Certain vector fields on 3-dimensional $N(k)$-contact metric manifolds

Debabrata Kar and Pradip Majhi

Abstract. The paper deals with the characterization of 3-dimensional $N(k)$-contact metric manifolds admitting Holomorphically Planar Conformal Vector (HPCV) fields. We obtain a necessary and sufficient condition for a 3-dimensional $N(k)$-contact metric manifold admitting HPCV field to be K-contact. Moreover, we prove that a 3-dimensional $N(k)$-contact metric manifold admitting a HPCV field is solenoidal under certain conditions.

M.S.C. 2010: 53C15, 53C25.
Key words: Killing vector field; closed conformal vector field; HPCV field; divergence; solenoidal; contact metric manifold; k-nullity distribution.

1 Introduction

In Riemannian manifolds (or pseudo-Riemannian manifolds) the Killing vector fields, named after Wilhelm Killing, have the quality of preserving the metric. Also the energy and momentum of a particle moving freely in flat spacetimes are conserved by the Killing vectors fields.

Definition 1.1. A smooth vector field X on a Riemannian manifold is called an infinitesimal isometry field or a Killing vector field if $\pounds_X g = 0.$, that is, the Lie derivative of the metric g vanishes in the direction of X.

The conformal vector fields generalize the Killing vector fields, which allow metrics preserve their conformal class.

Definition 1.2. A smooth vector field X on a Riemannian manifold is called a conformal vector field if there exists a smooth function ρ on M such that $\pounds_X g = 2\rho g$.

If ρ is constant, then X is called homothetic. Moreover, if the 1-form metrically equivalent to V is closed, then X is forenamed by a closed conformal vector field. The conformal vector fields have been investigated geometrically in ([4]-[7]).

Differential Geometry - Dynamical Systems, Vol.23, 2021, pp. 67-73.

Definition 1.3. A smooth vector field V on a contact metric manifold is aforesaid as holomorphically planar conformal vector field if it obeys the following condition:

hpcv1.1
$$\nabla_X V = aX + b\phi X, \tag{1.1}$$

where X is a smooth vector field and a, b are smooth functions.

In [11], Sharma introduced the notion of Holomorphically Planar Conformal Vector (HPCV) field.

The paper is arranged as follows: after introduction in Section 2, we discuss preliminaries of 3-dimensional $N(k)$-contact metric manifolds. In Section 3, we prove that any 3-dimensional $N(k)$-contact metric manifold admitting HPCV is K-contact, and conversely - under certain conditions. We also show that a HPCV field V is solenoidal in such a manifold with $\phi V = 0$, and obtain several related results.

2 3-dimensional N(k)-contact metric manifolds

A contact manifold is a smooth manifold M^{2n+1} of dimension $2n+1$, endowed with a 1-form η which globally satisfies $\eta \wedge (d\eta)^n \neq 0$. We know that there is an unique vector field ξ, called characteristic vector field so that $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. The same holds true in the case of a Riemannian metric g and a structural vector field ϕ of type (1,1) such that

hpcv2.1
$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

for any smooth vector fields X, Y. With the help of (2.1) we have

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

The manifold M^{2n+1} together with the structure tensor (η, ξ, ϕ, g) is called a contact metric manifold ([1], [2]).

In a contact metric manifold $M(\eta, \xi, \phi, g)$, we define a symmetric tensor h of type (1,1) as $h = \frac{1}{2}\pounds_\xi \phi$. The following identities hold true ([1], [2]):

hpcv2.3
$$h\xi = 0, \quad h\phi + \phi h = 0, \tag{2.2}$$

If ξ is a Killing vector field, then the manifold (M, η, ξ, ϕ, g) is known as a K-contact manifold. This is equivalent to the fact that $h = 0$. Finally, if the Riemannian curvature tensor R satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

or, equivalently, if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

then the manifold (M, η, ξ, ϕ, g) becomes Sasakian. We should mention that a Sasakian manifold is K-contact, but the converse is true for dimension three only.

The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by [12]

$$N(k) : p \to N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

where k is a real number. If $\xi \in N(k)$, then we call such manifold as a $N(k)$-contact metric manifold [12]. If $k = 1$, then the manifold becomes Sasakian and if $k = 0$, then the manifold is locally isometric to $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and it is flat for $n = 1$ [1].

However, for a $(2n+1)$-dimensional $N(k)$-contact metric manifold M, we have the following ([1], [2]):

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.3}$$

hpcv2.10

where $h = \frac{1}{2}\pounds_\xi \phi$,

$$h^2 = (k-1)\phi^2, \tag{2.4}$$

hpcv2.11

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{2.5}$$

hpcv2.12

$$\begin{aligned} S(X,Y) = \; & 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \\ & +[2nk - 2(n-1)]\eta(X)\eta(Y), \text{ for } n \geq 1. \end{aligned}$$

$$S(Y,\xi) = 2nk\eta(X),$$
$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y),$$
$$(\nabla_X h)(Y) = \{(1-k)g(X,\phi Y) + g(X,h\phi Y)\}\xi + \eta(Y)[h(\phi X + \phi hX)],$$

for any smooth vector fields X, Y, Z, where S is the Ricci tensor. $N(k)$-contact metric manifolds are studied by many authors, e.g., [8, 9, 10]).

The curvature tensor of a 3-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X,Y)Z = \; & [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ & -\tfrac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

where r is the scalar curvature and Q is called the Ricci operator, defined by $g(QX,Y) = S(X,Y)$.

In [3], Blair et al. proved that in a three dimensional contact metric manifold with ξ belonging to the k-nullity distribution, the following conditions hold:

$$QX = (\frac{r}{2} - k)X + (3k - \frac{r}{2})\eta(X)\xi,$$

$$S(X,Y) = \left(\frac{r}{2} - k\right) g(X,Y) + \left(3k - \frac{r}{2}\right) \eta(X)\eta(Y), \tag{2.6}$$

hpcv2.19

$$\nabla_X \xi = -(1+\alpha)\phi X, \tag{2.7}$$

hpcv2.20

where

$$\alpha = \pm\sqrt{1-k}. \tag{2.8}$$

hpcv2.21

3 3-dimensional N(k)-contact metric manifolds admitting HPCV field

This section is devoted to the study of a HPCV field V on a $N(k)$-contact metric manifold $M^3(\phi, \xi, \eta, g)$ of dimension 3. Then the equation (1.1) holds.

Theorem 3.1. *Let $M^3(\phi, \xi, \eta, g)$ be a 3-dimensional $N(k)$-contact metric manifold with $k < 1$ admitting a HPCV field V with $\phi V = 0$. Then M^3 is K-contact if and only if $\eta(V) = \frac{1}{1 \pm \sqrt{1-k}}$.*

Proof. Let us assume that

(3.1) hpcv3.1
$$\phi V = 0.$$

As a consequence of (3.1), we have

(3.2) hpcv3.2
$$V = \eta(V)\xi.$$

By covariantly differentiating (3.2) with respect to an arbitrary smooth vector field X and then using (1.1), (2.7) and (3.1), we get

(3.3) hpcv3.3
$$aX + b\phi X = a\eta(X)\xi - (1+\alpha)\eta(V)\phi X.$$

Taking the inner product of (3.3) with Y, we find

$$ag(X,Y) + bg(\phi X, Y) = a\eta(X)\eta(Y) - (1+\alpha)\eta(V)g(\phi X, Y).$$

Contracting X, Y in the above equation and using $\operatorname{Tr}\phi = 0$, we obtain

(3.4) hpcv3.5
$$a = 0.$$

Operating ϕ on (3.3) and using (2.1) yields

(3.5) hpcv3.6
$$a\phi X - bX + b\eta(X)\xi = (1+\alpha)\eta(V)X - (1+\alpha)\eta(V)\eta(X)\xi.$$

Taking the inner product of (3.5) with Y, we have

$$\begin{aligned} ag(\phi X, Y) - bg(X,Y) + b\eta(X)\eta(Y) = \; & (1+\alpha)\eta(V)g(X,Y) \\ & -(1+\alpha)\eta(V)\eta(X)\eta(Y). \end{aligned}$$

Contracting X, Y and then using (2.1) and (2.8), we obtain

(3.6) hpcv3.8
$$b = -(1 \pm \sqrt{1-k})\eta(V).$$

We see that b is constant. Then $b = -1$ if and only if $\eta(V) = \frac{1}{1 \pm \sqrt{1-k}}$ and by virtue of (3.6), the manifold becomes K-contact. Hence the proof is complete. □

Theorem 3.2. *Let $M^3(\phi, \xi, \eta, g)$ be a 3-dimensional $N(k)$-contact metric manifold admitting a HPCV field V with $\phi V = 0$. Then V becomes divergence free.*

Proof. Making use of (3.4) and (3.6) in (1.1), we get

$$\nabla_X V = -(1 \pm \sqrt{1-k})\eta(V)\phi X.$$

By taking the inner product of the last equation with an arbitrary smooth vector field Y, we have

(3.7) hpcv3.10 $$g(\nabla_X V, Y) = -(1 \pm \sqrt{1-k})\eta(V)g(\phi X, Y).$$

Contracting X, Y in (3.7), we infer $\operatorname{div} V = 0$. □

This Theorem leads to the following

Corollary 3.3. *Let V be a HPCV field on a 3-dimensional $N(k)$-contact metric manifold with $\phi V = 0$. Then V is solenoidal.*

Lemma 3.4. *Let V be a HPCV field on a 3-dimensional $N(k)$-contact metric manifold with $\phi V = 0$. Then the curvature tensor R of M^3 can be expressed as*

$$R(X,Y)V = b\{\eta(X)(Y + hY) - \eta(Y)(X + hX)\}.$$

Proof. Using $a = 0$ in (1.1), we get

(3.8) hpcv3.12 $$\nabla_X V = b\phi X.$$

By covariantly differentiating (3.8) with respect to an arbitrary smooth vector field Y, we get

(3.9) hpcv3.13 $$\nabla_Y \nabla_X V = b(\nabla_Y \phi X).$$

With the help of (1.1), (3.9) and using the fact that b is constant, we obtain

$$R(X,Y)V = b\{(\nabla_X \phi)Y - (\nabla_Y \phi)X\}.$$

Using (2.3) in the preceding equation yields

(3.10) hpcv3.15 $$R(X,Y)V = b\{\eta(X)(Y + hY) - \eta(Y)(X + hX)\}.$$

□

Theorem 3.5. *Let $M^3(\phi, \xi, \eta, g)$ be a 3-dimensional $N(k)$-contact metric manifold admitting a HPCV field V with $\phi V = 0$. Then either V is a closed conformal vector field or V is an eigenvector of h corresponding to the eigenvalue 0.*

Proof. By substituting $Y = hY$ in (3.10) and using (2.4) we have

(3.11) hpcv3.16 $$R(X, hY)V = b\eta(X)\{hY + (k-1)\phi^2 Y\}.$$

Taking the inner product of (3.11) with V and using (3.1), we get

$$b\eta(X)g(hY, V) = 0,$$

from which it follows that

(3.12) hpcv3.18 $$b\mathrm{hV} = 0.$$

From (3.12) we can say that either $b = 0$ or $hV = 0$, whence the claim is proved. □

Theorem 3.6. *Let $M^3(\phi, \xi, \eta, g)$ be a 3-dimensional $N(k)$-contact metric manifold admitting a HPCV field V with $\phi V = 0$. Then either M^3 is flat or V is an eigenvector of h corresponding to the eigenvalue* 0.

Proof. Taking the inner product of (3.10) with ξ and use of (2.2), we get

$$g(R(X, Y)V, \xi) = 0.$$

Again, using (2.5), we can easily obtain

$$g(R(X, Y)V, \xi) = -k\{\eta(Y)g(X, V) - \eta(X)g(Y, V)\}.$$

Comparing the last two equations we infer that

$$k\{\eta(Y)g(X, V) - \eta(X)g(Y, V)\} = 0.$$

Putting $X = hX$ and $Y = \xi$, we get khV $= 0$, which shows that either $k = 0$ or $hV = 0$. Thus the theorem is proved. □

Theorem 3.7. *Let $M^3(\phi, \xi, \eta, g)$ be a 3-dimensional $N(k)$-contact metric manifold admitting a HPCV field V with $\phi V = 0$. Then M^3 is either flat, or Sasakian.*

Proof. Taking the inner product of (3.10) with Z, we get

(3.13) hpcv3.23
$$\begin{aligned} g(R(X, Y)V, Z) = \quad & b\{\eta(X)g(Y, Z) + \eta(X)g(hY, Z) \\ & -\eta(Y)g(X, Z) - \eta(Y)g(hX, Z)\}. \end{aligned}$$

Contracting X, Z in (3.13) and using (3.6), we infer

$$S(Y, V) = 2(1 \pm \sqrt{1-k})\eta(V)\eta(Y).$$

By replacing Y by ξ in the foregoing equation, we obtain

(3.14) hpcv3.25
$$S(\xi, V) = 2(1 \pm \sqrt{1-k})\eta(V).$$

Also, by replacing Y by ξ in (2.6) and using (3.14), we have

$$k = 1 \pm \sqrt{1-k},$$

from which it follows that either $k = 0$, or $k = 1$, that is, the manifold is either flat, or Sasakian. □

Acknowledgements. The authors are grateful to the referees for their valuable suggestions and comments towards the improvement of the paper. The author Debabrata Kar is supported by CSIR, India (File no: 09/028(1007)/2017-EMR-1).

References

bl1 [1] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math. 509, Springer-Verlag, Berlin-New York, 1976.

bl2 [2] D. E. Blair, *Riemannian Geometry of contact and symplectic manifolds*, Birkhäuser, Boston, 2002.

bl3 [3] D. E. Blair, T. Koufogiorgos, R. Sharma, *A classification of* 3*-dimensional contact metric manifolds with* $Q\phi = \phi Q$, Kodai Math. J. 13 (1990), 391-401.

des1 [4] S. Desmukh, *Gradient conformal vector fields on a compact Riemannian manifold*, Colloq. Math. 112 (2008), 157-161.

?des2? [5] S.Desmukh, *Conformal vector fields and eigen vectors of Laplace operator*, Math. Phys. Anal. Geom. 15 (2012), 163-172.

?des3? [6] S. Desmukh, *Geometry of conformal vector fields*, Arab J. Math. Sci. 23 (2017), 44-73.

des4 [7] S. Desmukh, F. Alsolamy, *Conformal vector fields on a Riemannian manifold*, Balkan J. Geom. Appl. 19 (2014), 86-93.

de4 [8] A. De, J. B. Jun, *On N(k)-contact metric manifolds satisfying certain curvature conditions*, Kyungpook Math. J. 51, 4 (2011), 457-468.

ozgur1 [9] C. Özgür, *Contact metric manifolds with cyclic-parallel Ricci tensor*, Diff. Geom. Dynamical Systems, 4 (2002), 21-25.

ozgur2 [10] C. Özgür, S. Sular, *On N(k)-contact metric manifolds satisfying certain conditions*, SUT J. Math. 44, 1 (2008), 89-99.

sharma [11] R. Sharma, *Holomorphically planar conformal vector fields on almost Hermitian manifolds*, Contemp. Math. 337 (2003), 145-154.

tanno [12] S. Tanno, *Ricci curvature of contact Riemannian manifolds,* Tohoku Math. J. 40 (1988), 441-448.

Authors' address:

Debabrata Kar, Pradip Majhi
Department of Pure Mathematics,
University of Calcutta,
35, Ballygunge Circular Road,
Kolkata 700019,
West Bengal, India.
E-mail: debabratakar6@gmail.com , mpradipmajhi@gmail.com