Rigging and existence of stable currents in null hypersurface of Lorentzian manifold

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Abstract. In this paper, using Rigging techniques, we give some conditions on both the null hypersurface wich affects the existence of stable current screen (or more importantly the nonexistence) of a null hypersurface M of a Lorentzian manifold $\overline{M}$. We also gave the generalize version of the theorem relating the geometry to the nonexistence of stable currents in the null hypersurface case in Lorentzian Manifold.

M.S.C. 2010: 53C23, 53C25, 53C50.
Key words: Null hypersurface; rigging; closed normalization; associated Riemannian metric; Lorentzian manifold; stable currents.

1 Introduction

It is an interesting problem in geometry and topology to find sufficient conditions on the second fundamental form wich affects the existence, or the nonexistence of stable submanifolds. The existence theorem due to Federer and Fleming [9] states that, for any compact Riemannian manifold M, each non-trivial integral homology class in $H_p(M, \mathbb{Z})$ corresponds to a stable integral current; thus, the method can also be used to give pointwise conditions on the second fundamental form of a compact submanifold which forces some of the integral homology groups of M to vanish. The homology groups of a manifold are important topological invariants that provide an algebraic summary of the manifold. Among other things, these groups contain rich topological informationss about the connected components,holes, tunnels and dimension of the manifold and this theory has many applications ([9],[8], [11],[12], [14], [15], [4],[5],$\cdots$). Indeed, homology theory has applications in gene expression data, protein docking, image segmentation and root architecture [7]. There is a close relationship between homology theory and submanifold theory. By applying the Federer Fleming theorem and the techniques from the calculus of variations in the geometric measure theory, Lawson and Simons [12] investigated the topology and geometry of submanifolds of the sphere, and showed that appropriate assumptions on the extrinsic geometry of the submanifolds imply the vanishing of a given homology group. The study of nonexistence of stable submanifolds has received wide attention among mathematicians. The earliest result of this type known to us is the result of Synge that

Differential Geometry - Dynamical Systems, Vol.23, 2021, pp. 74-89.

a compact, orientable, even dimensional Riemannian manifold of positive sectional curvature has no stable closed geodesics. The first results about the nonexistence of stable submanifolds other than closed geodesics seem to be in the celebrated paper of Simons [15]. Lawson and Simons show that an Euclidean sphere S^n contains no stable submanifolds and that there are no oriented closed stable hypersurfaces in an oriented manifold with positive Ricci curvature. Ralph Howard and S.Walter Wei [11] verify this conjecture for several classes of positively curved manifolds. They show how the geometry of a Riemannian manifold M^n affects the existence, or the nonexistence of stable submanifolds by giving conditions on the second fundamental form of M^n immersed in a higher dimensional Euclidean space. In the present paper, we first consider the associated Riemannian metric of null hypersurface in Lorentzian manifold as in [2] but arising from a null rigging defined on a neighborhood of the null hypersurface. Using Nash's celebrated theorem in differential geometry and the method of stable currents due to Lawson and Simons [12] we give some conditions on the second fundamental form of the null hypersurface in Lorentzian manifold which affects the existence, or the nonexistence of stable currents. This is can give some topological implications on relation between the intrinsic and extrinsic geometries of a null hypersurface in Lorentzian manifold. It is here important to give the organisational structure of this paper : In section (2), we present the basic elements of null hypersurface emphasizing those parts of the subject need for our work. The relationship between the geometry and existence of stable current (screen) of a closed null hypersurface is given in (3.6).

2 Preliminaries on null hypersurface

Let (M, g) be a hypersurface of a $(n+2)-$dimensional Lorentzian manifold $(\overline{M}, \overline{g})$ of constant index $0 < \nu < n+2$. The normal bundle of M is the subbundle $TM^{\perp} = \{V \in \Gamma(T\overline{M}) : g(V, W) = 0 \ \forall \ W \in \Gamma(TM)\}$ of the tangent bundle TM. In the classical theory of non-degenerate hypersurfaces, tangent and normal bundles of a non-degenerate hypersurface are complementary with in the ambient bundle. Therefore by respective projections, one has fundamental equations such as the Gauss, the Codazzi, the Weingarten equations $,\cdots$ along with the second fundamental form, shape operator , induced connection, etc. On the contrary for null hypersurfaces, normal bundle is contained in the tangent bundle and classical methods fails to study null hypersurfaces. To deal with the problems, Bejancu and Duggal introduced the notions of screen distributions that is a rank n (see [6]) which we denote by $\mathcal{S}(N)$, such that

$$TM = \mathcal{S}(N) \oplus_{orth} TM^{\perp}. \tag{2.1}$$

With a choice of a screen distribution, one can induce geometric objects on lightlike submanifold in a manner which analogous to what is done in the classical theory of nondegenerate submanifolds .

Theorem 2.1. *[6] Let $(M^{n+1}, g, \mathcal{S}(N))$ be a null hypersurface equipped with a screen distribution of a $(n+2)-$dimensional Lorentzian manifold. Then, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M , such that for any non-zero section*

ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying

$$\bar{g}(N,\xi) = 1,\ \bar{g}(N,N) = \bar{g}(N,W) = 0 \ \ \forall\ W \in \Gamma(\mathcal{S}N)_{|\mathcal{U}}. \tag{2.2}$$

Then $T\overline{M}$ is decomposed as following:

$$T\bar{M} = \mathcal{S}(N) \oplus_{orth} \left(TM^{\perp} \oplus tr(TM)\right) = TM \oplus tr(TM). \tag{2.3}$$

Where $tr(TM)$ a (null) transversal vector bundle along M.

Given a vector field ζ in a neighborhood of M in $(\overline{M}, \overline{g})$, let θ denote the 1-form $\overline{g}$-metrically equivalent to ζ, i.e $\theta = \overline{g}(\zeta, .)$. Then, take

$$\eta = i^{\star\theta} \tag{2.4}$$

to be its restriction to M, the map $i : M \to \overline{M}$ being the inclusion map. The one-form η satisfies

$$\eta(X)(p) = \begin{cases} \neq 0 \forall X \in (T_pM) \\ = 0 \forall X \in \mathcal{S}_p(N). \end{cases} \tag{2.5}$$

The normalization N will said to be closed if the $1-$form θ is closed on M.

Definition 2.1 ([10])**.** A rigging for M is a vector field L defined on some open set containing M and such that $L_p \notin T_pM$ for each $p \in M$.

An outstanding property of a rigging is that it allows to define geometrical objects globally on M. We say that we have a null rigging in case the restriction of L to the null hypersurface is a null vector field.

Definition 2.2. ([10]) The rigged vector field of ζ is the g_η-metrically equivalent vector field to the $1-$form η and it is denoted by ξ.

As in the Lorentzian case, the following also holds:

Lemma 2.2. *([10]) The rigged vector field ξ is the unique null vector field in M such that $\overline{g}(\zeta, \xi) = 1$.*

A screen distribution on M is given by $\mathcal{S}(\zeta)$. and the corresponding null transverse vector field to $\mathcal{S}(\zeta)$ is

$$N = \zeta - \frac{1}{2}\overline{g}(\zeta,\zeta)\xi. \tag{2.6}$$

In [2], the authors, after fixing a pair of normalization, constructed an associated Riemannian metric to the " normalized null structure". These ideas have been generalized and improved in [10] where authors used riggings defined on neighborhood of the null hypersurface.

In fact, from (2.2) and (2.3) one shows that, conversely, a choice of a transversal bundle $tr(TM)$ determines uniquely the distribution $\mathcal{S}(N)$. A vector field N as in

(2.2) is called a null rigging of M. It is noteworthy that the choice of null transversal vector field N along M determines both the null transversal vector bundle, the screen distribution and a unique radical vector field ξ, say rigged vector field , satisfying (2.2).

Throughout the paper, we fix a null rigging N for M on $\overline{M}$. In particular this rigging fixes a unique null vector field $\xi \in \Gamma(TM^{\perp})$ called the associated rigged vector field, and a unique screen distribution $\mathcal{S}(N)$, all of them globally defined on M such that (2.1), (2.2) and (2.3) hold. From now on, we denote the normalized (or rigged) null hypersurface by the triplet (M, g, N) where $g = \overline{g}_{|M}$ is the first fundamental form and N a null rigging for M.

Let $\overline{\nabla}$ be the Levi-Civita connection on $(\overline{M}, \overline{g})$, and ∇ the rigged connection on (M, g) induced from $\overline{\nabla}$ through the projection along N. On a normalized null hypersurface (M, g, N), the Gauss and Weingarten type formulas are given by:

$$\overline{\nabla}_X Y = \nabla_X Y + B^N(X, Y)N, \tag{2.7}$$

$$\overline{\nabla}_X N = -A_N X + \tau^N(X)N, \tag{2.8}$$

$$\nabla_X PY = \nabla^*_X PY + C^N(X, PY)\xi, \tag{2.9}$$

$$\nabla_X \xi = -\overset{\star}{A}_\xi X - \tau^N(X)\xi, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$, where B^N (resp. C^N) is the local second fundamental form on M (resp. the screen distribution), A_N (resp. $\overset{\star}{A}_\xi$) is the shape operator on M (resp. the screen distribution), ∇^* denotes the Levi-Civita connection on the screen distribution, and P is the projection from sections of TM onto sections of $\mathcal{S}(N)$ with respect to (2.1). In general, the connection ∇ is not metric but satisfies

$$(\nabla_X g)(Y, Z) \quad = \quad B^N(X, Y)\eta(Z) + B^N(X, Z)\eta(Y). \tag{2.11}$$

The second fundamental forms are related to their shape operators by

$$B^N(X, Y) = g(\overset{\star}{A}_\xi X, Y), \ \overline{g}(\overset{\star}{A}_\xi X, N) = 0. \tag{2.12}$$

$$C^N(X, PY) = g(A_N X, PY), \ \overline{g}(A_N Y, N) = 0, \ \forall \ X, Y \in \Gamma(TM). \tag{2.13}$$

And

$$B^N(X, \xi) = 0 \ , \overset{\star}{A}_\xi \ \xi = 0. \tag{2.14}$$

τ^N is the $1-$form defined by $\tau^N(X) = \overline{g}(\overline{\nabla}_X N, \xi)$. The trace of $\overset{\star}{A}_\xi$ with respect to $\overline{g}$ is the lightlike (non normalized) mean curvature of M, explicitly given by

$$H_x = \sum_{i=2}^{n+1} \overline{g}(e_i, e_i)\overline{g}(\overset{\star}{A}_\xi (e_i), e_i),$$

being $(e_2,\ldots,e_{n+1})$ an orthonormal basis of $\mathcal{S}(N)$ at x. Denote by $\overline{R}$ and R the Riemann curvature tensors of $\overline{\nabla}$ and ∇, respectively. Recall the following Gauss-Codazzi equations [6] for all $X,Y,Z\in\Gamma(TM),\xi\in\Gamma(TM^{\perp})$.

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z,\xi) &= (\nabla_X B^N)(Y,Z)-(\nabla_Y B^N)(X,Z)\\ &+ B^N(Y,Z)\tau^N(X)-B^N(X,Z)\tau^N(Y).\end{aligned}\tag{2.15}$$

$$\begin{aligned}\bar{g}(\overline{R}(X,Y)Z,PW) &= g(R(X,Y)Z,PW)\\ &+ B^N(X,Z)C^N(Y,PW)-B^N(Y,Z)C^N(X,PW)\end{aligned}\tag{2.16}$$

$$\begin{aligned}\overline{g}(\bar{R}(X,Y)\xi,N) &= \overline{g}(R(X,Y)\xi,N)\\ &= C^N(Y,\overset{\star}{A}_\xi X)-C^N(X,\overset{\star}{A}_\xi Y)-2d\tau^N(X,Y).\end{aligned}\tag{2.17}$$

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)PZ,N) &= (\nabla_X C^N)(Y,PZ)-(\nabla_Y C^N)(X,PZ)\\ &+\tau^N(Y)C^N(X,PZ)-\tau^N(X)C^N(Y,PZ).\end{aligned}\tag{2.18}$$

Definition 2.3. A normalized null hypersurface (M,g,N) of a pseudo-Riemannian manifold $(\overline{M},\overline{g})$ is said to have a conformal screen [1] if there exists a non vanishing smooth function φ on M such that $A_N=\varphi A^{\star}_{\xi}$ holds.

A remarkable fact is that due to the degeneracy of the induced metric g on the null hypersurface M, it is not possible to define the natural dual(musical) isomorphisms $\flat$ and $\sharp$ between the tangent vector bundle TM and its dual $T^{\star}M$ following the usual Riemannian way. However, this construction is made possible by setting a rigging (normalization) N.

Definition 2.4. The associated metric to (M,g,N) is the metric g_η given by

$$g_\eta = g+\eta\otimes\eta.\tag{2.19}$$

In the latter case, $\Gamma(\mathcal{S}(N))$ coincides with $\Gamma(TM)$, and as a consequence the $1-$ forme η vanishes identically and the projection morphism P becomes the identity map on $\Gamma(TM)$. Cleary, g_η defines a non-degenerate metric on M which plays an important role in defining the usual differential operators gradient, divergence, Laplacien with respect to degenerate metric g on null hypersurface (we refer to [2] for further details). It is nothing to see that

$$g_\eta(\xi,\xi)=1, \eta(.)=g_\eta(\xi,.)\tag{2.20}$$

Lemma 2.3. *[2]. Let (M,g,N) be a normalized null hypersurface of pseudo-Riemannian manifold $(\overline{M}^{n+2},\overline{g})$, ∇^η the Levi-Civita connection of g_η and ∇ the induced connection on M due to N. Ten, for all $X,Y,Z\in\Gamma(TM)$, we have*

$$\begin{aligned}(\nabla_X g_\eta)(Y,Z) &= \eta(Y)[B^N(X,PZ)-C^N(X,PZ)]\\ &+\eta(Z)[B^N(X,PY)-C^N(X,PY)]\\ &+2\tau^N(X)\eta(Y)\eta(Z).\end{aligned}\tag{2.21}$$

We relate the main geometric objects of both null and associated non-degenerate geometry on the null hypersurface in the following.

Proposition 2.4. *[13] Let (M, g, N) be a normalized null hypersurface with rigged vector field ξ. Then, for all $X, Y \in \Gamma(TM)$, we have*

$$\begin{aligned} \nabla^{\eta}_X Y &= \nabla_X Y + \frac{1}{2}\big[2g(\overset{\star}{A}_{\xi} X, Y) - g(A_N X, Y) - g(A_N Y, X) \\ &\quad +\eta(X)\tau^N(Y) + \eta(Y)\tau^N(X)\big]\xi + \eta(X)(i_Y d\eta)^{\sharp_\eta} + \eta(Y)(i_X d\eta)^{\sharp_\eta}. \end{aligned} \tag{2.22}$$

In particular for a closed normalization,

$$\begin{aligned} \nabla^{\eta}_X Y &= \nabla_X Y + \frac{1}{2}\big[2g(\overset{\star}{A}_{\xi} X, Y) - g(A_N X, Y) - g(A_N Y, X) \\ &\quad +\eta(X)\tau^N(Y) + \eta(Y)\tau^N(X)\big]\xi. \end{aligned} \tag{2.23}$$

Let R^η and R denote the Riemann curvature tensors of ∇^η and ∇ respectively. From the proposition (2.4), we prove the following:

Proposition 2.5. *[13] Let (M, g, N) be a closed normalized null hypersurface with rigged vector field ξ. Then, for all $X, Y, W \in \Gamma(TM)$ we have,*

$$\begin{aligned} g_\eta(R^\eta(X,Y)Z, PW) &= g(R(X,Y)Z, PW) \\ &\quad + \frac{1}{2}\{\phi_{(X,Z)} B^N(Y, W) - \phi_{(Y,Z)} B^N(X, W)\}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} g_\eta(R^\eta(X,Y)Z, \xi) &= -g(R(X,Y)\xi, PZ) \\ &\quad - \frac{1}{2}\big[g(A_N\xi, Y)B^N(X, Z) - g(A_N\xi, X)B^N(Y, Z)\big] \\ &\quad - \frac{1}{2}\big[\tau^N(X)B^N(Y, Z) - \tau^N(Y)B^N(X, Z)\big], \end{aligned} \tag{2.25}$$

where $\phi(X, Z)$ is given by :

$$\begin{aligned} \phi_{(X,Z)} &= 2B^N(X, Z) - g(A_N X, Z) - g(A_N Z, X) \\ &\quad +\tau^N(X)\eta(Z) + \tau^N(Z)\eta(X). \end{aligned} \tag{2.26}$$

3 Relationship between the geometry and existence of stable currents of a null hypersurface in Lorentzian manifold

3.1 Current screen of a null hypersurface in a Lorentzian manifold

Let $(M, g, N) \to (\overline{M}^{n+2}, \bar{g})$ be a closed normalized null hypersurface of a $(n+2)-$dimensional Lorentzian manifold. Therefore, (M, g_η) is a $(n+1)-$dimensional Riemannian Manifold and by the Nash Theorem, can be isometrically immersed into a higher dimensional real Euclidean space $\mathbb{R}^{n+1+m}$. We will give conditions on both null hypersurface which implies that M^{n+1} has no stable submanifolds (or no stable integral currents). Let $\mathfrak{D}$ be a $p-$rectifiable subset called a $\mathfrak{H}^p_\eta-$borel subset on (M, g_η) with finite measure $\|\mathfrak{D}\|$ and $p \leq n+1$ so that $\int_M d\|\mathfrak{D}\|$ gives $\mu^p_\eta(\mathfrak{D})$ (the "volume" of $\mathfrak{D}$).

At $\mathfrak{H}^p_\eta-$ almost all points $x \in \mathfrak{D}$, there exists a $p-$dimensional subspace say $T_x\mathfrak{D}$ of T_xM. One can then views $\mathfrak{D}$ as a $p-$dimensional submanifold of M. (See [8, 3.3] for more details.)

Let $\zeta : \mathfrak{D} \to \bigwedge^p TM$ be a $\mathfrak{H}^p_\eta$-mesurable section of the p^{th} exterior product of the tangent bundle of M, ζ_x is a simple vector of unit length which represents $T_x(\mathfrak{D})$. The pair $\mathfrak{S} = (\mathfrak{D}, \zeta)$ is called an oriented $p-$rectifiable set.

One can then define a functional from the set $\Omega^p(M)$ of $p-$forms on M by taking

$$\mathfrak{S}(\omega) = \int_{\mathfrak{D}} \omega(\zeta_x)\ d\|\mathfrak{D}\|(x),$$

for all $p-$form $\omega \in \Omega^p(M)$. Thus $\mathfrak{S} = (\mathfrak{D}, \zeta)$ is a $p-$current as a continuous linear functional on $\Omega^p(M)$ with its usual topology.

We assume from now on that the $p-$current $\mathfrak{S}$ is a submanifold of the screen distribution. This means that at $\mathfrak{H}^p_\eta-$almost all point $x \in \mathfrak{D}$, $T_x\mathfrak{D}$ is a sub-bundle of $\mathcal{S}(N)_x$. When $p = n$, $\mathfrak{D}$ is an integral submanifold of the screen distribution and we say $\mathfrak{S}$ is a **current screen** of (M, g_η).

3.2 The screen Dirac measure on null hypersurface

Definition 3.1. The Dirac measure δ_M on the hypersurface M is the one-form distribution given by

$$\langle \delta_M(V), \Phi\rangle = \int_M i^\star(j_V\Phi), \Phi \in \Omega^n(\overline{M}) \tag{3.1}$$

where $i^\star : \Omega^{n-1}(\overline{M}) \to \Omega^{n-1}(M)$ is the pullback induced by the inclusion$i$ and j_V the interior product of a $n-$form $\in \Omega^n(\overline{M})$ given by $j_V\Phi(V_1, \cdots, V_{n-1}) = \Phi(V, V_1, \cdots, V_{n-1})$.

Lemma 3.1. *$\delta_M(V) = 0$ if $V|_M = 0$.*

Proof. Let $p \in M$ and $(\mathcal{U}, \psi = (x_1, \cdots, x_n))$ be coordinates around p adapted to the null hypersurface. Then $\partial_{x_j}|_M \in \Gamma(\mathcal{U} \cap M, TM|_{U\cap M})$ for $j = 2, \cdots, n$ and we can write $V = \displaystyle\sum_{j=2}^n V^j \frac{\partial}{\partial_{x_j}}|_M$. Take $\Phi = f dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \in \Omega^n_c(\mathcal{U})$, one has $i_{\partial_j}\Phi(V_1, \cdots, V_{n-1}) = f(dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n)(\partial_j, V_1, \cdots, V_{n-1}) = 0$ for $V_1, \cdots V_{n-1} \in \Gamma(M, TM)$ $2 \leq j \leq n$ and thus $\delta_M(\partial_{x_j}) = 0$ for $j = 2, \cdots, n$ and thus $i^\star(j_V\Phi)$ are zero on M) which implies that which implies that supp$\delta_M \subset M$ and $\delta_M(V) = 0$.

□

Definition 3.2. The Screen Dirac measure on null hypersurface is denoted by $\delta_M(\eta)$. Where η is one-form satisfying (2.5) and (2.20).

Proposition 3.2. *Let $(\overline{M}^{n+2}, \overline{g})$ be an oriented semi-Riemanian manifold and (M, g_η) be a closed Riemannian hypersurface with $\eta(\xi) = 1$ and $\eta(U) = 0 \forall U \in \mathcal{S}(N)$. Then $\delta_M = \delta_M(\eta)$.*

Proof. Showing $\delta_M = \delta_M(\eta)$ is equivalent to showing $\delta_M(V) = \delta_M(\eta)(V) \forall V \in \Gamma(T\overline{M})$. Also the support of both of $\delta_M(\eta)$ and δ_M is contained in M so it suffices to show their equality locally around M. To do this let $p \in M$ and $(\mathcal{U}, \psi)$ be canonical coordinates around p. Then every $\phi \in \Omega^n_c(\mathcal{U})$ can be written as

$$\phi = f(|det(g_{\eta ij})^{\frac{1}{2}}_{1 \le i,j \le n+1}|)dx^1 \wedge \cdots dx^{n+1},$$

with a unique $f \in D(\mathcal{U})$ (smooth function with compact support in M). Let

$$\widetilde{\Omega}_{g_\eta} = |det(g_{\eta_{ij}})^{\frac{1}{2}}_{1 \le i,j \le n+1}|dx^2 \wedge \cdots dx^{n+1}.$$

Every $u \in T(\mathcal{U})$ can be written as $u = u^i \partial_i$ and $\eta = dx^1$ on $\mathcal{U} \cap M$. We want to show that

$$\langle \delta_M(u), \phi \rangle = \langle \delta_M(\eta)(u), f \rangle. \tag{3.2}$$

This allows us to calculate
$i^\star(j_u(dx^1 \wedge dx^2 \wedge \cdots dx^{n+1})) = i^\star(j_u(dx^1) \wedge dx^2 \wedge \cdots dx^{n+1})) - i^\star(dx^1)(j_u \wedge (dx^2 \wedge \cdots dx^{n+1}))$
$= u^1_{|M} i^\star(dx^2 \wedge \cdots dx^{n+1}) - 0 = \eta(u)(dx^2 \wedge \cdots dx^{n+1})_{|M}$. Thus the left hand side of (3.2) becomes

$$\begin{aligned}
\int_{M \cap \mathcal{U}} i^\star(j_u \phi) &= \int_{M \cap \mathcal{U}} f_{|M} \eta(u) |det(g_{\eta_{ij}})^{\frac{1}{2}}_{1 \le i,j \le n+1}| dx^1 \wedge \cdots dx^{n+1}_{|M} \\
&= = \int_{M \cap \mathcal{U}} f_{|M} \eta(u) |det(g_{\eta_{ij}})^{\frac{1}{2}}_{2 \le i,j \le n+1}| dx^2 \wedge \cdots dx^{n+1}_{|M} \\
&= \int_{M \cap \mathcal{U}} f_{|M} \eta(u) \widetilde{\Omega}_{g_\eta} \\
&= \langle \delta_M(\eta)(u), f \rangle
\end{aligned}$$

□

We now summarize the part of the variational theory of currents we need. Let X be a smooth parallel vector field on $\mathbb{R}^{n+1+m}$ and let $\phi^{X^T}_t$ be the flow of X. Then for small t we can deform $\mathfrak{D}$ along the flow of X^T (obtained by taking the orthogonal projections of X onto tangent spaces of M^{n+1}) to get a new current ${\phi_t}^{X^T}_\star \mathfrak{D}$. In this case the formula for the second variation $\frac{d^2}{dt^2}|_{t=0}\ \mu^p_\eta({\phi_t}^{X^T}_\star \mathfrak{D})$ can be greatly simplified by use of the fundamental equations of Gauss and Weingarten in submanifold theory. The resulting formula still has one term (involving the covariant derivative of the Weingarten map) that is hard to understand. It is a well-known fact that $\overset{\star}{A}_\xi$ is diagonalizable. Let $(\mathfrak{n}_0 = \xi, \mathfrak{n}_1, \ldots, \mathfrak{n}_l, e_1, \ldots, e_p)$ be a quasi-orthonormal basis of eigenvectors of $\overset{\star}{A}_\xi$ (with $l+p = n$) in a neighborhood of a point $x \in \mathfrak{D}$ with corresponding eigenfunctions $\nu_0 = 0, \nu_1, \ldots, \nu_l, \lambda_1, \ldots, \lambda_p$ and such that $T_x\mathfrak{D} = span(e_1, \ldots, e_p)$.

We shall agree on the following range of indices $a, b = 0, \dots l; i, j = 1, \dots, n$. Let us consider the functional

$$
\begin{aligned}
T\mathfrak{D}(X^T) &= \Big(\sum g_\eta(\nabla^\eta_{e_i} X^T, e_i)\Big)^2 + 2\sum g_\eta(\nabla^\eta_{e_i} X^T, \mathfrak{n}_a)^2 + \sum g_\eta(R^\eta(X^T, e_i)X^T, e_i) \\
&\quad - \sum \|\nabla^\eta_{e_i} X^T\|^2_\eta, \qquad (3.3)
\end{aligned}
$$

for a vector field X on M.

Definition 3.3. A current $\mathfrak{S}$ is said to be stable if for each vector field X^T there is an $\varepsilon > 0$ such that

$$\mu^p_\eta(\phi^X_{t\star}\mathfrak{D}) \geqslant \mu^p_\eta(\mathfrak{D}),$$

for $|t| < \varepsilon$. For stable currents $\mathfrak{D}$ there is the stability inequality $\frac{d^2}{dt^2}|_{t=0}\, \mu^p_\eta(\phi_{t\star}\mathfrak{D}) \geqslant 0$.

Lemma 3.3. *[12] If $\mathfrak{S}$ is a stable $p-$dimensional current, then for any vector field X which is the gradient of a function on M, we have*

$$\int_M T_{\mathfrak{D}}(X)\ d\|\mathfrak{D}\| \geqslant 0. \qquad (3.4)$$

Using Lemma 3.3 we proof the following results.

Theorem 3.4. *Let $f : (M, g, N)$ be a closed normalization for a null hypersurface with rigged vector field ξ of a $(n+2)-$dimensional Lorentzian manifold not simply connected where the first De Rham cohomology group $H^1(M, \mathbb{R})$ is trivial or the one form η be exact. If $\mathfrak{S}$ as defined above is a stable $p-$current, then*

$$\int_M 2\sum_{i<j} \lambda_i\lambda_j + \sum \bar{g}(\overline{R}(\xi, e_i)\xi, e_i)\ d\|\mathfrak{D}\| \geqslant 0.$$

In particular if $\mathfrak{D}$ is a stable current screen (just $p = n$) then

$$\int_M \Big(tr(\overset{\star}{A}_\xi)\Big)^2 - tr\left(\overset{\star}{A}{}^2_\xi\right) + \overline{Ric}(\xi, \xi)\ d\|\mathfrak{D}\| \geqslant 0.$$

Proof. Since N is closed, by Poincaré Lemma, there exists locally a function f such that $df = \eta$. By relation (2.20) $\eta = g_\eta(\xi, \cdot)$ which implies that ξ is the gradient of f. Using (2.23), we have $\nabla^\eta_{e_i}\xi = -A^\star_\xi e_i = -\lambda_i e_i$. Therefore,

$$g_\eta(\nabla^\eta_{e_i}\xi, e_i) = -g(\overset{\star}{A}_\xi\ e_i, e_i) = -\lambda_i; \qquad (3.5)$$

$$g_\eta(\nabla^\eta_{e_i}\xi, \mathfrak{n}_a) = -g(\overset{\star}{A}_\xi\ e_i, \mathfrak{n}_a) = 0; \qquad (3.6)$$

and

$$\|\nabla^\eta_{e_i}\xi\|^2_{g_\eta} = g(A^\star_\xi e_i, A^\star_\xi e_i) = \lambda_i^2. \qquad (3.7)$$

Using the relations (2.25) and (2.16) we have

$$g_\eta(R^\eta(\xi, e_i)\xi, e_i) = g(R(\xi, e_i)\xi, e_i) = \bar{g}(\overline{R}(\xi, e_i)\xi, e_i). \qquad (3.8)$$

From here, remarking that $(\sum \lambda_i)^2 - \sum \lambda_i^2 = 2\sum_{i<j} \lambda_i\lambda_j$ one obtains

$$T_{\mathfrak{D}}(\xi) = 2\sum_{i<j} \lambda_i\lambda_j + \sum \bar{g}(\overline{R}(\xi, e_i)\xi, e_i). \tag{3.9}$$

□

In the following we give the conditions on the nonexistence of stable currents screen of a closed normalized null hypersurface equipped with the associated Riemannian metric. As the consequence, there doesn't stable current on a closed normalized null compact hypersurface with geodesic rigged vector field ξ with

$$\int_M \left(tr(\overset{\star}{A}_\xi)\right)^2 - tr\left(\overset{\star}{A}_\xi^2\right) + \overline{Ric}(\xi,\xi)\ d\|C\| < 0.$$

Lemma 3.5. *[3] Let $f : (M, g, N)$ be a normalization for a null hypersurface of a Lorentzian manifold $(\overline{M}^{n+2}, \bar{g})$. Then we have: $\overline{Ric}(\xi) = \xi.(H) + \tau(\xi)H - \| \overset{\star}{A}_\xi \|^2$.*

Theorem 3.6. *Let $f : (M, g, N) \to (\overline{M}^{n+2}, \bar{g})$ be a closed normalized null compact hypersurface with geodesic rigged vector field ξ, of a $(n+2)-$dimensional Lorentzian manifold. If*

$$\int_M \left(tr(\overset{\star}{A}_\xi)\right)^2 - tr\left(\overset{\star}{A}_\xi^2\right) + \xi.(H) + \tau^N(\xi)H - \| \overset{\star}{A}_\xi \|^2\ d\|\mathfrak{D}\| < 0. \tag{3.10}$$

Then, there is no stable current screen in (M, g_η).

Theorem 3.7. *Let $i : (M, g, N) \to (\overline{M}^3, \bar{g})$ be a be a proper totally umbilical closed rigged null compact surface with an umbilical factor ρ and geodesic rigged vector field ξ of a $3-$dimensional Lorentzian manifold such that $\xi.\rho \leq 0$. Then, the associated Riemannian surface (M, g_η) cannot contain a stable current screen.*

Proof. From (3.10), we have

$$\left(tr(\overset{\star}{A}_\xi)\right)^2 - tr\left(\overset{\star}{A}_\xi^2\right) + \xi.(H) + \| \overset{\star}{A}_\xi \|^2 = -2\rho^2 + n\xi.\rho < 0, \tag{3.11}$$

that leads to results $-2\rho^2 \geqslant -n\xi.\rho$. The result follows from Theorem 3.6. □

In the following, we give an example with respect to Theorem 3.6.

Example 3.4. Let f be the immersion

$$\begin{array}{rcl} f : M_0^{n+1} & \longrightarrow & \mathbb{R}_1^{n+2} \\ (x^1, ..., x^n) & \longmapsto & [x^1, ..., x^n, (x^1)^2 + ... + (x^n)^2], \epsilon = \pm 1 \end{array}$$

and the null hypersurface $M_0^{n+1} = \{x = (x_0, ..., x_{n+1}) \in \mathbb{R}_1^{n+2}, -x_0^2 + \sum_{a=1}^{n+1} x_a^2 = 0\}$.

Locally, M_0 is the graph $x_0 = \varepsilon(x^2 + y^2 + z^2)^{\frac{1}{2}}$ and it is an obvious fact that this is a null hypersurface. Let N be the null rigging of M defined by

$$N = -x_0\partial_0 + \sum_{a=1}^{n+1} x_a\partial_a. \tag{3.12}$$

and the rigged field

$$\xi_0 = \frac{1}{2x_0^2}\left(x_0\partial_0 + \sum_{a=1}^{n+1} x_a\partial_a\right). \tag{3.13}$$

Let P and D denote the projection morphism of the tangent bundle TM on to $\mathcal{S}(N)$ and the Levi-Civita connection $\mathbb{R}_1^{n+2}$ respectively. Then,

$$D_X\xi \quad = \quad \left[2x_0^2X.(\frac{1}{2x_0^2}) + \frac{1}{2x_0^2}\eta(X)\right]\xi + \frac{1}{2x_0^2}PX \ \forall X \in \Gamma(TM). \tag{3.14}$$

By a direct computation , we have

$$\overset{\star}{A}_\xi X = -\frac{1}{2x_0^2}PX \Rightarrow \overset{\star}{A}_\xi = -\frac{1}{2x_0^2}, \| \overset{\star}{A}_\xi \|^2 = \frac{-1}{4x_0^4}, tr\ \overset{\star}{A}_\xi = \frac{-n}{2x_0^2}, tr(\overset{\star}{A}_\xi)^2 = \frac{n}{4x_0^4} \tag{3.15}$$

and

$$\tau^N(X) = -\left[2x_0^2X.(\frac{1}{2x_0^2}) + \frac{1}{2x_0^2}\eta(X)\right] \tag{3.16}$$

$$X.\left(\frac{1}{2x_0^2}\right) = \left(X_0\partial_0 + \sum_{a=1}^{n+1} X_a\partial_a\right).\left(\frac{1}{2x_0^2}\right) = X_0\partial_0(\frac{1}{2x_0^2}) = -\frac{dx_0}{x_0^3}(X) \tag{3.17}$$

Take $X = X_0\partial_0 + \sum_{b=1}^{n+1} X_b\partial_b$, we compute

$$\eta(X) = \langle N, X\rangle \tag{3.18}$$

$$= \langle -x_0\partial_0 + \sum_{a=1}^{n+1} x_a\partial_a, X_0\partial_0 + \sum_{b=1}^{n+1} X_b\partial_b\rangle == 2x_0dx_0(X) \tag{3.19}$$

and the $1-$form η is given by

$$\eta = 2x_0dx_0. \tag{3.20}$$

It is obvious that $d\eta = 0$ which shows that η is closed and the equality gives

$$\tau^N(X) = -\left[2x_0^2(-\frac{dx_0}{x_0^3}) + \frac{1}{2x_0^2}2x_0dx_0\right](X). \tag{3.21}$$

Thus

$$\tau^N = \frac{dx_o}{x_0}. \tag{3.22}$$

The mean curvature H on $M_0 \subset \mathbb{R}_1^{n+2}$ is given by:

$$H = -\frac{n}{2(n+1)}\frac{1}{x_0^2}. \tag{3.23}$$

We compute

$$\xi.H = \frac{n}{2(n+1)}\frac{1}{x_0^4} \tag{3.24}$$

and

$$\left(tr(\overset{\star}{A}_\xi)\right)^2 - tr\left(\overset{\star}{A}{}_\xi^2\right) + \xi.(H) + \| \overset{\star}{A}_\xi \|^2 = \frac{1}{x_0^4}\frac{n-1}{4(n+1)} \tag{3.25}$$

which show the existence of stable integral currents in M_0 with $n \geq 2$.

To prove the generalized version of the Lawson-Simons theorem [12], we need the following lemma

Lemma 3.8. *[5] Let $a_1, \cdots, a_m, b_1, \cdots, b_m$ be real numbers satisfying $\sum_i a_i^2 = \sum_i b_i^2 = 1$ and $\sum_i a_i b_i = 0$. and $\sum_j a_j b_j = 0$. Then we have the following: For given real numbers $\lambda_1, \cdots, \lambda_m$, $2\big(\sum_i \lambda_i a_i b_i\big)^2 - \sum_i \lambda_i a_i^2 \sum_i \lambda_i b_i^2 \leq 2\lambda^2$*

Theorem 3.9. *Let N be a $n-$dimensional compact submanifold of M^{m+1} which is a $m+1$ -dimensional closed null hypersurface of a $m+2$ -dimensional Lorentzian manifold $\mathbb{R}_1^{m+2}$. If*

$$\sum_{i,l}\left[2\|h'(e_i, e_l)\|^2 - \langle h'(e_i, e_i), h'(e_l, e_l)\rangle\right] < 2p(p-n)\lambda(x)^2 \tag{3.26}$$

$\forall\{e_i, e_l\} \in \mathcal{S}_x^0(1)$, $e_l \in O_\eta(e_i)$ is satisfied for every $x \in N$ any orthonormal basis $\{e_i, e_l\}$ of $T_xN, i = 1, \cdots, p; l = p+1, \cdots, n$, there is no stable integral $p-$ current in N. Here h' is the second fundamental form of N in M^{m+1}, and $\lambda(x)^2$ is the maximum at x of the square of principal curvatures of M^{m+1}.

Proof. Let h' be the second fundamental form of N in M and $\{e_i, e_l\}$ an orthonormal basis of the tangent space T_xN for a point $x \in N$ where $1 \leq i \leq p$ and $p+1 \leq l \leq n$. Since N is a submanifold of M^{m+1}, it is also a submanifold in $\mathbb{R}_1^{m+2}$. Let D be the flat connection of $\mathbb{R}_1^{m+2}$, ∇' and ∇ the Levi-Civita connection N with respect to M^{m+1} and $\mathbb{R}^{n+2}$ respectively. Let $\Gamma(T^\perp(\mathbb{R}_1^{m+2}, N)), \Gamma(T^\perp(\mathbb{R}_1^{m+2}, M^{m+1}))$ and $\Gamma(T^\perp(M^{m+1}, N))$ the respective spaces of normal vector fields of N^n, M^{m+1} and N with respectively to $\mathbb{R}_1^{m+2}, \mathbb{R}_1^{m+2}$ and M^{m+1}. The shape operator A_ζ determined by $\zeta \in \Gamma(T^\perp(\mathbb{R}_1^{m+2}, N^n))$ is given by

$$A_\zeta Y = -(D_Y\zeta)^T, \tag{3.27}$$

where $Y \in \Gamma(TN)$. If $\zeta \in \Gamma(T^\perp(M^m, N^n))$, then

$$A_\zeta Y = -(D_Y\zeta)^T = -\big(\overline{\nabla}_Y\zeta + \overline{h}(Y,\zeta)\big)^T = -\big(\overline{\nabla}_Y\zeta\big)^T = A'_\zeta Y \tag{3.28}$$

where $Y \in \Gamma(TN), \overline{\nabla}$ and $\overline{h}$, are the Levi-Civita connection and the second fundamentalnwith respect to $\mathbb{R}_1^{m+2}$ and A'_ζ is the shape operator determined by $\zeta \in \Gamma(T \perp (M^{m+1}, N))$. If ζ is the normal vector to M^{m+1}, then

$$A_\zeta Y = \overline{A}_\zeta Y. \tag{3.29}$$

$\overline{A}_\zeta$ is called shape operator determined by $\zeta \in \Gamma\big(T^\perp(\mathbb{R}_1^{m+2}, M^{m+1}\big)$. Let $A_\eta \mathrm{Y} = (\overline{A}_\eta \mathrm{Y})^T$ and let S be an $p-$ oriented Current. For $x \in N$, we have a tangent $p-$space $T_xS \subset T_xN$. Let $T_xN\{e_i, e_l\}$ be an orthonormal basis such that $\{e_i\}$ is an orthonormal basis of T_xS. Suppose that Suppose $\{\zeta_u\}$ is an orthonormal basis of $\Gamma(T_x^\perp(\mathbb{R}_1^{m+2}, N))$. Let $A_u = A_{\zeta_u}$. Then $\{e_i, e_l, e_{\zeta_u}\}$ is an orthonormal basis of $\mathbb{R}_1^{m+2}$. We have the bilinear form

$$\begin{aligned} I(W,W) &= \int_M \mathbb{Q}(W)d\|S\| \\ &= \sum_{i,j=1}^{p} \langle h(e_i,e_i), W^\perp\rangle\langle h(e_j,e_j), W^\perp\rangle + \sum_{i=1}^{p}\sum_{l=p+1}^{n} \langle h(e_i,e_l), W^\perp\rangle\langle h(e_i,e_l), W^\perp\rangle \\ &+\sum_{i=1}^{p}\langle R(W^\perp,e_i)W^\perp,e_i)\rangle - \sum_{i,j=1}^{p}\langle h(e_i,e_j), W^\perp\rangle\langle h(e_i,e_j), W^\perp\rangle. \end{aligned} \tag{3.30}$$

$$\begin{aligned} tr\mathbb{Q} &= \sum_{i,j=1}^{p}\sum_{\zeta_u=p+1}^{m+2} \langle h(e_i,e_i),\zeta_u\rangle\langle h(e_j,e_j),\zeta_u\rangle \\ &+\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_u=p+1}^{m+2} \langle h(e_i,e_l),\zeta_u\rangle\langle h(e_i,e_l),\zeta_u\rangle \\ &-\sum_{i,j=1}^{p}\langle R(e_i,e_j)e_j,e_i\rangle - \sum_{i,l=1}^{p}\langle R(e_i,e_l)e_l,e_i\rangle \\ &-\sum_{i,j=1}^{p}\sum_{\zeta_u=p+1}^{m+2} \langle h(e_i,e_j),\zeta_u\rangle\langle h(e_i,e_j),\zeta_u\rangle \\ &\geq 0. \end{aligned} \tag{3.31}$$

$$
\begin{aligned}
tr\mathbb{Q} &= 2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_u=p+1}^{m+2}\langle h(e_i,e_l),\zeta_u\rangle\langle h(e_i,e_l),\zeta_u\rangle \\
&\quad -\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_u=p+1}^{m+2}\langle h(e_i,e_i),\zeta_u\rangle\langle h(e_l,e_l),\zeta_u\rangle \\
&= 2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_u=p+1}^{m+2}\langle A_{\zeta_u}e_i,e_l)\rangle\langle A_{\zeta_u}e_i,e_l)\rangle \\
&\quad -\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_u=p+1}^{m+2}\langle A_{\zeta_u}e_i,e_i\rangle\langle A_{\zeta_u}e_l,e_l\rangle. \qquad (3.32)
\end{aligned}
$$

At a point $x \in N$, we take an orthonormal basis $\{\zeta_1,\cdots,\zeta_{m+1-n},\zeta\}$ of $T_x^{\perp}(\mathbb{R}_1^{m+2},N)$ $\{\zeta_v\}$ and ζ are the ortonormal bases of $T_x^{\perp}(M^{m+1},N)$ and $T_x^{\perp}(\mathbb{R}_1^{m+2},M^{m+1})$ respectively and $v=1,\cdots,m+1-n$.

Let

$$\overline{A}_\eta(\tilde{e}_a) = -\lambda_a\tilde{e}_a \qquad (3.33)$$

for $a=1,\cdots,m+1$ where $\{\tilde{e}_a\}$ is an orthonormal basis of $T_x^{\perp}M^{m+1}$ and $\overline{A}_\eta$ is the so-called shape operator determined by $\zeta \in \Gamma(T^{\perp}(\mathrm{R}^{m+2},M^{m+1}))$.

By (3.28,3.29),we have

$$
\begin{aligned}
tr\mathbb{Q} &= 2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle A_{\zeta_v}e_i,e_l)\rangle^2 - \sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle A_{\zeta_v}e_i,e_i\rangle\langle A_{\zeta_v}e_l,e_l\rangle \\
&\quad +2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta}\langle A_{\zeta}e_i,e_l)\rangle^2 - \sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta}\langle A_{\zeta}e_i,e_i\rangle\langle A_{\zeta}e_l,e_l\rangle \\
&= 2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle A'_{\zeta_v}e_i,e_l)\rangle^2 - \sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle A'_{\zeta_v}e_i,e_i\rangle\langle A'_{\zeta_v}e_l,e_l\rangle. \\
&\quad +2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta}\langle \overline{A}_{\zeta}e_i,e_l)\rangle^2 - \sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta}\langle \overline{A}_{\zeta}e_i,e_i\rangle\langle \overline{A}_{\zeta}e_l,e_l\rangle \\
&= 2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle h'(e_i,e_l),\zeta_v\rangle^2 \\
&\quad -\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle h'(e_i,e_i),\zeta_v\rangle\langle h'(e_l,e_l),\zeta_v\rangle. \\
&\quad +2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta}\langle \overline{A}_{\zeta}e_i,e_l)\rangle^2 - \sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta}\langle \overline{A}_{\zeta}e_i,e_i\rangle\langle \overline{A}_{\zeta}e_l,e_l\rangle. \qquad (3.34)
\end{aligned}
$$

Suppose that $e_i = e_i^a \tilde{e}_a, e_\alpha = e_\alpha^a \tilde{e}_a$. From (3.33) in (3.34), the relation (3.34) becomes

$$\begin{aligned} tr\mathbb{Q} &= 2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle h'(e_i,e_l),\zeta_v\rangle^2 - \sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle h'(e_i,e_i),\zeta_v\rangle\langle h'(e_l,e_l),\zeta_v\rangle \\ &+\sum_{i=1}^{p}\sum_{l=p+1}^{n} 2\big(\sum_a e_i^a e_l^a \lambda_a)^2\big) - \sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_a (e_i^a)^2\lambda_a \sum_a (e_i^a)^2\lambda_a \end{aligned} \tag{3.35}$$

Using the lemma 3.8, (3.35) becomes

$$\begin{aligned} tr\mathbb{Q} &= 2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle h'(e_i,e_l),\zeta_v\rangle^2 - \sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle h'(e_i,e_i),\zeta_v\rangle\langle h'(e_l,e_l),\zeta_v\rangle \\ &+2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\lambda^2 \geq 0 \\ &= 2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle h'(e_i,e_l),\zeta_v\rangle^2 - \sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle h'(e_i,e_i),\zeta_v\rangle\langle h'(e_l,e_l),\zeta_v\rangle \\ &+2p(n-p)\lambda^2 \geq 0. \end{aligned} \tag{3.36}$$

Using Lawson theorem, if

$$\begin{aligned} 2\sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle h'(e_i,e_l),\zeta_v\rangle^2 - \sum_{i=1}^{p}\sum_{l=p+1}^{n}\sum_{\zeta_v=1}^{m+1-n}\langle h'(e_i,e_i),\zeta_v\rangle\langle h'(e_l,e_l),\zeta_v\rangle \\ \leq -2p(n-p)\lambda^2(x), \end{aligned}$$

we conclude that there is no stable integral $p-$ current in N. □

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