X), \tag{3.62}$$

$$\tilde{g}(h(X,Y),L)=\tilde{g}(Y,A_LX-D^l(X,L)). \tag{3.63}$$

Thus we drive (i) $\iff$ (ii). Choosing $Y\in\Gamma(D^{\perp})$ there is a vector field $U\in\Gamma(tr(TN'))$ such that $Y=\tilde{J}U$,
from (2.5) and (2.6), we get

$$\begin{aligned}\tilde{g}(h(X,Y),E)&=\tilde{g}(\overline{\nabla}_XY,E),\\&=\tilde{g}(\overline{\nabla}_X\tilde{J}U,E),\\&=\tilde{g}(\overline{\nabla}_XU,\tilde{J}E),\end{aligned}$$

$$=-\tilde{g}(A_UX,\tilde{J}E). \tag{3.64}$$

$$\begin{aligned}\tilde{g}(h(X,Y),L)&=\tilde{g}(\overline{\nabla}_XY,L),\\&=\tilde{g}(\overline{\nabla}_X\tilde{J}U,L),\\&=\tilde{g}(\overline{\nabla}_XU,\tilde{J}L),\end{aligned}$$

$$=-\tilde{g}(A_UX,\tilde{J}L). \tag{3.65}$$

This is (i) $\iff$ (iii). □

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