On some types of submanifolds of generalized Sasakian space forms

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Abstract. The object of the present paper is to study submanifolds of generalized Sasakian space forms. We characterize anti-invariant submanifolds whose second fundamental form satisfy some conditions. We also study totally umbilical submanifolds of generalized Sasakian space forms and show that totally umbilical submanifolds are either totally geodesic or isomorphic to a sphere or homothetic to a Sasakian manifold. Finally, we study totally umbilical anti-invariant submanifolds of generalized Sasakian space forms.

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1 Introduction

A Sasakian manifold $\tilde{M}$ of dimension greater than 3 with constant ϕ-sectional curvature c is called Sasakian-space-form [19]. The curvature tensor $\tilde{R}$ of such a manifold is given by

$$\tilde{R} = \left(\frac{c+3}{4}\right) R_1 + \left(\frac{c-1}{4}\right) R_2 + \left(\frac{c-1}{4}\right) R_3, \tag{1.1}$$

where R_1, R_2, R_3 are given by

$$R_1(X,Y)U = g(Y,U)X - g(X,U)Y,$$
$$R_2(X,Y)U = g(X,\phi U)\phi Y - g(Y,\phi U)\phi X + 2g(X,\phi Y)\phi U,$$
$$R_3(X,Y)U = \eta(X)\eta(U)Y - \eta(Y)\eta(U)X + g(X,U)\eta(Y)\xi - g(Y,U)\eta(X)\xi,$$

for any vector field X, Y, U on the manifold.
As a natural generalization of Sasakian space forms Alegre, Blair and Carriazo introduced in [1] the notion of generalized Sasakian space forms. They also gave several examples of generalized Sasakian space forms. An almost contact metric manifold is

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called generalized Sasakian-space-form if there exist three smooth functions f_1, f_2, f_3 such that

$$\tilde{R} = f_1 R_1 + f_2 R_2 + f_3 R_3,$$

where R_1, R_2, R_3 are defined in (1.1). It is obvious that Sasakian space forms are natural examples of generalized Sasakian space forms, with constant functions

$$f_1 = \frac{c+3}{4}, \quad f_2 = \frac{c-1}{4}, \quad f_3 = \frac{c-1}{4}.$$

Generalized Sasakian space forms have been studied by several authors, see ([2], [5], [8], [16], [24]).

A submanifold of an almost contact metric manifold is called invariant if the tensor field ϕ maps tangent vector to tangent vector. It is called anti-invariant if ϕ maps tangent vector to normal vector. The totally geodesic submanifolds are simplest submanifolds. So, there is a natural trend to verify whether invariant submanifolds are totally geodesic. Invariant submanifolds of Sasakian manifolds were studied by M. Kon [17]. Invariant submanifolds have been studied by many authors, see ([11], [15], [24], [26], [27]). Similarly the study of anti-invariant submanifolds are also significant. So, it is natural to ask is there any relation between anti-invariant submanifolds and totally geodesic submanifolds? In this paper we shall try to give the answer. Many authors have studied anti-invariant submanifold, for example see ([23], [29], [30], [31]).

Motivated by above works in this present paper we would like to study anti-invariant submanifold of generalized Sasakian space forms. Present paper is organized as follows:

Section 1 contains introduction. We provide preliminaries in Section 2 and give an example of generalized Sasakian-space-form. After the introduction and preliminaries we study anti-invariant submanifold with semiparallel, pseudo-parallel second fundamental form in Section 3. Section 4 contains the study of totally umbilical submanifolds. Finally, we study totally umbilical anti-invariant submanifolds of generalized Sasakian space forms in Section 5.

2 Preliminaries

Let us consider an odd dimensional almost contact smooth manifold $\tilde{M}$ with an almost contact structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, η is a one-form, ξ is a vector field and g is a Riemannian metric on the manifold $\tilde{M}$. Such manifolds satisfy the following relations [6]

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \eta(X) = g(X, \xi),$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \phi\xi = 0, \tag{2.1}$$

$$\eta o \phi = 0, \quad g(X, \phi Y) = -g(\phi X, Y), \quad (\nabla_X \eta)Y = g(\nabla_X \xi, Y)$$

For any vector field X, Y on the manifold $\tilde{M}$. An almost contact manifold is contact manifold if there exists a two-form Φ, for any X, Y such that

$$d\eta(X, Y) = \Phi(X, Y) = g(X, \phi Y).$$

Moreover, if ξ is Killing vector field then the manifold is known as K-contact manifold. We know that a contact metric manifold is K-contact if and only if $\tilde{\nabla}_X \xi = -\phi X$. Again an almost contact manifold is Sasakian manifold if and only if [18]

$$(\tilde{\nabla}_X \phi)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.2}$$

On a generalized Sasakian-space-form, we have

$$\tilde{R}(X,Y)\xi = (f_1 - f_3)\big(\eta(Y)X - \eta(X)Y\big) \tag{2.3}$$

$$\tilde{R}(\xi,Y)Z = (f_1 - f_3)\big(g(Y,Z)\xi - \eta(Z)Y\big) \tag{2.4}$$

$$\begin{aligned} &\tilde{S}(\xi,\xi) = 2n(f_1 - f_3), \\ &\tilde{S}(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - \big(3f_2 + (2n-1)f_3\big)\eta(X)\eta(Y), \end{aligned} \tag{2.5}$$

$$\tilde{R}(X,\xi)\xi = (f_1 - f_3)\big(X - \eta(X)\xi\big), \tag{2.6}$$

$$\begin{aligned} &\tilde{r} = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \\ &\tilde{Q}(X) = (2nf_1 + 3f_2 - f_3)X - \{3f_2 + (2n-1)f_3\}\eta(X)\xi \end{aligned} \tag{2.7}$$

for any vector field X, Y on the manifold, where $\tilde{R}$, $\tilde{S}$, $\tilde{r}$, and $\tilde{Q}$, are curvature tensor, Ricci tensor, scalar curvature and Ricci operator on $\tilde{M}$ respectively.
In a K-contact manifold we have [6]

$$(\tilde{\nabla}_X \phi)(Y) = \bar{R}(\xi, X)Y, \tag{2.8}$$

for any vector field X, Y and $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$. Using (2.4) and the previous equation we have in a generalized Sasakian-space-form $\tilde{M}(f_1, f_2, f_3)$

$$(\tilde{\nabla}_X \phi)(Y) = (f_1 - f_3)\big(g(X,Y)\xi - \eta(Y)X\big). \tag{2.9}$$

From the above equation we get

$$\tilde{\nabla}_X \xi = -(f_1 - f_3)\phi X. \tag{2.10}$$

Let M^{2m+1} $(m < n)$ be the submanifold of the manifold $\tilde{M}^{2n+1}$. Let ∇ be the Levi-Civita connection of M. Then for any vector field X, Y on the manifold, the second fundamental form σ is defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y). \tag{2.11}$$

For any vector field V of normal bundle $T^{\perp}M$, we have

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp} V, \tag{2.12}$$

where $\nabla^{\perp}$ is the induced connection on the normal bundle $T^{\perp}M$. The relation between the second fundamental form σ and the shape operator A_V is

$$g(A_V X, Y) = g(\sigma(X,Y), V).$$

For any vector field X, on the manifold we can right

$$\phi X = TX + NX, \tag{2.13}$$

where TX is the tangential component of ϕX and NX is the normal component of ϕX. Similarly for any vector field V in normal bundle we have

$$\phi V = tV + nV, \tag{2.14}$$

where tV and nV are the tangential and normal components of ϕV.

The submanifold M is said to be invariant if N is identically zero, that is $\phi X \in TM$, for any vector field X. On the other hand M is said to be an anti-invariant submanifold if T is identically zero, that is $\phi X \in T^{\perp}M$, for any vector field X.
From (2.1) and (2.13), we get

$$g(TX, Y) = -g(X, TY), \tag{2.15}$$

for any X, Y on the manifold.

On a Riemannian manifold $\tilde{M}$, for a $(0, k)$-type tensor field $T(k \geq 1)$ and a $(0, 2)$-type tensor field E, we denote by $Q(E, T)$ a $(0, k+2)$-type tensor field [28] defined as follows

$$\begin{aligned} Q(E,T)(X_1, X_2, ..., X_k; X, Y) = & - T((X \wedge_E Y)X_1, X_2, ..., X_n) \\ & - T(X_1, (X \wedge_E Y)X_2, ..., X_k) - ... \\ & - T(X_1, ..., (X \wedge_E Y)X_k), \end{aligned} \tag{2.16}$$

where $(X \wedge_E Y)U$ is defined by

$$(X \wedge_E Y)U = E(Y, U)X - E(X, U)Y,$$

for all vector field $X, Y, U \in TM$.

In 1996, A. Lotta introduced the notion of slant submanifolds of an almost contact metric manifold in his paper [20]. A submanifold M is said to be slant if for any $X \in T_xM$, linearly independent on ξ, the angle between ϕX and $X \in T_xM$ is constant $\theta \in [0, \pi/2]$. The angle θ is called slant angle of M in $\tilde{M}$. Moreover if $\theta = 0$, then the submanifold is called invariant, and if $\theta = \pi/2$, then the submanifold is called anti-invariant. A slant submanifold which is neither invariant nor anti-invariant is called proper slant submanifold. For a slant submanifold the slant angle θ is defined by

$$\cos\theta = \frac{g(\phi X, TX)}{|\phi X||TX|}. \tag{2.17}$$

For any vector field $X \in TM$, equations (2.1) and (2.13) gives

$$T^2X + NTX + tNX + nNX = -X + \eta(X)\xi.$$

Comparing tangential and normal components, we obtain

$$T^2X + tNX = -X + \eta(X)\xi, \tag{2.18}$$

and

$$NTX + nNX = 0. \tag{2.19}$$

Now for any vector field $V \in T^{\perp}M$, and using the equations (2.1), (2.14) we get

$$TtV + NtV + tnV + n^2V = -V.$$

Comparing tangential and normal components we have

$$TtV + tnV = 0, \tag{2.20}$$

and

$$NtV + n^2V = -V. \tag{2.21}$$

By the virtue of (2.9), (2.11), (2.13) and comparing tangential and normal components we obtain

$$(\nabla_X T)Y - t\sigma(X,Y) - A_{NY}X = (f_1 - f_3)\big(g(X,Y)\xi - \eta(Y)X\big), \tag{2.22}$$

and

$$(\nabla_X N)Y\sigma(X,TY) - n\sigma(X,Y) = 0. \tag{2.23}$$

Similarly, using (2.9), (2.11), (2.14) and comparing tangential and normal components we obtain

$$(\nabla_X t)V - A_{nV}X + TA_VX = 0, \tag{2.24}$$

and

$$(\nabla^{\perp}_X n)V - \sigma(X,tV) + NA_VX = 0, \tag{2.25}$$

Example 2.1. In [1] it has been proven that $\mathbb{R} \times_f \mathbb{C}^m$ is a generalized Sasakian-space-form with

$$f_1 = -\frac{f'^2}{f^2}, \qquad f_2 = 0, \qquad f_3 = -\frac{f'^2}{f^2} + \frac{f''}{f},$$

where $f = f(t)$, $t \in \mathbb{R}$, $f(t) > 0$, and m is a positive integer.
If we take $m = 2$, we see that $\mathbb{R} \times_f \mathbb{C}^2$ is a five-dimensional generalized Sasakian-space-form.
In [9] the authors proved that the immersion defined by

$$x(t,u,v) = (t, u\cos\theta, u\sin\theta, v, 0)$$

defines a three-dimensional, totally geodesic and θ-slant ([10]) submanifold M in $\mathbb{R} \times_f \mathbb{C}^2$, for $\theta \in [0, \pi/2)$.

3 Anti-invariant submanifolds of generalized Sasakian space forms with some conditions

Let M be a submanifold of a generalized Sasakian-space-form $\tilde{M}$. A submanifold M of the manifold $\tilde{M}$ is invariant if ϕ map tangent vector to tangent vector, that is $\phi(TM) \subset TM$, where TM is the tangent space of M and the submanifold is anti-invariant if $\phi(TM) \subset T^{\perp}M$, where $T^{\perp}M$ is orthogonal to M. In this section we would like to study anti-invariant submanifold of generalized Sasakian space forms.

We have the following Proposition of [8]

Proposition 3.1. *In a submanifold of a generalized Sasakian-space-form the following relations hold*

$$\nabla_X \xi = -(f_1 - f_3)TX, \tag{3.1}$$

$$\sigma(X,\xi) = -(f_1 - f_3)NX, \tag{3.2}$$

$$\nabla_\xi \xi = 0, \tag{3.3}$$

$$\sigma(\xi,\xi) = 0, \tag{3.4}$$

$$-A_V \xi = -(f_1 - f_3)tV, \tag{3.5}$$

$$\nabla^{\perp}_{\xi} V = -(f_1 - f_3)nV, \tag{3.6}$$

for any X tangent to M and V normal to M.

From the previous proposition we can say in an anti-invariant submanifold

$$\nabla_X \xi = 0, \tag{3.7}$$

$$\sigma(X,\xi) = -(f_1 - f_3)\phi(X), \tag{3.8}$$

$$\sigma(\xi,\xi) = 0. \tag{3.9}$$

Definition 3.1. The submanifold M of the manifold $\tilde{M}$ is called semiparallel [13] if

$$\tilde{R}(X,Y).\sigma(U,V) = 0, \tag{3.10}$$

for any vector field $X, Y, U, V \in TM$.

Here $\tilde{R}(X,Y).\sigma(U,V)$ is defined by

$$\tilde{R}(X,Y).\sigma(U,V) = R^{\perp}(X,Y)\sigma(U,V) - \sigma(R(X,Y)U,V) - \sigma(U,R(X,Y)V).$$

Theorem 3.2. *If an anti-invariant submanifold M of a generalized Sasakian-space-form $\tilde{M}$, is semiparallel, then the submanifold is an integral curve of the Reeb vector field ξ, whenever $f_1 \neq f_3$.*

Proof. Here the submanifold M is semiparallel, hence from (3.10) we get

$$\tilde{R}(X,Y).\sigma(U,V) = 0.$$

Putting $Y = U = V = \xi$ in the above and using (3.9) we obtain

$$\sigma(R(X,\xi)\xi,\xi) + \sigma(\xi,R(X,\xi)\xi) = 0.$$

Now using (2.6) in the foregoing equation we have

$$(f_1 - f_3)\phi X = 0.$$

Since $f_1 \neq f_3$ then we have from above $\phi X = 0$. This completes the proof. □

From the previous theorem we have the following result

Corollary 3.3. *If M be an anti-invariant submanifold of a generalized Sasakian-space-form $\tilde{M}$, is semiparallel, then there does not exists a non-zero Legendre curve on the manifold, whenever $f_1 \neq f_3$.*

Definition 3.2. The submanifold M of the manifold $\tilde{M}$ is said to be pseudo-parallel ([3], [4]) if

$$\tilde{R}(X,Y).\sigma(U,V) = fQ(g,\sigma)(X,Y)(U,V), \tag{3.11}$$

for any vector field X, Y on the submanifold and f denotes real valued function on M and the operator Q is defined in (2.16).

Theorem 3.4. *If an anti-invariant submanifold M of a generalized Sasakian-space-form $\tilde{M}$ is pseudo-parallel, then the submanifold is an integral curve of the Reeb vector field ξ, whenever $f \neq f_1 - f_3$.*

Proof. The submanifold is pseudo-parallel, hence then form (3.11) we have

$$\tilde{R}(X,Y).\sigma(U,V) = fQ(g,\sigma)(X,Y;U,V),$$

By (2.16) the above equation gives

$$\begin{aligned} R^{\perp}(X,Y)\sigma(U,V) \qquad & -\sigma(R(X,Y)U,V) - \sigma(U,R(X,Y)V) = \\ & -f\big(g(Y,U)\sigma(X,V) - g(X,U)\sigma(Y,V) \\ & +g(Y,V)\sigma(U,X) - g(X,V)\sigma(U,Y)\big). \end{aligned}$$

Putting $Y = U = V = \xi$ and using (2.6) and (3.9), we have the form from above

$$(f_1 - f_3)\phi X = f\sigma(X,\xi).$$

Now using (3.2) in the foregoing equation, we obtain

$$(f_1 - f_3 - f)\phi X = 0.$$

By the assumed condition $f \neq f_1 - f_3$, we have $\phi X = 0$. This proves the theorem. □

From the previous theorem we have the following corollary

Corollary 3.5. *If M be an anti-invariant submanifold of a generalized Sasakian-space-form $\tilde{M}$ is pseudo-parallel, then there does not exist a non-zero Legendre curve on the manifold, whenever $f \neq f_1 - f_3$.*

4 Totally umbilical submanifolds of a generalized Sasakian space forms

Let M be an n-dimensional totally umbilical submanifold of a generalized Sasakian-space-form $\tilde{M}$. Then the second fundamental form σ of M is given by $\sigma(X,Y) = g(X,Y)H$ for any vector field X, Y on the submanifold and H is mean curvature vector defined by

$$H = \sum_{i=1}^{n} \sigma(e_i, e_i)$$

If we take $\alpha = ||H||^2$, then for the totally umbilical submanifold M with mean curvature vector H parallel in the normal bundle, we have $X.\alpha = 0$, for any vector field X, that is, α is constant.

If $\alpha \neq 0$, define a unit vector e in the normal bundle, by setting $H = \sqrt{\alpha}e$. The normal bundle can be split into the direct sum $\alpha = \{e\} \oplus \{e\}^{\perp}$, where $\{e\}^{\perp}$ is the orthogonal compliment of the line sub-bundle $\{e\}$ spanned by e. For any vector field X set

$$\phi X = \psi(X) - A(X)e + P(X), \qquad \phi e = t + F, \tag{4.1}$$

where $\psi(x)$ is the tangential components of ϕX, $A(X)$ and $P(X)$ are the $\{e\}$ and $\{e\}^{\perp}$ components, respectively. t and F are the $\{e\}$ and $\{e\}^{\perp}$ components of ϕe, respectively.

Proposition 4.1. *Let M be a totally umbilical submanifold of a generalized Sasakian-space-form $\tilde{M}$ with curvature vector parallel to the normal bundle. If $\alpha \neq 0$, then for any vector field X we have*
i) $\tilde{\nabla}_X e = -\sqrt{\alpha}X$,
ii) $\nabla_X t = -\sqrt{\alpha}\psi(X) - \big(f_1 - f_3\big)\eta(e)X$,
iii) $\nabla^{\perp}_X F = -\sqrt{\alpha}P(X)$.

Proof. Taking the inner product with respect to Y in both sides of equation (2.12), we obtain

$$\tilde{\nabla}_X V = -g(H,V)X + \nabla^{\perp}_X V.$$

Putting $V = e$, in above equation we obtain

$$\tilde{\nabla}_X e = -\sqrt{\alpha}X.$$

Thus (i) is proved.
Next putting $Y = e$, in the equation (2.2), and using the equation (4.1) we obtain

$$\begin{aligned}\nabla_X t + \nabla^{\perp}_X F \quad &+ \quad \sqrt{\alpha}(\psi(X) - A(X)e + P(X)) + \sigma(X, \phi(e)) \\ &= \quad -\big(f_1 - f_3\big)\eta(e)X.\end{aligned}$$

Comparing the tangential part we have

$$\nabla_X t = -\sqrt{\alpha}\psi(X) - \big(f_1 - f_3\big)\eta(e)X.$$

Thus (ii) is proved. Now comparing $\{e\}^{\perp}$ component and using the result $A(X) = g(X,t)$ we obtain

$$\nabla^{\perp}_{X} F = -\sqrt{\alpha} P(X).$$

Thus (iii) is proved. □

Proposition 4.2. *Let M be a totally umbilical submanifold of a generalized Sasakian-space-form $\tilde{M}$ with mean curvature vector parallel in the normal bundle. If $\alpha \neq 0$, and $\xi \perp e$, then, setting $\xi = \xi_1 + \xi_2$, where ξ_1 is the tangential component and ξ_2 is the $\{e\}^{\perp}$-component of ξ, we have*

(i) $\nabla_X \xi_1 = -(f_1 - f_3)\psi(X)$,

(ii)$(\nabla_X \psi)Y = \big((f_1 - f_3) - \frac{\alpha}{(f_1 - f_3)}\big)\big(g(X,Y)\xi_1 - \eta(Y)X\big)$.

Proof. Putting $\xi = \xi_1 + \xi_2$ in the equation (2.11) and (4.1) we have

$$\nabla_X \xi_1 + \nabla_X \xi_2 + \sigma(X,\xi) = (f_1 - f_3)\big(\psi X - A(X)e + P(X)\big).$$

Comparing tangential part we have (i), and comparing e component, we have $\sigma(X,\xi) = (f_1 - f_3)A(X)e$ that is,

$$\sqrt{\alpha}\eta(X) = (f_1 - f_3)A(X), \qquad \sqrt{\alpha}\xi_1 = (f_1 - f_3)t. \tag{4.2}$$

Now using the equations (2.2) and (4.1) we have

$$\begin{aligned} \nabla_X(\psi Y) - \nabla_X(AY)e - A(Y)(\nabla_X e) - \psi(\nabla_X Y) + A(\nabla_X Y)e - \\ P(\nabla_X Y) + (\nabla_X PY) = (f_1 - f_3)(g(X,Y)\xi - \eta(Y)X). \end{aligned}$$

Using the Proposition 4.1, we obtain from the above equation

$$\begin{aligned} (\nabla_X \psi)Y + (\nabla_X P)Y + \sqrt{\alpha} g(X,Y)(t + F) - \frac{\sqrt{\alpha}\eta(Y)X}{f_1 - f_3} \\ = (f_1 - f_3)(g(X,Y)\xi - \eta(Y)X). \end{aligned}$$

Comparing the tangential parts we obtain (ii). □

Theorem 4.3. *If M be a totally umbilical submanifold of a generalized Sasakian-space-form $\tilde{M}$ with mean curvature vector parallel in the normal bundle then one of the following holds:*

(i) M is totally geodesic;

(ii) M is isometric to a sphere;

(iii) M is homothetic to a Sasakian manifold, whenever $f_1 \neq f_3$.

Proof. Since H is parallel in the normal bundle and α is a constant. If $\alpha = 0$, then $H = 0$, and consequently $\sigma(X,Y) = 0$, for any vector field X, Y on the manifold. Thus the submanifold M is totally geodesic, which proves the first part of the theorem. Next we assume that $\alpha \neq 0$. Define a smooth function $f : M \to R$ by $f = g(e,\xi)$, for any vector field X. Then Proposition 4.1, and the equations (2.10), (2.11), (2.12), imply that

$$\begin{aligned} Xf &= g(\nabla_X \xi, e) + g(\xi, \nabla_X e) \\ &= (f_1 - f_3)g(X,t) - \sqrt{\alpha} g(\xi, X). \end{aligned}$$

So, by using the equation (4.1) and the above equation and the Proposition 4.2, we have

$$XYf - (\nabla_X Y)f = -(f_1 - f_3)^2 f g(X,Y),$$

$$g(\nabla_X \mathrm{grad} f, Y) = -(f_1 - f_3)^2 f g(X,Y). \tag{4.3}$$

Taking the trace of this equation we have

$$\Delta f = -(f_1 - f_3)^2 n f. \tag{4.4}$$

If f is non-constant function, then the above equation is the differential equation in [21], which is necessary and sufficient condition for the submanifold M to be isometric to a sphere of radius $\frac{1}{f_1 - f_3}$.
If f is a constant, then equation (4.4) gives $-n(f_1 - f_3)^2 f = 0$, if we take $f_1 \neq f_3$ and consequently $f = 0$, that is $\xi \perp e$.
Again we define a smooth function $G : M \to R$ by

$$G = \frac{1}{2} tr.\psi^2. \tag{4.5}$$

Note that (4.1) gives $g(\psi Y, X) = -g(\psi X, Y)$, for any vector field X, Y on the submanifold.
Consider ω be a one-form, defined by $\omega = dG$. For each $p \in M$ we can choose a local orthonormal frame $\{e_1, ..., e_n\}$ of M such that $\nabla e_i(p) = 0$. Thus, for any vector field Z, we have

$$\omega(Z) = ZG = \sum_{i=1}^{n} g((\nabla_Z \psi)(e_i), \psi(e_i)).$$

Using Proposition 4.2, we obtain

$$\omega(Z) = 2\big((f_1 - f_3) - \frac{\alpha}{(f_1 - f_3)}\big) g(\psi Z, \xi_1). \tag{4.6}$$

The first covariant derivative of (4.6) is

$$\begin{aligned}(\nabla\omega)(Y,Z) &= 2\big((f_1 - f_3) - \frac{\alpha}{(f_1 - f_3)}\big)(f_1 - f_3) g(\psi Y, \psi Z) \\ &\quad + 2\big((f_1 - f_3) - \frac{\alpha}{(f_1 - f_3)}\big)^2 [g(\xi_1, \xi_1) g(Y,Z) - g(Y,\xi) g(Z,\xi_1)],\end{aligned}$$

and consequently formed the above equation

$$\begin{aligned}(\nabla^2\omega)(X,Y,Z) &+ \big((f_1 - f_3) - \frac{\alpha}{(f_1 - f_3)}\big)(f_1 - f_3)(2g(Y,Z)\omega(X) \\ &+ g(X,Y)\omega(Z) + g(X,Z)\omega(Y)) = 0.\end{aligned} \tag{4.7}$$

Equation (4.7) is the differential equation in [14] which, G being non-constant, is the necessary and sufficient condition for M to be isometric to a sphere. This again leads to case (ii). Suppose G is constant function. Then (4.7) gives $\psi(\xi_1) = 0$. Define another smooth function $G_1 : M \to R$ by

$$G_1 = g(\xi_1, \xi_1).$$

Then using the Proposition 4.2, we get $X.\alpha = 0$ for any vector field X. That is ξ_1 has constant length. Taking the covariant derivative in (i) of Proposition 4.2 and using (ii), we get

$$\nabla_X \nabla_Y \xi_1 - \nabla_{\nabla_X Y} \xi_1 = \big((f_1 - f_3) - \frac{\alpha}{(f_1 - f_3)}\big)(f_1 - f_3)\big(g(X, Y)\xi_1 - g(Y, \xi_1)X\big). \tag{4.8}$$

From (i) of Proposition 4.2, it follows that ξ_1 is a Killing vector field if we take $(f_1 - f_3) \neq \frac{\alpha}{(f_1 - f_3)}$. ξ_1 is a Killing vector field of constant length, which satisfies the above equation which is a result of Okumura [22] states that, if $\xi_1 \neq 0$, then M is homothetic to a Sasakian manifold, which is (iii). Clearly $\xi_1 \neq 0$, more details see [12]. □

5 Totally umbilical anti-invariant submanifolds of generalized Sasakian space forms

Proposition 5.1. *If M is a totally umbilical anti-invariant submanifold of a generalized Sasakian space forms $\tilde{M}$ with mean curvature vector H, then $\nabla^{\perp}_X H = 0$, for any vector field X on the submanifold.*

Proof. Consider the Codazzi equation

$$R^{\perp}(X, Y)Z = (\tilde{\nabla}_X \sigma)(Y, Z) - (\tilde{\nabla}_Y \sigma)(X, Z).$$

Putting $Z = \xi$ in the above equation we have

$$R^{\perp}(X, Y)\xi = \eta(Y)\nabla^{\perp}_X H - \eta(X)\nabla^{\perp}_Y H.$$

Now from (2.3) we can say $R^{\perp}(X, Y)\xi = 0$, then we have from above

$$\eta(Y)\nabla^{\perp}_X H - \eta(X)\nabla^{\perp}_Y H = 0.$$

Finally we put $Y = \xi$ in the foregoing equation we obtain $\nabla^{\perp}_X H = 0$. □

From Theorem 4.3 and Proposition 5.1 we have the following:

Theorem 5.2. *If M be a totally umbilical anti-invariant submanifold of a generalized Sasakian-space-form, then one of the following holds:*
(i) M is totally geodesic;
(ii) M is isometric to a sphere;
(iii) M is homothetic to a Sasakian manifold, whenever $f_1 \neq f_3$.

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