Certain vector fields on 3-dimensional $N(k)$-contact metric manifolds

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Abstract. The paper deals with the characterization of 3-dimensional $N(k)$-contact metric manifolds admitting Holomorphically Planar Conformal Vector (HPCV) fields. We obtain a necessary and sufficient condition for a 3-dimensional $N(k)$-contact metric manifold admitting HPCV field to be $K$-contact. Moreover, we prove that a 3-dimensional $N(k)$-contact metric manifold admitting a HPCV field is solenoidal under certain conditions.

Key words: Killing vector field; closed conformal vector field; HPCV field; divergence; solenoidal; contact metric manifold; $k$-nullity distribution.

1 Introduction

In Riemannian manifolds (or pseudo-Riemannian manifolds) the Killing vector fields, named after Wilhelm Killing, have the quality of preserving the metric. Also the energy and momentum of a particle moving freely in flat spacetimes are conserved by the Killing vectors fields.

Definition 1.1. A smooth vector field $X$ on a Riemannian manifold is called an infinitesimal isometry field or a Killing vector field if $\mathcal{L}_X g = 0$, that is, the Lie derivative of the metric $g$ vanishes in the direction of $X$.

The conformal vector fields generalize the Killing vector fields, which allow metrics preserve their conformal class.

Definition 1.2. A smooth vector field $X$ on a Riemannian manifold is called a conformal vector field if there exists a smooth function $\rho$ on $M$ such that $\mathcal{L}_X g = 2\rho g$.

If $\rho$ is constant, then $X$ is called homothetic. Moreover, if the 1-form metrically equivalent to $V$ is closed, then $X$ is forenamed by a closed conformal vector field. The conformal vector fields have been investigated geometrically in ([4]-[7]).
**Definition 1.3.** A smooth vector field $V$ on a contact metric manifold is aforesaid as holomorphically planar conformal vector field if it obeys the following condition:

$$\nabla_X V = aX + b\phi X,$$

(1.1)

where $X$ is a smooth vector field and $a, b$ are smooth functions.

In [11], Sharma introduced the notion of Holomorphically Planar Conformal Vector (HPCV) field.

The paper is arranged as follows: after introduction in Section 2, we discuss preliminaries of 3-dimensional $N(k)$-contact metric manifolds. In Section 3, we prove that any 3-dimensional $N(k)$-contact metric manifold admitting HPCV is $K$-contact, and conversely - under certain conditions. We also show that a HPCV field $V$ is solenoidal in such a manifold with $\phi V = 0$, and obtain several related results.

## 2 3-dimensional $N(k)$-contact metric manifolds

A contact manifold is a smooth manifold $M^{2n+1}$ of dimension $2n + 1$, endowed with a 1-form $\eta$ which globally satisfies $\eta \wedge (d\eta)^n \neq 0$. We know that there is a unique vector field $\xi$, called characteristic vector field so that $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. The same holds true in the case of a Riemannian metric $g$ and a structural vector field $\phi$ of type $(1,1)$ such that

$$\phi \xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

(2.1)

The manifold $M^{2n+1}$ together with the structure tensor $(\eta, \xi, \phi, g)$ is called a contact metric manifold ([1], [2]).

In a contact metric manifold $M(\eta, \xi, \phi, g)$, we define a symmetric tensor $h$ of type (1,1) as $h = \frac{1}{2} \xi \phi$. The following identities hold true ([1], [2]):

$$h\xi = 0, \quad h\phi + \phi h = 0,$$

(2.2)

If $\xi$ is a Killing vector field, then the manifold $(M, \eta, \xi, \phi, g)$ is known as a $K$-contact manifold. This is equivalent to the fact that $h = 0$. Finally, if the Riemannian curvature tensor $R$ satisfies

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

or, equivalently, if

$$\nabla_X \phi Y = g(X,Y)\xi - \eta(Y)X,$$

then the manifold $(M, \eta, \xi, \phi, g)$ becomes Sasakian. We should mention that a Sasakian manifold is $K$-contact, but the converse is true for dimension three only.

The $k$-nullity distribution $N(k)$ of a Riemannian manifold is defined by [12]

$$N(k) : p \to N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$
where $k$ is a real number. If $\xi \in N(k)$, then we call such manifold as a $N(k)$-contact metric manifold [12]. If $k = 1$, then the manifold becomes Sasakian and if $k = 0$, then the manifold is locally isometric to $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and it is flat for $n = 1$ [1].

However, for a $(2n + 1)$-dimensional $N(k)$-contact metric manifold $M$, we have the following ([1], [2]):

$$\nabla_X \phi Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \quad (2.3)$$

where $h = \frac{1}{2} \mathcal{L}_v \phi$, $h^2 = (k - 1)\phi^2$, $\eta(Y) = \xi - \eta(Y)(X + hX)$.

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y], \quad (2.5)$$

$$S(X, Y) = 2(n - 1)g(X, Y) + 2(n - 1)g(hX, Y)$$

$$+ [2nk - 2(n - 1)]\eta(X)\eta(Y), \quad (2.6)$$

for any smooth vector fields $X, Y, Z$, where $S$ is the Ricci tensor. $N(k)$-contact metric manifolds are studied by many authors, e.g., [8, 9, 10]).

In [3], Blair et al. proved that in a three dimensional contact metric manifold with $\xi$ belonging to the $k$-nullity distribution, the following conditions hold:

$$QX = \left(\frac{r}{2} - k\right)X + \left(3k - \frac{r}{2}\right)\eta(X)\xi, \quad (2.6)$$

$$S(X, Y) = \left(\frac{r}{2} - k\right)g(X, Y) + \left(3k - \frac{r}{2}\right)\eta(X)\eta(Y), \quad (2.7)$$

$$\nabla_X \xi = -(1 + \alpha)\phi X, \quad (2.8)$$

where

$$\alpha = \pm \sqrt{1 - k}. \quad (2.8)$$
3 3-dimensional N(k)-contact metric manifolds admitting HPCV field

This section is devoted to the study of a HPCV field $V$ on a $N(k)$-contact metric manifold $M^3(\phi, \xi, \eta, g)$ of dimension 3. Then the equation (1.1) holds.

**Theorem 3.1.** Let $M^3(\phi, \xi, \eta, g)$ be a 3-dimensional $N(k)$-contact metric manifold with $k < 1$ admitting a HPCV field $V$ with $\phi V = 0$. Then $M^3$ is $K$-contact if and only if $\eta(V) = \frac{1}{1 \pm \sqrt{1 - k}}$.

**Proof.** Let us assume that

(3.1) $\phi V = 0$.

As a consequence of (3.1), we have

(3.2) $V = \eta(V)\xi$.

By covariantly differentiating (3.2) with respect to an arbitrary smooth vector field $X$ and then using (1.1), (2.7) and (3.1), we get

(3.3) $aX + b\phi X = a\eta(X)\xi - (1 + \alpha)\eta(V)\phi X$.

Taking the inner product of (3.3) with $Y$, we find

$$ag(X, Y) + bg(\phi X, Y) = a\eta(X)\eta(Y) - (1 + \alpha)\eta(V)g(\phi X, Y).$$

Contracting $X, Y$ in the above equation and using $\text{Tr} \phi = 0$, we obtain

(3.4) $a = 0$.

Operating $\phi$ on (3.3) and using (2.1) yields

(3.5) $b\phi X - bX + b\eta(X)\xi = (1 + \alpha)\eta(V)X - (1 + \alpha)\eta(V)\eta(X)\xi$.

Taking the inner product of (3.5) with $Y$, we have

$$ag(\phi X, Y) - bg(X, Y) + b\eta(X)\eta(Y) = (1 + \alpha)\eta(V)g(X, Y) - (1 + \alpha)\eta(V)\eta(X)\eta(Y).$$

Contracting $X, Y$ and then using (2.1) and (2.8), we obtain

(3.6) $b = -(1 \pm \sqrt{1 - k})\eta(V)$.

We see that $b$ is constant. Then $b = -1$ if and only if $\eta(V) = \frac{1}{1 \pm \sqrt{1 - k}}$ and by virtue of (3.6), the manifold becomes $K$-contact. Hence the proof is complete. $\square$

**Theorem 3.2.** Let $M^3(\phi, \xi, \eta, g)$ be a 3-dimensional $N(k)$-contact metric manifold admitting a HPCV field $V$ with $\phi V = 0$. Then $V$ becomes divergence free.
Proof. Making use of (3.4) and (3.6) in (1.1), we get
\[ \nabla_X V = -(1 \pm \sqrt{1 - k}) \eta(V) \phi X. \]

By taking the inner product of the last equation with an arbitrary smooth vector field \( Y \), we have
\[ g(\nabla_X V, Y) = -(1 \pm \sqrt{1 - k}) \eta(V) g(\phi X, Y). \]

Contracting \( X, Y \) in (3.7), we infer \( \text{div} \ V = 0 \).
\[ \square \]

This Theorem leads to the following

**Corollary 3.3.** Let \( V \) be a HPCV field on a 3-dimensional \( N(k) \)-contact metric manifold with \( \phi V = 0 \). Then \( V \) is solenoidal.

**Lemma 3.4.** Let \( V \) be a HPCV field on a 3-dimensional \( N(k) \)-contact metric manifold with \( \phi V = 0 \). Then the curvature tensor \( R \) of \( M^3 \) can be expressed as
\[ R(X, Y)V = \{ \eta(X)(Y + hY) - \eta(Y)(X + hX) \}. \]

Proof. Using \( a = 0 \) in (1.1), we get
\[ \nabla_X V = b\phi X. \]

By covariantly differentiating (3.8) with respect to an arbitrary smooth vector field \( Y \), we get
\[ \nabla_Y \nabla_X V = b(\nabla_Y \phi X). \]

With the help of (1.1), (3.9) and using the fact that \( b \) is constant, we obtain
\[ R(X, Y)V = b\{ (\nabla_X \phi)Y - (\nabla_Y \phi)X \}. \]

Using (2.3) in the preceding equation yields
\[ R(X, Y)V = b\{ \eta(X)(Y + hY) - \eta(Y)(X + hX) \}. \]
\[ \square \]

**Theorem 3.5.** Let \( M^3(\phi, \xi, \eta, g) \) be a 3-dimensional \( N(k) \)-contact metric manifold admitting a HPCV field \( V \) with \( \phi V = 0 \). Then either \( V \) is a closed conformal vector field or \( V \) is an eigenvector of \( h \) corresponding to the eigenvalue 0.

Proof. By substituting \( Y = hY \) in (3.10) and using (2.4) we have
\[ R(X, hY)V = b\eta(X)\{ hY + (k - 1)\phi^2 Y \}. \]

Taking the inner product of (3.11) with \( V \) and using (3.1), we get
\[ b\eta(X)g(hY, V) = 0, \]
from which it follows that
\[ bhV = 0. \]

From (3.12) we can say that either \( b = 0 \) or \( hV = 0 \), whence the claim is proved. \[ \square \]
Theorem 3.6. Let $M^3(\phi, \xi, \eta, g)$ be a 3-dimensional $N(k)$-contact metric manifold admitting a HPCV field $V$ with $\phi V = 0$. Then either $M^3$ is flat or $V$ is an eigenvector of $h$ corresponding to the eigenvalue 0.

Proof. Taking the inner product of (3.10) with $\xi$ and use of (2.2), we get
\[ g(R(X,Y)V, \xi) = 0. \]
Again, using (2.5), we can easily obtain
\[ g(R(X,Y)V, \xi) = -k\{\eta(Y)g(X,V) - \eta(X)g(Y,V)\}. \]
Comparing the last two equations we infer that
\[ k\{\eta(Y)g(X,V) - \eta(X)g(Y,V)\} = 0. \]
Putting $X = hX$ and $Y = \xi$, we get $khV = 0$, which shows that either $k = 0$ or $hV = 0$. Thus the theorem is proved.

Theorem 3.7. Let $M^3(\phi, \xi, \eta, g)$ be a 3-dimensional $N(k)$-contact metric manifold admitting a HPCV field $V$ with $\phi V = 0$. Then $M^3$ is either flat, or Sasakian.

Proof. Taking the inner product of (3.10) with $Z$, we get
\[ g(R(X,Y)V, Z) = b\{\eta(X)g(Y,Z) + \eta(X)g(hY,Z) - \eta(Y)g(X,Z) - \eta(Y)g(hX,Z)\}. \]
Contracting $X, Z$ in (3.13) and using (3.6), we infer
\[ S(Y,V) = 2(1 \pm \sqrt{1 - k})\eta(V)\eta(Y). \]
By replacing $Y$ by $\xi$ in the foregoing equation, we obtain
\[ S(\xi,V) = 2(1 \pm \sqrt{1 - k})\eta(V). \]
Also, by replacing $Y$ by $\xi$ in (2.6) and using (3.14), we have
\[ k = 1 \pm \sqrt{1 - k}, \]
from which it follows that either $k = 0$, or $k = 1$, that is, the manifold is either flat, or Sasakian.

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References

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