Second order parallel tensor on generalized f.pk-space form and hypersurfaces of generalized f.pk-space form

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Abstract. The purpose of the present paper to study a second order symmetric parallel tensor in generalized f.pk-space form. Second order symmetric parallel tensor in f.pk-space form is combination of the associated metric tensor and 1-forms of structure vector fields. We prove that there does not exist second order skew-symmetric parallel tensor in f.pk-space form. We also deduce that there is no parallel hypersurface in a generalized f.pk-space form but there is semi-parallel hypersurfaces in a generalized f.pk-space form.

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1 Introduction

In 1923, Eisenhart [10] proved that if a positive definite Riemannian manifold admits a second order parallel symmetric tensor other than a constant multiple of the metric tensor, then it is reducible. In 1926, Levy [17] proved that a second order parallel symmetric non-singular (with non-vanishing determinant) tensor in a real space form is proportional to the metric tensor. As an improvement of the result of Levy, Sharma [19] proved that any second-order parallel tensor not necessarily symmetric in a real-space form of dimension greater than 2 is proportional to the Riemannian metric. Later on, many authors [3, 7, 15, 18, 20, 21, 22] studied second-order parallel tensor on various spaces and obtained the important results. In [13], Gherib and Belkhelfa studied the second order symmetric parallel tensors on generalized Sasakian space form and proved that it is proportional to the metric tensor. Recently, Belkhelfa and Mahi [4] proved that second order symmetric parallel tensors on $\mathcal{S}$-space form is linear combination of the metric tensor and 1-forms of structure vector fields with constant coefficients. They also prove the existence of semi-parallel hypersurfaces in $\mathcal{S}$-space form.

On the other side, Falcitelli and Pastore [11] extend the notion of generalized Sasakian-space form to the generalized f.pk-space form and discussed the constancy of the φ-sectional curvature. They also interrelated generalized f.pk-space forms with

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generalized Sasakian and generalized complex space-forms. Also the $\mathcal{S}$-space form, Sasakian space forms are the particular cases of the generalized f.pk-space forms. The purpose of this paper is to study second order parallel tensors on the generalized f.pk-space form. We prove that the second order symmetric parallel tensors on a generalized f.pk-space form is linear combination of the associated positive-definite metric tensor and 1-forms of structure vector fields. We prove the non-existence of any non-zero second order skew-symmetric parallel tensor. We also find out that there do not exist parallel hypersurfaces of a generalized f.pk-space forms $M^{2n+s}(F_1, F_2, \mathcal{F})$, tangent to the structure vector fields. Even there exist semi-parallel hypersurface of a generalized f.pk-space forms $M^{2n+s}(F_1, F_2, \mathcal{F})$, tangent to the structure vector fields.

2 Preliminaries

An f.pk-manifold [5, 9, 14] is a manifold M^{2n+s} on which is defined an f-structure [23] (also known as f-structure with complemented frames or globally framed f-stuctures or f-structures with parallelizable kernel, i.e., f.pk-structures), that is a $(1,1)$-tensor fields φ satisfying

$$\varphi^3 + \varphi = 0,$$

of rank $2n$ such that the subbundle $\ker\varphi$ is parallelizable. Then there exists a global frame $\{\xi_i\}, i = 1, \ldots, s$ for the subbundle $\ker\varphi$, with dual 1-forms η^i, satisfying

$$\varphi^2 = -I + \eta^i \otimes \xi_i, \tag{2.1}$$

$$\eta^i(\xi_j) = \delta^i_j, \quad \varphi\,\xi_i = 0, \quad \eta^i \circ \varphi = 0. \tag{2.2}$$

An f.pk-structure on a manifold M^{2n+s} is said to be normal if the Nijenhuis tensor field $N = [\varphi, \varphi] + 2d\eta^i \otimes \xi_i$ vanishes, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. Consider a Riemannian metric g on M^{2n+s} associated with an f.pk-structure (φ, ξ_i, η^i) such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y), \tag{2.3}$$

for any vector fields X, Y on $\Gamma(TM)$. Then an f.pk-structure is known as a metric f.pk-structure. A manifold with a metric f.pk-structure is known as a metric f.pk-manifold.

Let Φ be the fundamental 2-form on M^{2n+s} defined by

$$\Phi(X, Y) = g(X, \varphi Y) = -g(\varphi X, Y),$$

for any vector fields X, Y on $\Gamma(TM)$. A metric f.pk-structure is said to be a $\mathcal{K}$-structure [5] if it is normal and the fundamental 2-form Φ is closed, a manifold with a $\mathcal{K}$-structure is known as $\mathcal{K}$-manifold. A $\mathcal{K}$-structure is said to be a $\mathcal{S}$-structure if $d\eta^i = \Phi$ for all $i \in \{1, \ldots, s\}$, a manifold with an $\mathcal{S}$-structure is known as $\mathcal{S}$-manifold. A $\mathcal{K}$-structure is said to be a $\mathcal{C}$-structure if $d\eta^i = 0$ for all $i \in \{1, \ldots, s\}$, a manifold with a $\mathcal{C}$-structure is known as $\mathcal{C}$-manifold. Obviously, if $s = 1$, a $\mathcal{K}$-manifold is a quasi Sasakian manifold, a $\mathcal{C}$-manifold is a cosymplectic manifold and a $\mathcal{S}$-manifold is a Sasakian manifold.

The Levi-Civita connection ∇ [5, 9] of a metric f.pk-manifold satisfies

$$\begin{aligned} 2g(\ (\nabla_X\varphi)Y\ ,\ Z\ ) &= 3d\Phi(X,\varphi Y,\varphi Z) - 3d\Phi(X,Y,Z) \\ &\quad + g(N(Y,Z),\varphi X) + 2d\eta^j(\varphi Y,Z)\eta^j(X) \\ &\quad - 2d\eta^j(\varphi Z,Y)\eta^j(X) + 2d\eta^j(\varphi Y,X)\eta^j(Z) \\ &\quad - 2d\eta^j(\varphi Z,X)\eta^j(Y). \end{aligned}$$

In particular, for $\mathcal{S}$-manifolds [5], we have $\nabla_X\xi_i = -\varphi X, \quad i = 1,\dots,s.$

A plane section Π in T_pM spanned by X and φX, where X is a tangent vector orthogonal to structure vector fields, is known as φ-section. The sectional curvature of φ-section is called a φ-sectional curvature. A metric f.pk-manifold $(M^{2n+s},\varphi,\xi_i,\eta^i,g)$ has pointwise constant φ-sectional curvature if at any point $p \in M^{2n+s}$, $R_p(X, \varphi X, X, \varphi X)$ does not depend on the φ-section spanned by $\{X,\varphi X\}$.

3 Generalized f.pk-space forms

Definition 3.1. [11]. A generalized f.pk-space form $M^{2n+s}(F_1,F_2,\mathcal{F})$, is a metric f.pk-manifold $(M^{2n+s},\varphi,\xi_i,\eta^i,g)$ which admits smooth functions $F_1,F_2,\mathcal{F}$ such that its curvature tensor field satisfies

$$\begin{aligned} R(X,Y)Z &= F_1\left(g(\varphi X,\varphi Z)\varphi^2Y - g(\varphi Y,\varphi Z)\varphi^2X\right) \\ &\quad + F_2\left(g(Z,\varphi Y)\varphi X - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\right) \\ &\quad + \sum_{i,j=1}^{s} F_{ij}\left(\eta^i(X)\eta^j(Z)\varphi^2Y - \eta^i(Y)\eta^j(Z)\varphi^2X\right. \\ &\quad \left. + g(\varphi Y,\varphi Z)\eta^i(X)\xi_j - g(\varphi X,\varphi Z)\eta^i(Y)\xi_j\right), \end{aligned} \tag{3.1}$$

The φ-sectional curvature of a generalized f.pk-space form $M^{2n+s}(F_1,F_2,\mathcal{F})$ is pointwise constant, which is $c = F_1 + 3F_2$.

Remark 3.2. For $s \geq 2$, an $\mathcal{S}$-maniolfd with pointwise constant φ-sectional curvature c is known as $\mathcal{S}$-space form [5, 16]. If the given structure is $\mathcal{S}$-structure then we obtain a $\mathcal{S}$-space form by (3.1) with $F_1 = \dfrac{c+3s}{4}, F_2 = \dfrac{c-s}{4}$ and $F_{ij} = 1$ for all $i,j \in \{1,\dots,s\}$.

Remark 3.3. For $s = 1$, we obtain a generalized Sasakian-space form $M^{2n+1}(f_1,f_2,f_3)$ with $f_1 = F_1$, $f_2 = F_2$ and $f_3 = F_1 - F_{11}$. If the given structure is Sasakian then (3.1) holds with $F_{11} = 1, F_1 = (c+3)/4, F_2 = (c-1)/4$ and $f_3 = F_1 - F_{11} = (c-1)/4 = f_2$. If the given structure is Kenmotsu, then (3.1) holds with $F_{11} = -1, F_1 = (c-3)/4, F_2 = (c+1)/4$ and $f_3 = F_1 - F_{11} = (c+1)/4 = f_2$. If the given structure is cosymplectic, then (3.1) holds with $F_{11} = 0, F_1 = c/4, F_2 = c/4$ and $f_3 = F_1 - F_{11} = c/4$.

Let M^{2n+s} be a metric f.pk-manifold with its metric tensor g and Levi-Civita connection ∇. Let R denote the Riemann curvature tensor of M^{2n+s}. If H is a $(0,2)$-tensor which is parallel with respect to ∇ then we can show easily that

$$H(R(X,Y)Z,W) + H(Z,R(X,Y)W) = 0. \tag{3.2}$$

Theorem 3.1. *A second order parallel symmetric tensor in a generalized f.pk-space form $M^{2n+s}(F_1, F_2, \mathcal{F})$ is a linear combination of the associated positive definite metric tensor and 1-forms of structure vector fields if $s \geq 2$ with assumption that $\mathcal{F}$ has at least one non-zero function.*

Proof. Using (3.1) in (3.2) and taking $Y = \xi_k$ $(k \in \{1, \dots s\})$ and $Z = W$, we get

$$H\left(\sum_{j=1}^{s} F_{kj}\left(g(\varphi X, \varphi Z)\xi_j + \eta^j(Z)\varphi^2 X\right), Z\right) = 0 \tag{3.3}$$

On solving and taking $Z = \xi_r(r \in \{1, \dots s\})$, we get

$$H(X, \xi_r) = \sum_{\alpha,\beta=1}^{s} \eta^\alpha(X) H(\xi_\beta, \xi_r), \quad F_{kr} \neq 0. \tag{3.4}$$

Using (3.4) in (3.3), we have

$$\begin{aligned} H(X, Z) = \;& g(X, Z) \sum_{\gamma=1}^{s} H(\xi_j, \xi_\gamma) \\ & - \sum_{\alpha,\beta,\gamma=1}^{s} \eta^\alpha(X)\eta^\beta(Z) H(\xi_j, \xi_\gamma) + \sum_{\alpha,\beta,\gamma=1}^{s} \eta^\alpha(X)\eta^j(Z) H(\xi_\beta, \xi_\gamma). \end{aligned} \tag{3.5}$$

Remark 3.4. For $s = 1$, the (3.5) reduces to $H(X, Z) = g(X, Z)H(\xi, \xi)$.

Remark 3.5. For an $\mathcal{S}$-maniolfd, $H(\xi_j, \xi_\gamma)$ are constant for all $j, \gamma \in \{1, \dots s\}$. Therefore we can say that a second order parallel symmetric tensor in a $\mathcal{S}$-space form is a linear combination of the associated positive definite metric tensor and 1-forms of structure vector fields with constant coefficients if $s \geq 2$ [4].

Remark 3.6. For $s = 1$ and if the manifold is Sasakian then a second order parallel symmetric tensor in a Sasakian-space form or generalized Sasakian space form is proportional to the associated positive definite metric tensor.

As an application of Theorem 3.1, now we consider the Ricci tensor of the manifold. Since we know that the Ricci tensor S of the manifold is symmetric $(0, 2)$-tensor.

Definition 3.7. A non-flat semi-Riemannian manifold M is said to be pseudo-Ricci symmetric [6] if the Ricci tensor S of type $(0, 2)$ of the manifold is non-zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = 2\alpha(X)S(Y, Z) + \alpha(Y)S(X, Z) + \alpha(Z)S(X, Y),$$

for all vector fields X, Y, Z, where α is a non-zero 1-form. If $\alpha = 0$, then the manifold reduces to Ricci symmetric manifold.

Definition 3.8. A semi-Riemannian manifold M is said to be Ricci-semisymmetric [8] if its Ricci tensor S satisfies $R{\cdot}S = 0$, that is,

$$S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0,$$

for all vector fields X, Y, Z, W.

Definition 3.9. A semi-Riemannian manifold M is said to be Ricci flat if its Ricci tensor S satisfies $S = 0$.

Corollary 3.2. *A Ricci symmetric generalized f.pk-space form $M^{2n+s}(F_1, F_2, \mathcal{F})$ is a linear combination of the associated positive definite metric tensor and 1-forms of structure vector fields if $s \geq 2$ with assumption that $\mathcal{F}$ has at least one non-zero function.*

In [1, Theorem 2.2], it was shown that if M is pseudo Ricci symmetric manifold such that $\mathrm{div}R = 0$, then M is Ricci flat. We know that Ricci flat condition implies Ricci symmetric but the converse does not hold. Therefore, we can state the following

Corollary 3.3. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a pseudo-Ricci symmetric generalized f.pk-space form such that $\mathcal{F}$ has at least one non-zero function and $s \geq 2$. Then the linear combination of the associated positive definite metric tensors and 1-forms of the structure vector fields is zero.*

We further consider skew-symmetric parallel tensors on generalized f.pk-space forms.

Theorem 3.4. *There does not exist a non-zero second order parallel skew-symmetric tensor in a generalized f.pk-space form $M^{2n+s}(F_1, F_2, \mathcal{F})$ with the assumption that $\mathcal{F}$ has at least one non-zero function.*

Proof. Consider (3.2) with H as a non-zero second order parallel skew-symmetric tensor, using (3.1) and taking $Y = W = \xi_k$ $(k \in \{1, \dots s\})$ with the assumption that $\mathcal{F}$ has at least one non-zero function, we obtain

$$H(Z, X) = \eta^k(Z)H(X, \xi_k) + \eta^k(X)H(Z, \xi_k). \tag{3.6}$$

Let A be the dual $(1,1)$-type tensor which is metrically equivalent to H, that is, $H(X, Y) = g(AX, Y)$. Then (3.6) is equivalent to

$$AX = -g(AX, \xi_k)\xi_k + \eta^k(X)A\xi_k. \tag{3.7}$$

Now, taking the inner product of (3.7) with ξ_k, we have $H(X, \xi_k) = 0$. Therefore (3.7) reduces to

$$AX = \eta^k(X)A\xi_k. \tag{3.8}$$

Taking the inner product of (3.8) with Y, we have $H(X, Y) = 0$.

4 Parallel submanifolds of f.pk-space forms

Let G be an immersed hypersurface of $M^{2n+s}(F_1, F_2, \mathcal{F})$. Then the formulas of Gauss and Weingarten are

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X.Y)N,$$
$$\tilde{\nabla}_X N = -SX,$$

where X and Y are tangent vector fields, N a unit normal vector normal to G, h the second fundamental form and S the shape operator of G. Note that h and S

are related by $h(X,Y) = g(SX,Y)$. In a hypersurface, the $(0,4)$ tensor field $\tilde{R}.h$ is defined by

$$\tilde{R}.h(X,Y,Z,W) = -h(\tilde{R}(X,Y)Z,W) - h(Z,\tilde{R}(X,Y)W) \tag{4.1}$$

A hypersurface is called parallel [2] if $\tilde{\nabla}h = 0$.

Theorem 4.1. *Let G be an hypersurface of a generalized f.pk-space forms $M^{2n+s}(F_1,F_2,\mathcal{F})$, tangent to the structure vector fields with $F_2 \neq 0$, then G is not parallel.*

Proof. Let G be a parallel hypersurface of a generalized f.pk-space forms $M^{2n+s}(F_1,F_2,\mathcal{F})$, tangent to the structure vector fields. Let h be the second fundamental form of G. Consider N be the unit normal of G in M and $W = -\varphi N$. Since $g(\xi_i,N) = \eta_i(N) = 0$ for all i, therefore

$$g(W,W) = g(\varphi N,\varphi N) = 1,$$
$$g(N,W) = g(N,\varphi N) = 0,$$
$$\varphi W = N.$$

Let $X \in \Gamma(TG)$, then we have

$$\varphi X = TX + w(X)N \tag{4.2}$$

where w and T are tensor fields on G of type $(0,1)$ and $(1,1)$, respectively and TX is also the tangent part of φX.

Setting $w \neq 0$, by (4.2) we have $w(X) = g(X,W)$.

Now, using (3.1), (4.2) and Codazzi equation, we infer

$$\begin{aligned} 0 &= \tilde{\nabla}_X h(Y,Z) - \tilde{\nabla}_Y h(X,Z) = \left(\tilde{R}(X,Y)Z\right)^{\perp} \\ &= F_2\left(g(Z,\varphi Y)\varphi X - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\right)^{\perp} \end{aligned}$$

Since $F_2 \neq 0$, we obtain

$$\left(g(Z,\varphi Y)w(X) - g(Z,\varphi X)w(Y) + 2g(X,\varphi Y)w(Z)\right)N = 0.$$

Putting $Z = W$ in the above equation, we get

$$2g(X,TY) = 0,$$

which implies that $TY = 0$, so $\dim\varphi(T_p(G)) = 1$ for all $p \in G$.

Since $T_p(M) = T_p(G) \oplus (T_p(G))^{\perp}$ and $rank\varphi = 2n$, therefore we obtain

$$2n - 1 \leq \dim\varphi\left(T_p(G)\right)^{\perp} \leq 2n,$$

which is impossible since $n > 1$.

Remark 4.1. This result is true for S-space form [4] and generalized Sasakian-space form [12].

Theorem 4.2. *There are semi-parallel hypersurfaces, tangent to the structure vector fields in a generalized f.pk-space forms $M^{2n+s}(F_1,F_2,\mathcal{F})$, assuming that $\mathcal{F}$ has at least one non-zero function.*

Proof. Let G be not semi-parallel hypersurfaces, tangent to the structure vector fields in a generalized f.pk-space forms $M^{2n+s}(F_1, F_2, \mathcal{F})$. It means that

$$\tilde{R}.h(X,Y,Z,W) = -h(\tilde{R}(X,Y)Z,W) - h(Z,\tilde{R}(X,Y)W) \neq 0,$$

where h is the second fundamental form of G, which is a symmetric $(0,2)$ -tensor field on G. By using the same argument of Theorem 3.1, we get

$$\begin{aligned} H(X,Z) \neq \quad & g(X,Z)\sum_{\gamma=1}^{s} H(\xi_j,\xi_\gamma) - \sum_{\alpha,\beta,\gamma=1}^{s} \eta^\alpha(X)\eta^\beta(Z)H(\xi_j,\xi_\gamma) \\ & + \sum_{\alpha,\beta,\gamma=1}^{s} \eta^\alpha(X)\eta^j(Z)H(\xi_\beta,\xi_\gamma), \end{aligned}$$

which is impossible.

Remark 4.2. Even for S-space form [4], this result is not true.

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