Riemann soliton on non-Sasakian (κ, μ)-contact manifolds

Shruthi Chidananda and V. Venkatesha

Abstract. The aim of the present paper is to study Riemann soliton on non-Sasakain (κ, μ)-contact manifolds. First, we prove that, if a non-Sasakian (κ, μ)-contact manifold M satisfies Riemann soliton for V as an infinitesimal contact transformation, then M is η-Einstein. Next, we show that, if M is three dimensional and represents a Riemann soliton for potential vector field V then M is flat. An example has been constructed to verify this result. Further, we show that, if M admits a gradient almost Riemann soliton then either the potential function f is constant, or M is flat for $dimM = 3$ and M is locally isometric to $E^{n+1} \times S^n(4)$ for $dimM > 3$. Finally, we prove that, if a (κ, μ)-contact manifold is non-Sasakian then there do not exist an almost Riemann soliton for V collinear with ξ.

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1 Introduction

For a Riemannian manifold (M, g), a Riemann flow is the solution for the non-linear evolution PDE defined by [12]

$$\frac{\partial G_{klmn}}{\partial t}(x,t) = -2R_{klmn}(g)(x,t), \qquad t \in [0,1]. \tag{1.1}$$

A Riemann soliton is the generalized fixed points of the Riemann flow and it is defined as;

Definition 1.1. A Riemannian manifold (M, g) is said to have Riemann soliton or a Riemannian metric g is said to be Riemann soliton if there exists a vector field V, a constant λ on M such that,

$$\begin{aligned} 2g(R(X,Y)Z,W) &+ 2\lambda\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ &+ \{(\pounds_V g)(X,W)g(Y,Z) + (\pounds_V g)(Y,Z)g(X,W) \\ &- (\pounds_V g)(X,Z)g(Y,W) - (\pounds_V g)(Y,W)g(X,Z)\} = 0, \end{aligned} \tag{1.2}$$

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where $\pounds_V$ denotes Lie-derivative along V and R is the Riemannian curvature tensor.

Moreover, on contracting the foregoing equation over X with respect to an orthonormal basis yields

$$(\pounds_V g)(Y, Z) = -\frac{1}{2n-1}\{2S(Y, Z) + 4n\lambda g(Y, Z) + 2\Delta V g(Y, Z)\}, \tag{1.3}$$

where ΔV denotes the divergence of V.
If the soliton parameter λ appearing in (1.2) is in $C^\infty(M)$, then g is said to be an almost Riemann soliton. Further, if the potential vector field V appearing in (1.2) is a gradient of some function $f \in C^\infty(M)$ and λ is in $C^\infty(M)$, then we say that g is a gradient almost Riemann soliton and the equation (1.2) becomes

$$\begin{aligned} &g(R(X, Y)Z, W) + \lambda\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\ &+ \{g(\nabla_X Df, W)g(Y, Z) + g(\nabla_Y Df, Z)g(X, W) \\ &- g(\nabla_X Df, Z)g(Y, W) - g(\nabla_Y Df, W)g(X, Z)\} = 0. \end{aligned} \tag{1.4}$$

Similarly, equation (1.3) for $V = Df$ becomes

$$\nabla_Y Df = -\frac{1}{2n-1}\{Q + 2n\lambda + \Delta V\}Y. \tag{1.5}$$

In an analog with Ricci soliton, Riemann soliton was recently introduced by Hirica and Udrista in [9]. In the beginning, the notion of Riemann soliton was studied in the context of Sasakian geometry and it was named as Sasaki-Riemann soliton. In [8], Venkatesha et.al., gave attention to Riemann soliton over a contact manifold, narrowing it down to K-contact manifolds. Further, in [13], they have considered Riemann soliton in an almost Kenmotsu manifolds and obtained some important results. In [1], [2], Blaga studied Riemann and almost Riemann soliton in (α, β)-contact metric manifolds and in Riemannian manifolds respectively. In [6], Krishnendu De and U. C. De studied an almost Riemann soliton in a non-cosymplectic normal almost contact metric manifold.
As we know, based on the κ value, (κ, μ)-contact manifolds can be divided into two major groups, namely, if $\kappa = 1$, then (κ, μ)-contact manifolds are said to be Sasakian and for $\kappa < 1$ manifolds have belonged to a non-Sasakian (non K-contact) geometry. We learned from the preceding literature review that the study of Riemann soliton is primarily focused on Sasakian and K-contact geometry. Motivated by this, we planned to study Riemann soliton in the context of non-Sasakian (κ, μ)-contact manifolds.
The structure of the present paper is as follows: In section 2, we recall some definitions and formulas related to (κ, μ)-contact structure. In section 3, we prove that, if a non-Sasakian (κ, μ)-contact manifold M has a Riemann soliton for V satisfies $\pounds_V \eta = \rho\eta$, then M is η-Einstein. Also, for a three-dimensional case, we show that if M has Riemann soliton, then M is flat. Further, we constructed an example of a three dimensional non-Sasakian (κ, μ)-contact manifold which admits Riemann soliton. Finally, in the last section, we consider an almost Riemann soliton on a non-Sasakian (κ, μ)-contact manifold M for (i)$V = Df$, (ii)$V = \sigma\xi$ and obtain some important results.

2 Preliminaries

If an odd dimensional smooth manifold M carries a global 1-form η satisfying $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, then M is said to be a contact manifold [3]. If M is contact manifold then for a 1-form η, there always exists a vector field ξ, a $(1,1)$-tensor φ and an associated Riemannian metric g such that

$$\begin{cases} d\eta(X,Y) &= g(X,\varphi Y), \\ g(X,\xi) &= \eta(X), \\ \varphi^2 X &= -X + \eta(X)\xi, \end{cases} \tag{2.1}$$

these also imply

$$\varphi\xi = 0, \qquad \eta\circ\varphi = 0, \qquad g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

where X and Y are the vector fields on M. The structure (φ,ξ,η,g) together with a group of equations (2.1) and (2.2) is called a contact metric structure. Any differentiable manifold M together with this contact metric structure is called a contact metric manifold. Define a tensor h by $2h = \pounds_\xi\varphi$ be a self-adjoint symmetric tensor on a contact manifold and it satisfies

$$\varphi h = -h\varphi, \quad trace(h) = trace(\varphi h) = 0. \tag{2.3}$$

A contact manifold M is said to be K-contact if and only if $h = 0$. On a contact metric structure, if ξ is Killing i.e., $\pounds_\xi g = 0$, then M is said to have a K-contact metric structure. For a contact manifold

$$\nabla_X\xi = -\varphi X - \varphi hX, \tag{2.4}$$

where ∇ denotes the Riemannian connection of the Riemannian metric g.
A contact manifold M is said to be Sasakian if the almost complex structure J on $M\times\mathbb{R}$ defined by $J(Y, fd/dt) = (\varphi Y - f\xi, \eta(Y)d/dt)$ is integrable.
If the characteristic vector field ξ of a contact manifold M belongs to the (κ,μ)-nullity distribution i.e.,

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{2.5}$$

for some real constants κ and μ, then M is said to be a (κ,μ)-contact manifold [4]. On a (κ,μ)-contact manifold we have the following:

$$Q\xi = 2n\xi, \tag{2.6}$$

$$h^2 = (\kappa-1)\varphi^2 X. \tag{2.7}$$

It follows from the equation (2.7) that $\kappa \leq 1$, equality holds when $\kappa = 1$, i.e., M is Sasakian. When $\kappa < 1$, M is non-Sasakian (κ,μ)-contact manifold. The (κ,μ)-nullity distribution on a contact manifold determines the following conditions for $\kappa < 1$ case

$$\begin{aligned} S(X,Y) = {} & (2n-2-n\mu)g(X,Y) + (2-2n+2n\kappa+n\mu)\eta(X)\eta(Y) \\ & + (2n-2+\mu)g(hX,Y), \end{aligned} \tag{2.8}$$

$$r = 2n(2n-2+\kappa-n\mu). \tag{2.9}$$

Moreover, it shows that the scalar curvature r is constant on M. Further, from [10], [11], [7] for $\kappa < 1$, we have

$$(\nabla_X\varphi)Y = g(X+hX,Y)\xi - \eta(Y)\{X+hX\}, \tag{2.10}$$

$$(\nabla_X h)Y = \{g(h^2X,\varphi Y) + g(X,h\varphi Y)\}\xi + \eta(Y)\{h\varphi X + h\varphi hX\} - \mu\eta(X)\varphi hY, \tag{2.11}$$

$$(\nabla_X h\varphi)Y = g(X,hY)\xi + g(hX,hY)\xi + \eta(Y)hX + \eta(Y)h^2X + \mu\eta(X)hY, \tag{2.12}$$

$$R(\xi,Y)Z = \kappa\{g(Y,Z)\xi - \eta(Z)Y\} + \mu\{g(hY,Z)\xi - \eta(Z)hY\}. \tag{2.13}$$

Also from equation (2.8), we derive

$$(\nabla_X Q)Y = (2-2n+2n\kappa+n\mu)\{g(Y,\nabla_X\xi)\xi + \eta(Y)\nabla_X\xi\} + (2n-2+\mu)(\nabla_X h)Y. \tag{2.14}$$

Recall that on any three-dimensional (κ,μ)-contact manifold M, the Ricci operator is given by

$$S(X,Y) = \left(\frac{r}{2}-\kappa\right)g(X,Y) + \left(3\kappa-\frac{r}{2}\right)\eta(X)\eta(Y) + \mu g(hX,Y). \tag{2.15}$$

As a result, if M is non-Sasakian, then the preceding relation yields

$$(\nabla_X Q)Y = \left(3\kappa-\frac{r}{2}\right)(g(Y,-\varphi X-\varphi hX)\xi - \eta(Y)\{\varphi X+\varphi hX\}) + \mu(\nabla_X h)Y. \tag{2.16}$$

The Riemannian curvature tensor of a three-dimensional Riemannian manifold is given by [5]

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \tag{2.17}$$

So, use of (2.15) in the above equation, determines the curvature tensor R of a 3-dimensional (κ,μ)-contact manifold as follows:

$$\begin{aligned} R(X,Y)Z =& 2\left(\frac{r}{2}-\kappa\right)\{g(Y,Z)X - g(X,Z)Y\} + \left(3\kappa-\frac{r}{2}\right)\{g(Y,Z)\eta(X)\xi \\ &- g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\ &+ \mu\{g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(X,hZ)Y\} \\ &- \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{2.18}$$

From K. Yano [14], we have the following Integral formulas

$$2g((\pounds_V\nabla)(X,Y),Z) = (\nabla_X\pounds_V\nabla)(Y,Z) + (\nabla_Y\pounds_V\nabla)(X,Z) - (\nabla_Z\pounds_V\nabla)(X,Y), \tag{2.19}$$

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) - (\nabla_Y\pounds_V\nabla)(X,Z). \tag{2.20}$$

Before going to our main results, we prove following result

Proposition 2.1. *For a (κ, μ)-contact manifold we have, either $\mu = 0$, or*

$$\sum_{i=1}^{2n+1} g((\pounds_V h)e_i, e_i) = 0. \tag{2.21}$$

Proof. Finding the term $(\pounds_V R)(X, \xi)\xi$ in a (κ, μ)-contact manifold results

$$\begin{aligned}(\pounds_V R)(X, \xi)\xi = \kappa\{-\eta(\pounds_V \xi)X - ((\pounds_V \eta)X)\xi\} + \mu\{(\pounds_V h)X \\ -\eta(\pounds_V \xi)hX - \eta(X)(\pounds_V h)\xi\} - R(X, \xi)\pounds_V \xi.\end{aligned}$$

Thus, contracting the foregoing relation by using (2.8), over X with respect to an orthonormal basis, completes the proof. □

Definition 2.1. On a contact manifold M, a vector field V is said to be an infinitesimal contact transformation if there exists a smooth function ρ on M such that

$$(\pounds_V \eta)X = \rho\eta(X), \tag{2.22}$$

for all $X \in \mathcal{X}(M)$.

3 Riemannian soliton

Theorem 3.1. *Let M be a non-Sasakian (κ, μ)-contact manifold and if M admits a Riemann soliton for V an infinitesimal contact transformation, then M is η-Einstein.*

Proof. Let M has a Riemannian soliton for an infinitesimal contact transformation V. Then the equation (1.3) on (ξ, ξ) provides

$$\eta(\pounds_V \xi) = \frac{1}{2n-1}\{2n\kappa + 2n\lambda + \Delta V\}. \tag{3.1}$$

Since M is non-Sasakian, the scalar curvature r is constant on M. Use of this in the contraction of (1.3) over X, Y shows ΔV is also constant on M. Hence, from the above condition and from the equation (2.22) we obtain $\rho = \frac{-1}{2n-1}\{2n\kappa + 2n\lambda + \Delta V\}$. Obviously, this shows ρ is constant on M. Thus, applying d on both sides of the equation (2.22) leads to

$$(\pounds_V d\eta)(X, Y) = (\pounds_V g)(X, \varphi Y) + g(X, (\pounds_V \varphi)Y) = \rho g(X, \varphi Y). \tag{3.2}$$

Next, taking $Y = \varphi Y$ in (1.3) and on simplifying we get

$$\begin{aligned}\rho g(X, \varphi Y) - g(X, (\pounds_V \varphi)Y) =& \frac{-1}{2n-1}\{2g(QX, \varphi Y) + 4n\lambda g(X, \varphi Y) \\ &+ 2\Delta V g(X, \varphi Y)\}.\end{aligned} \tag{3.3}$$

Symmetrizing the preceding equation results

$$g(X, (\pounds_V \varphi)Y) + g(Y, (\pounds_V \varphi)X) = \frac{2}{2n-1}\{g(Q\varphi X, Y) - g(\varphi QX, Y)\}. \tag{3.4}$$

At this point, we let $\{e_i\}$ be a φ-basis $(e_1, ...e_n, e_{n+1} = \varphi e_1, ..., e_{2n} = \varphi e_n, \xi)$ such that $he_i = (\sqrt{1-\kappa})e_i$, $h\varphi e_i = -(\sqrt{1-\kappa})\varphi e_i$. Then by this setting, from (1.3) and (2.8), one can easily obtain

$$\sum_{i=1}^{2n}(\pounds_V g)(e_i, e_i) - (\pounds_V g)(\varphi e_i, \varphi e_i) = 0. \tag{3.5}$$

Now, in (3.4), we replace Y by he_i and X by φe_i and then sum from $i = 1$ to $2n$ with the help of (2.8) and (3.5), we get

$$\frac{4n(\kappa-1)}{2n-1}(2n-2+\mu) = 0. \tag{3.6}$$

This proves from (2.8) that M is η-Einstein. Hence the theorem is proved. □

Next, with the help of this result here, we are going to prove the following result:

Theorem 3.2. *If a three dimensional non-Sasakian (κ, μ)-contact manifold M whose metric represents a Riemann soliton for a potential vector field V, then M is flat.*

Proof. Making use of equation (2.19), from (1.2), we obtain the computation formula (2.16) as follows

$$\begin{aligned} g((\pounds_V \nabla)(X, Y), Z) =& 2\left(3\kappa - \frac{r}{2}\right)\{g(Y, \varphi hX)\eta(Z) + g(\varphi X, Z)\eta(Y) \\ &+ g(\varphi Y, Z)\eta(X)\} - \mu\{g((\nabla_X h)Y \\ &+ (\nabla_X h)Y, Z) - g((\nabla_Z h)X, Y)\}. \end{aligned} \tag{3.7}$$

With the help of the above equation and by (2.11), we find

$$(\nabla_X \pounds_V \nabla)(Y, \xi) = 2\left(3\kappa - \frac{r}{2}\right)(\nabla_X \varphi)Y - 2\mu(\nabla_X \varphi)Y + \mu^2(\nabla_X \varphi h)Y \tag{3.8}$$

$$\begin{aligned} &-2\left(3\kappa - \frac{r}{2}\right)\{-g(hX, Y)\xi - g(Y, h^2X)\xi - \eta(Y)(\varphi^2 X - hX)\} \\ &+2\mu\{g(X, hY) + g(h^2X, Y)\}\xi - 2\mu(\kappa-1)\eta(Y)(\varphi^2 X - hX) \\ &+\mu^2\{\eta(Y)hX + \eta(Y)h^2X - g(hX, Y)\xi - g(h^2X, Y)\xi\}, \end{aligned} \tag{3.9}$$

and avail of this in (2.20) gives

$$\begin{aligned} (\pounds_V R)(X, Y)\xi =& 2\left(3\kappa - \frac{r}{2}\right)\{(\nabla_X \varphi)Y - (\nabla_Y \varphi)X\} - 2\mu(\kappa-1)\{(\nabla_X \varphi)Y \\ &- (\nabla_Y \varphi)X\} + \mu^2\{(\nabla_X \varphi h)Y - (\nabla_Y \varphi h)X\} \\ &- 2\left(3\kappa - \frac{r}{2}\right)\{\eta(X)(\varphi^2 Y - hY) - \eta(Y)(\varphi^2 X - hX)\} \\ &- 2\mu(\kappa-1)\{\eta(Y)(\varphi^2 X - hX) - \eta(X)(\varphi^2 Y - hY)\} \\ &+ \mu^2\{\eta(Y)hX + \eta(Y)h^2X - \eta(X)hY - \eta(X)h^2Y\}. \end{aligned} \tag{3.10}$$

Contracting the above condition over X with respect to an orthonaormal basis $\{e_i\}$ (i=1,......,2n+1) provides

$$(\pounds_V S)(Y, \xi) = -8\left(3\kappa - \frac{r}{2}\right)\eta(Y) + 8\mu(\kappa-1)\eta(Y) - 4\mu^2(\kappa-1)\eta(Y), \tag{3.11}$$

and putting $Y = \varphi Y$, we get

$$(\pounds_V S)(\varphi Y, \xi) = 0. \tag{3.12}$$

Again in (3.11), for $Y = \xi$, we find

$$(\pounds_V S)(\xi, \xi) = -8\left(3\kappa - \frac{r}{2}\right) + 8\mu(\kappa - 1) - 4\mu^2(\kappa - 1). \tag{3.13}$$

Taking the Lie-derivative of equation (2.15) along V gives us

$$\begin{aligned}(\pounds_V S)(X, Y) = &\left(\frac{r}{2} - \kappa\right)(\pounds_V g)(X, Y) + \left(3\kappa - \frac{r}{2}\right)(\pounds_V \eta \times \eta)(X, Y)\\ &+ \mu\{(\pounds_V g)(hX, Y) + g(\pounds_V h)X, Y)\}.\end{aligned} \tag{3.14}$$

Similarly, the Lie-derivative of the equation (2.18) along the vector field V yields

$$\begin{aligned}(\pounds_V R)(X,Y)Z = &\, 2\left(\frac{r}{2} - \kappa\right)\{(\pounds_V g)(Y, Z)X - (\pounds_V g)(X, Z)Y\}\\ &+ \left(3\kappa - \frac{r}{2}\right)\{(\pounds_V g)(Y, Z)\eta(X)\xi + (\pounds_V \eta)Xg(Y, Z)\xi\\ &+ (\pounds_V \eta)Yg(X, Z)\xi - (\pounds_V g)(X, Z)\eta(Y)\xi + g(Y, Z)\eta(X)\pounds_V \xi\\ &- g(X, Z)\eta(Y)\pounds_V \xi + (\pounds_V \eta \times \eta)(Y, Z)X - (\pounds_V \eta \times \eta)(X, Z)Y\}\\ &+ \mu\{(\pounds_V g)(Y, Z)hX - (\pounds_V g)(X, Z)hY + g(Y, Z)(\pounds_V h)X\\ &- g(X, Z)(\pounds_V h)Y + (\pounds_V g)(hY, Z)X - (\pounds_V g)(hX, Z)Y\\ &+ g((\pounds_V h)Y, Z)X - g((\pounds_V h)X, Z)Y\}.\end{aligned} \tag{3.15}$$

Contracting (3.15) over X and then fetching $Z = \xi$, $Y = \varphi Y$ and from (3.12), we have

$$(2\kappa + \mu)g(\varphi Y, \pounds_V \xi) - \mu g(h\varphi Y, \pounds_V \xi)\} = 0, \tag{3.16}$$

in (3.16) placing $Y = hY$ finds

$$(2\kappa + \mu)g(\varphi hY, \pounds_V \xi) - \mu(\kappa - 1)g(\varphi Y, \pounds_V \xi)\} = 0. \tag{3.17}$$

Later, comparing the equation (3.16) with (3.17) gives

$$\kappa\{2g(\varphi hY, \pounds_V \xi) - \mu g(\varphi Y, \pounds_V \xi) - 2g(\varphi Y, \pounds_V \xi)\} = 0. \tag{3.18}$$

Thus, from the preceding relation, three cases arise.
Case(i): If we assume in (3.18) that $\kappa = 0$ and $2g(\varphi hY, \pounds_V \xi) - \mu g(\varphi Y, \pounds_V \xi) - 2g(\varphi Y, \pounds_V \xi) \neq 0$ on M. Then from the equation (3.13) for $\kappa = 0$, we obtain $-16\mu + 4\mu^2 = 0$. Therefore, from this, if $\mu = 0$ then M is flat; and if $\mu \neq 0$ then we find $\mu = 4$. Further, in (3.15), taking $X = hX$ and contracting over X finds

$$\sum_{i=1}^{3} g((\pounds_V R)(he_i, \xi)\xi, e_i) = 4\mu(\kappa - 1)\eta(\pounds_V \xi). \tag{3.19}$$

Later, in (3.10) considering $X = hX$ and contracting over X gives that

$$\sum_{i=1}^{3} g((\pounds_V R)(he_i, \xi)\xi, e_i) = 8\left(3\kappa - \frac{r}{2}\right)(\kappa - 1) - 8\mu(\kappa - 1)^2 - 4\mu^2(\kappa - 1) + 2\mu^3(\kappa - 1). \tag{3.20}$$

On equating the last two equations and by using $\mu = 4$, $\kappa = 0$ values we obtain $\eta(\pounds_V \xi) = 8$. Thus, by contracting the equation (3.10) over X by taking $X = h^2 X$ and $Y = \xi$ and making use of $r = -2\mu$, we obtain

$$\sum_{i=1}^{3} (\pounds_V R)(h^2 e_i, \xi)\xi, e_i) = -16\mu + 4\mu^2. \tag{3.21}$$

Substituting X=$h^2 e_i$ and $Y = Z = \xi$, in (3.15) and then summing it over $i = 1$ to 3 yields

$$\sum_{i=1}^{3} (\pounds_V R)(h^2 e_i, \xi)\xi, e_i) = 32\mu. \tag{3.22}$$

On equating the last two equations, we obtain $4\mu^2 - 48\mu = 0$. Since $\mu = 4$, which leads to a contradiction. Hence, by this we come to the conclusion that for $\kappa = 0$, μ is also a zero constant. Therefore, in this case M is flat.
Case(ii): If $\kappa \neq 0$ and

$$2g(\varphi hY, \pounds_V \xi) - \mu g(\varphi Y, \pounds_V \xi) - 2g(\varphi Y, \pounds_V \xi) = 0. \tag{3.23}$$

Because we assumed $\kappa \neq 0$, if we let $\mu = 0$, we get $\kappa = 0$ by equating (3.19) with (3.20). So for $\kappa \neq 0$, μ is also a non zero constant. As a result, using this in (3.17) leads to deducing

$$g(h\varphi Y, \pounds_V \xi) = \frac{(2\kappa + \mu)}{\mu} g(\varphi Y, \pounds_V \xi). \tag{3.24}$$

On substituting above condition in (3.23), we get

$$\{4\kappa + 4\mu + \mu^2\} g(\varphi Y, \pounds_V \xi) = 0. \tag{3.25}$$

In the above equation, if we consider $g(\varphi Y, \pounds_V \xi) = 0$, then we get $g(Y, \pounds_V \xi) = \eta(\pounds_V \xi)\eta(Y)$. Use of this in the equation (1.3) for $Z = \xi$ finds $(\pounds_V \eta)Y = \{2\kappa + 4\lambda + 2\Delta V\}\eta(Y)$. This shows V is an infinitesimal contact transformation. Thus, from the Theorem 3.1, we get $\mu = 0$, which is a contradiction. So let's proceed with $4\kappa + 4\mu + \mu^2 = 0$. In equation (3.13) multiplying μ on both sides, we get

$$-4\kappa\mu\eta(\pounds_V \xi) = -8\left(3\kappa - \frac{r}{2}\right)\mu + 8\mu^2(\kappa - 1) - 4\mu^3(\kappa - 1). \tag{3.26}$$

On comparing (3.19) with (3.20) yields

$$4\mu\kappa\eta(\pounds_V \xi) = 8\left(3\kappa - \frac{r}{2}\right)\kappa - 8\mu(\kappa - 1)\kappa - 4\mu^2\kappa + 2\mu^3\kappa. \tag{3.27}$$

On adding the last two equations and by using $\kappa = -\frac{4\mu+\mu^2}{4}$ we find $\mu = 0$. But which contradicts our assumption that $\kappa \neq 0$. Hence, from (3.18) we can say κ must be a zero constant on M.

case(iii): Both $\kappa = 2g(\varphi hY, \pounds_V\xi) - \mu g(\varphi Y, \pounds_V\xi) - 2g(\varphi Y, \pounds_V\xi) = 0$. Simply this follows the case (i) and gives $\kappa = \mu = 0$. This completes the proof. □

Example 3.1. Let us consider a three dimensional manifold

$$M = \{(X, Y, Z) \in R^3, Z \neq 0\}, \tag{3.28}$$

where (X, Y, Z) is the canonical coordinates on R^3. Let $\{e_1, e_2, e_3\}$ be the set of orthogonal vector fields. Let we define the Riemannian metric g by

$$g(e_i, e_j) = \begin{cases} 1 \ \ if \ \ i = j \\ 0 \ \ if \ \ i \neq j, \end{cases}$$

and the vector fields e_1, e_2, e_3 holds the following Lie algebra structure

$$[e_1, e_3] = -c_2e_2, \ \ [e_2, e_3] = 2e_1, \ \ [e_1, e_2] = c_3e_3,$$

where c_2, c_3 are real constants on R. Now define a 1-form η such that $\eta(X) = g(X, e_1)$, for any arbitrary vector field X on $\mathcal{X}(M)$ and a $(1, 1)$-tensor φ defined by $\varphi e_1 = 0$, $\varphi e_2 = e_3$, $\varphi e_3 = e_2$. Then from the linearity property of g and φ we have

$$\eta(e_1) = 1, \ \ \varphi^2X = -X + \eta(X)\xi, \ \ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{3.29}$$

Let ∇ be the unique Riemannian connection with respect to the metric g and from the Koszula's formula, we have the following expressions

$$\begin{aligned} &\nabla_{e_1}e_1 = 0, \ \nabla_{e_1}e_2 = \frac{1}{2}(c_2 + c_3 - 2)e_3, \ \nabla_{e_1}e_3 = -\frac{1}{2}(c_2 + c_3 - 2)e_2, \\ &\nabla_{e_2}e_1 = \frac{1}{2}(c_2 - c_3 + 2)e_3, \ \nabla_{e_2}e_2 = 0, \ \nabla_{e_2}e_3 = -\frac{1}{2}(c_2 - c_3 - 2)e_1, \\ &\nabla_{e_3}e_1 = \frac{1}{2}(c_2 - c_3 + 2)e_2, \ \nabla_{e_3}e_2 = -\frac{1}{2}(c_2 - c_3 + 2)e_1, \ \nabla_{e_3}e_3 = 0. \end{aligned} \tag{3.30}$$

As a result of the preceding expressions, we obtain $d\eta(X, Y) = g(X, \varphi Y)$ for any X, $Y \in \mathcal{X}(M)$. As a result, the structure (φ, ξ, η, g) is a contact structure on M for $\xi = e_1$. As we know that on contact manifold $\nabla_X\xi = -\varphi X - \varphi hX$, from this and from the above conditions, we find out $he_1 = 0$, $he_2 = \frac{c_3-c_2}{2}e_2$, and $he_3 = -\frac{c_3-c_2}{2}e_3$. Let us find the Riemannian curvature tensor by the preceding equation as follows:

$$\begin{aligned} &R(e_2, e_3)e_1 = 0, \\ &R(e_2, e_1)e_1 = \{1 - \frac{(c_3 - c_2)^2}{4}\}e_2 + (2 - c_2 - c_3)he_2, \\ &R(e_3, e_1)e_1 = \{1 - \frac{(c_3 - c_2)^2}{4}\}e_3 + (2 - c_2 - c_3)he_3. \end{aligned}$$

So the above equations show that M is a (κ, μ)-contact manifold for $\kappa = 1 - \frac{(c_3-c_2)^2}{4}$ and $\mu = 2-c_2-c_3$. Moreover, M is Sasakian if and only if $c_2 = c_3$ and here we assume that M is non-Sasakian. Remaining curvature tensors are calculated as follows:

$$\begin{aligned}
R(e_1, e_2)e_3 &= 0,\\
R(e_2, e_3)e_3 &= \frac{1}{4}\{(c_2 - c_3)^2 - 4\}e_2 + \{c_2 + c_3 - 2\}e_2,\\
R(e_1, e_3)e_3 &= \frac{1}{4}\{(c_3 - 2)^2 - c_2^2\}e_1 - \frac{c_2}{2}\{c_2 - c_3 - 2\}e_1,\\
R(e_1, e_3)e_2 &= 0,
\end{aligned}$$

and applying these, we obtain

$$\begin{aligned}
S(e_1, e_1) &= 2\left(1 - \frac{(c_2 - c_3)^2}{4}\right),\\
S(e_2, e_2) &= \frac{1}{2}\{2 - c_2 - c_3\}\{c_3 - c_2 - 2\},\\
S(e_3, e_3) &= \frac{1}{2}\{2 - c_2 - c_3\}\{c_2 - c_3 + 2\}.
\end{aligned}$$

A vector field $V = a_2e_2$, for $a_2 \in R$ satisfies the soliton equation (1.2) for $\lambda = c_3 = 0$, and $c_2 = 2$. Hence, from the Riemannian curvature tensor expressions we get $\kappa = \mu = 0$. Therefore, this verifies the above theorem.

4 Almost Riemannian soliton

In this section, we consider an almost Riemann soliton for V is (i) gradient of a function f and (ii) collinear with ξ.

Theorem 4.1. *If a non-Sasakian (κ, μ)-contact metric g satisfies an almost Riemann soliton for $V = Df$, then either $\kappa = \mu = 0$, or f is constant, or both.*

Proof. Since $V = Df$, we can obtain the Riemannian curvature tensor by using the equation (1.5), as follows

$$\begin{aligned}
(4.1)\qquad 2R(X, Y)Df =& \frac{1}{2n-1}\{-2(\nabla_X Q)Y + 2(\nabla_Y Q)X - 4n(X\lambda)Y + 4n(Y\lambda)X\\
&- 2(X\Delta V)Y + 2(Y\Delta V)X\}.
\end{aligned}$$

Substituting $Y = \xi$ in (4.1) and by (2.14), we obtain

$$\begin{aligned}
(4.2)\quad & 2\kappa\{g(X, Df)\eta(Y) - (\xi f)g(X, Y)\} + 2\mu\{g(hX, Df)\eta(Y) - (\xi f)g(hX, Y)\} =\\
& -4n\kappa g(\varphi X, Y) - 4n\kappa g(\varphi hX, Y) + 2g(Q\varphi hX, Y) - 2(2(n-1) + \mu)\mu g(h\varphi X, Y)\\
& +4n(X\lambda)\eta(Y) - 4n(\xi\lambda)g(X, Y) + 2(X\Delta V)\eta(Y) - 2(\xi\Delta V)g(X, Y).
\end{aligned}$$

Asymmetrizing the preceding relation and then taking $X = \varphi X$, $Y = \varphi Y$ gives

$$(4.3)\qquad \{-8n\kappa + 4[2(n-1) - n\mu] + 4[2(n-1) + \mu](\kappa - 1)\}g(\varphi X, Y) = 0.$$

Again symmetrizing (4.2), then taking X by φhX and $Y = \varphi Y$ and contracting over X, Y yields

$$\mu(\xi f) = 0. \tag{4.4}$$

This implies, either $\xi f = 0$ or $\mu = 0$, or both are zero.
Case(i); if $\xi f = 0$, $\mu \neq 0$, then taking covariant derivative of ξf with respect to an orbitrary vector field X gives

$$g(-\varphi X - \varphi hX, Df) + g(\xi, \nabla_X Df) = 0. \tag{4.5}$$

In (4.5), with the help of (1.5) and by taking φX instead of X finds

$$g(X, Df) - g(hX, Df) = 0. \tag{4.6}$$

Replacing X by hX in (4.6) we have

$$g(hX, Df) + (\kappa - 1)g(X, Df) = 0. \tag{4.7}$$

Solving the last two equations results in $\kappa(Xf) = 0$. In this if $\kappa = 0$ then by (4.3) we get $\mu = 0$ which contradicts our assumption that $\mu \neq 0$. Therefore, $Xf = 0$ which shows f is constant on M.
Next, *case(ii)*; if $\xi f \neq 0$ and $\mu = 0$ then by (4.3) we obtain $\kappa = 0$.
Finally, if $\xi f = \mu = 0$, then by summing up the above two cases, we conclude that on M, either $\kappa = \mu = 0$, or f is constant or both. Hence the theorem is proved. □

At the end of this paper, we are going to study an almost Riemann soliton for $V = \sigma\xi$. Let us consider a Riemann soliton on a non-Sasakian (κ, μ)-contact manifold for $V = \sigma\xi$, where σ is a smooth function on M. Then from (1.3), we have

$$\begin{aligned}(X\sigma)\eta(Y) + (Y\sigma)\eta(X) + 2\sigma g(h\varphi X, Y) =& \frac{1}{2n-1}\{-2g(QX, Y) - 4n\lambda g(X, Y) \\ &- 2\Delta V g(X, Y)\}.\end{aligned} \tag{4.8}$$

Putting $X = \varphi X$ and $Y = hY$ in (4.8), and contracting it over X, Y gives

$$2\sigma(\kappa - 1)2n = 0. \tag{4.9}$$

Since M is non-Sasakian, if V is a non-zero vector field, then the foregoing relation leads to a contradiction. Hence, we can state the following theorem.

Theorem 4.2. *If M is a non-Sasakian (κ, μ)-contact manifold, then M cannot have an almost Riemann soliton for a non-zero vector V collinear with ξ.*

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Author's address:

Shruthi Chidananda, Venkatesha Venkatesha
Department of Mathematics, Kuvempu University, Shankaraghatta, India.
E-mail: c.shruthi28@gmail.com , vensmath@gmail.com