Weak $M$–projective symmetries for a Sasakian manifold

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Abstract. In this study, we deal with the $M$–projective curvature tensor on a Sasakian manifold. After defining both weakly $M$–projective symmetric Sasakian manifolds and weakly $M$–projective Ricci symmetric Sasakian manifolds, we obtained all the associated 1-forms and the relation among them. Moreover, we investigated a weakly $M$–projective Ricci symmetric Sasakian manifold having a cyclic parallel $M$–projective Ricci tensor.

Key words: Sasakian manifold, $M$–projective curvature tensor, $M$–projective Ricci tensor.

1 Introduction

In the 19th century, the notion of contact transformation was introduced by Lie [12] to study systems of differential equations from a geometrical point of view. This notion can be seen as the beginning of contact geometry. The subject has manifold connections with other fields of pure mathematics, and a significant place in applied areas such as optics, control theory or mechanics, etc... until Chern [5] and Gray [9] contributed to the development of this geometry, contact geometry has not been receiving attention. Then the notion of a Sasakian manifold was introduced by Sasaki [16]. This manifold is a contact manifold with a Sasakian metric which is a special kind of Riemannian metric $g$. It forms an important class of a contact manifold and has been studied by many authors (see, [1, 7, 10, 11, 14, 21, 22]).

Pokhariyal and Mishra [15] have introduced the $M$– projective curvature tensor of type $(0,4)$ on a Riemannian manifold $(M^n, g)$ in the form:

$$W(Y, Z, U, V) = R(Y, Z, U, V) - \frac{1}{2(n-1)} (S(Z, U)g(Y, V) - S(Y, U)g(Z, V) + g(Z, U)S(Y, V) - g(Y, U)S(Z, V)),$$

so that

$$W(Y, Z, U, V) = g(W(Y, Z)U, V), \quad R(Y, Z, U, V) = g(R(Y, Z)U, V).$$
where \(W(Y, Z)U, R(X, Y)Z\) and \(S(X, Y)\) are the \(M\)–projective curvature tensor of type \((1, 3)\), the curvature tensor and the Ricci tensor of \((M^n, g)\), respectively. The Ricci tensor \(S\) is defined by \(S(X, Y) = g(LX, Y)\), where \(L\) is the Ricci operator. The \(M\)–projective curvature tensor is important in the terms of relativity [15].

Furthermore, the other properties of this tensor have been extensively studied on the various manifolds by several authors (see, [4, 13, 14, 24, 25]).

In 1992, Tamássy and Binh [19] introduced the notion of weakly symmetric manifolds. A non-flat Riemannian manifold \((M^n, g)\) is called a \textit{weakly-symmetric manifold} if there exist 1–forms \(\alpha, \beta, \gamma, \delta\) and \(\sigma\) satisfying the condition

\[
(1.2) \quad (\nabla_X R)(Y, Z, U, V) = \alpha(X) R(Y, Z, U, V) + \beta(Y) R(X, Z, U, V) + \gamma(Z) R(Y, X, U, V) + \delta(U) R(Y, Z, X, V) + \sigma(V) R(Y, Z, U, X)
\]

for all vector fields \(X, Y, Z, U, V \in \chi(M)\), where \(\nabla\) is the covariant derivative with respect to the Levi-Civita connection. Then, De and Bandyopadhyay [6] proved that in such a manifold 1–forms \(\beta\) and \(\delta\) are equal to \(\gamma\) and \(\sigma\), respectively. Thus, the number of different 1–forms in the above relation is reduced from five to three.

In 1993, again Tamássy and Binh [20] introduced the notion of weakly Ricci symmetric manifolds. A Riemannian manifold \((M^n, g)\) is called a \textit{weakly Ricci symmetric manifold} if its Ricci tensor \(S\) of type \((0, 2)\) is not identically equal to zero and satisfies the condition

\[
(1.3) \quad (\nabla_X S)(Z, U) = \rho(X) S(Z, U) + \mu(Z) S(X, U) + \nu(U) S(Z, X),
\]

where 1–forms \(\rho, \mu\) and \(\nu\) are not zero.

In 1998, Gray [8] introduced two classes of Riemannian manifolds determined by the covariant differentiation of Ricci tensor \(S\). One of these classes consisting of all Riemannian manifolds whose Ricci tensor \(S\) is cyclic parallel, i.e.

\[
(1.4) \quad (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.
\]

Recently, Chaubey [3] introduced the notion of weakly \(M\)–projectively symmetric manifolds. A non-\(M\)–projectively flat Riemannian manifold \((M^n, g)\) \((n > 2)\) is called a \textit{weakly \(M\)–projectively symmetric manifold} if the \(M\)–projective curvature tensor \(W\) of type \((0, 4)\) satisfies the relation

\[
(1.5) \quad (\nabla_X W)(Y, Z, U, V) = \alpha(X) W(Y, Z, U, V) + \beta(Y) W(X, Z, U, V) + \gamma(Z) W(Y, X, U, V) + \delta(U) W(Y, Z, X, V) + \sigma(V) W(Y, Z, U, X)
\]

for all vector fields \(X, Y, Z, U, V \in \chi(M)\), where \(\alpha, \beta, \gamma, \delta\) and \(\sigma\) are 1–forms. Chaubey proved in the same study that equation \((1.5)\) reduces to the form, [3]

\[
(1.6) \quad (\nabla_X W)(Y, Z, U, V) = \alpha(X) W(Y, Z, U, V) + \beta(Y) W(X, Z, U, V) + \beta(Z) W(Y, X, U, V) + \delta(U) W(Y, Z, X, V) + \delta(V) W(Y, Z, U, X)
\]

for all \(X, Y, Z, U, V \in \chi(M)\) and non-zero 1–forms \(\alpha, \beta, \delta\) called associated 1–forms.
In this study, we deal with both weakly $M$–projective symmetric Sasakian manifolds and weakly $M$–projective Ricci symmetric Sasakian manifolds. In Section 2 some preliminary results are reviewed. In Section 3 weakly $M$–projectively symmetric manifolds are defined and all the 1–forms of such a manifold are obtained. Moreover, weakly $M$–projective Ricci symmetric Sasakian manifolds are defined and all the 1–forms of such a manifold are determined. Then, weakly $M$–projective Ricci symmetric Sasakian manifolds having a cyclic parallel $M$–projective Ricci tensor $W^*$ are investigated.

2 Sasakian manifolds

In this section, some basic concepts of Sasakian manifolds which will be used through the paper are considered.

Let $(M^{2m+1}, g)$ be an almost contact metric manifold equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$. We have

\begin{align}
\phi^2(X) &= -X + \eta(X)\xi, \\
\phi \xi &= 0,
\end{align}

\begin{align}
\eta(\xi) &= 1, \\
g(X, \xi) &= \eta(X), \\
\eta(\phi X) &= 0,
\end{align}

\begin{align}
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y),
\end{align}

\begin{align}
g(\phi X, Y) &= -g(X, \phi Y),
\end{align}

\begin{align}
(\nabla_X \eta)(Y) &= g(\nabla_X \xi, Y),
\end{align}

where $\phi$ is a (1,1) tensor field, $\eta$ is a 1–form, $\xi$ is the corresponding vector field to the 1–form $\eta$ and $g$ is Riemannian metric.

An almost contact metric manifold $(M^n, g)$ is called a contact metric manifold if

\begin{align}
d\eta(X, Y) &= g(X, \phi Y),
\end{align}

for all $X, Y \in \chi(M)$. A contact metric manifold is said to be a $K$–contact manifold if the vector field $\xi$ is a Killing vector field. The basic property of a $K$-contact manifold is that,

\begin{align}
\nabla_X \xi = -\phi X.
\end{align}

It is said to be a Sasakian manifold if

\begin{align}
(\nabla_X \phi)(Y) &= g(X, Y)\xi - \eta(Y)X,
\end{align}

for all $X, Y \in \chi(M)$. It is well-known that Sasakian manifold is always $K$–contact Riemannian manifold.

The following relations also can be stated in a Sasakian manifold:

\begin{align}
(\nabla_X \eta)(Y) &= g(X, \phi Y),
\end{align}
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\[(2.10)\]
\[R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X,\]
\[(2.11)\]
\[S(X, \xi) = (n - 1)\eta(X),\]
\[(2.12)\]
\[R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi,\]
for all $X, Y \in \chi(M)$ (see, [2, 17, 18]).

3 Main Theorems

In this section, Weakly $M$–projective symmetric Sasakian manifolds and weakly $M$–projective Ricci symmetric Sasakian manifolds are examined as two subsections. Furthermore, weakly $M$–projective Ricci symmetric Sasakian manifolds having a cyclic parallel $M$–projective Ricci tensor $W^*$ is considered. The main theorems and corollaries about them are also specified.

3.1 Weakly $M$–projective symmetric Sasakian manifolds

Let us now define weakly $M$–projective symmetric Sasakian manifolds and obtain both all the associated 1-forms and the relation among them.

**Definition 3.1.** A Sasakian manifold $(M^n, g)$ ($n = 2m + 1$) is said to be weakly $M$–projective symmetric if its $M$–projective curvature tensor of type $(0, 4)$ is not identically equal to zero and satisfies equation (1.6).

Let $e_i$ ($1 \leq i \leq n$) be an orthonormal basis of the tangent space at each point of the manifold. Then, equation (1.1) becomes

\[(3.1)\]
\[W^*(Z, U) = \sum_{i=1}^{n} W(e_i, Z, U, e_i) = \frac{1}{2(n-1)} (nS(Z, U) - rg(Z, U)),\]

where $r$ is a scalar curvature of $(M^n, g)$. The tensor $W^*$ is called the $M$–projective Ricci tensor, which is a symmetric tensor of type $(0, 2)$.

Considering $Y = V = e_i$ into equation (1.6), using equations (1.1) and (3.1), and summing over $i$, $(1 \leq i \leq n)$, we obtain

\[(3.2)\]
\[n (\nabla_X S)(Z, U) - (\nabla_X r) g(Z, U) = \alpha(X) \{nS(Z, U) - rg(Z, U)\} + \beta(Z) \{nS(X, U) - rg(X, U)\} + \delta(U) \{nS(Z, X) - rg(Z, X)\} + 2(n - 1) \{\beta(R(X, Z)U + \delta(R(X, U)Z) - \{(\beta(Z) + \delta(X))S(Z, U) - S(X, U)\beta(Z) } - \delta(LU)g(Z, X) + (\beta(LX) + \delta(LX))g(Z, U) - g(X, U)\beta(LZ) - \delta(U)S(Z, X)\}.

Substituting $\xi$ for $X, Z$ and $U$ into equation (3.2) and then using properties (2.1), (2.2), (7), (11) and (12), we achieve

\[(3.3)\]
\[[r - n(n - 1)] (\alpha(\xi) + \beta(\xi) + \delta(\xi)) = dr(\xi).\]
Firstly, assumed that $r \neq n(n - 1)$. Then, we can rewrite equation (3.3) as follows
\begin{equation}
\alpha(\xi) + \beta(\xi) + \delta(\xi) = \frac{dr(\xi)}{r - n(n - 1)}.
\end{equation}

Hence this result proved above is put in the following lemma.

**Lemma 3.1.** In a weakly $M$–projective symmetric Sasakian manifold $(M^n, g)$, $r \neq n(n - 1)$, the associated 1–forms $\alpha$, $\beta$ and $\delta$ satisfy equation (3.4).

Now, considering $X = Z = \xi$ into equation (3.2) and using properties (2.1), (2.2), (2.4), (2.7), (2.9), (2.10), (2.11) and (2.12), we obtain
\begin{equation}
\delta(U) = \left( -(\alpha(\xi) + \beta(\xi)) + \frac{dr(\xi)}{r - n(n - 1)} \right) \eta(U).
\end{equation}

By using equations (3.4) and (3.5), we achieve
\begin{equation}
\delta(U) = \delta(\xi) \eta(U).
\end{equation}

Considering $X = U = \xi$ into equation (3.2) and using equations (2.1), (2.2), (2.4), (2.7), (2.9), (2.10), (2.11) and (2.12) provide
\begin{equation}
\beta(Z) = \left( -(\alpha(\xi) + \delta(\xi)) + \frac{dr(\xi)}{r - n(n - 1)} \right) \eta(Z).
\end{equation}

Hence, using equation (3.4) and the above equation, we obtain
\begin{equation}
\beta(Z) = \beta(\xi) \eta(Z).
\end{equation}

Similarly, substituting $\xi$ for $Z$ and $U$ into equation (3.2), we achieve
\begin{equation}
\alpha(X) = -(\beta(\xi) + \delta(\xi)) \eta(X) + \frac{dr(X)}{r - n(n - 1)}.
\end{equation}

From equations (3.4) and (3.9), we thus obtain
\begin{equation}
\alpha(X) = \left( \alpha(\xi) - \frac{dr(\xi)}{r - n(n - 1)} \right) \eta(X) + \frac{dr(X)}{r - n(n - 1)}.
\end{equation}

From equations (3.6), (3.8) and (3.10) the following expression is yielded
\begin{equation}
\alpha(X) + \beta(X) + \delta(X) = \frac{dr(X)}{r - n(n - 1)}
\end{equation}
for all $X \in \chi(M)$.

We can then state the following theorem.

**Theorem 3.2.** Let $(M^n, g)$ be a weakly $M$–projective symmetric Sasakian manifold, $r \neq n(n - 1)$. Then,
\begin{enumerate}
\item The associated 1–forms $\alpha$, $\beta$ and $\delta$ are expressed as equations (3.10), (3.8) and (3.6), respectively.
\end{enumerate}
2. The sum of the associated 1-forms $\alpha$, $\beta$ and $\delta$ is expressed as (3.11).

Again, considering $X = Z = \xi$, $X = U = \xi$, $Z = U = \xi$ into equation (3.2), respectively, and using properties (2.1), (2.2), (2.4), (2.7), (2.9), (2.10), (2.11) and (2.12), we obtain the following relations:

\begin{align}
(3.12) \quad & [r - n(n - 1)] \delta(U) = (-[\alpha(\xi) + \beta(\xi)] [r - n(n - 1)] + dr(\xi)) \eta(U), \\
(3.13) \quad & [r - n(n - 1)] \beta(Z) = (-[\alpha(\xi) + \delta(\xi)] [r - n(n - 1)] + dr(\xi)) \eta(Z), \\
(3.14) \quad & [r - n(n - 1)] \alpha(X) = -[r - n(n - 1)] [\beta(\xi) + \delta(\xi)] \eta(X) + dr(X).
\end{align}

Summing (3.12), (3.13) and (3.14) and using equation (3.3), we achieve

\begin{equation}
(3.15) \quad [r - n(n - 1)] [\alpha(X) + \beta(X) + \delta(X)] = dr(X).
\end{equation}

If the scalar curvature $r$ equals $n(n - 1)$ in equation (3.15), then the above equation is automatically satisfied.

We have thus proved the following

**Theorem 3.3.** There always exists a weakly $M$–projective symmetric Sasakian manifold whose scalar curvature $r$ equals $n(n - 1)$.

### 3.2 Weakly $M$–projective Ricci symmetric Sasakian manifolds

Let us define weakly M-projective Ricci symmetric Sasakian manifolds and give both all the associated 1-forms and the relation among them. Moreover, weakly $M$–projective Ricci symmetric Sasakian manifolds having a cyclic parallel $M$–projective Ricci tensor $W^*$ are investigated.

**Definition 3.2.** A Sasakian manifold $(M^n, g)$ $(n = 2m + 1)$ is said to be weakly $M$–projective Ricci symmetric if its $M$–projective Ricci tensor $W^*$ of type $(0, 2)$ is not identically equal to zero and satisfies the following condition

\begin{equation}
(3.16) \quad (\nabla_X W^*)(Z, U) = \rho(X)W^*(Z, U) + \mu(Z)W^*(X, U) + \nu(U)W^*(Z, X).
\end{equation}

Using equations (3.1) and (3.16), we obtain

\begin{equation}
(3.17) \quad n (\nabla_X S)(Z, U) - (\nabla_X r) g(Z, U) = \rho(X) \{nS(Z, U) - rg(Z, U)\} + \mu(Z) \{nS(X, U) - rg(X, U)\} + \nu(U) \{nS(Z, X) - rg(Z, X)\}.
\end{equation}

Substituting $\xi$ for $X$, $Z$ and $U$ into equation (3.17) and then using properties (2.1), (2.2), (2.7) and (2.11) yield

\begin{equation}
(3.18) \quad [r - n(n - 1)] [\rho(\xi) + \mu(\xi) + \nu(\xi)] = dr(\xi).
\end{equation}

Firstly, assumed that $r \neq n(n - 1)$. From equation (3.18), it follows that

\begin{equation}
(3.19) \quad \rho(\xi) + \mu(\xi) + \nu(\xi) = \frac{dr(\xi)}{r - n(n - 1)}.
\end{equation}

Then, the following lemma holds true:
Lemma 3.4. In a weakly $M$–projective Ricci symmetric Sasakian manifold $(M^n, g)$, $r \neq n(n - 1)$, the associated 1–forms $\alpha$, $\beta$, and $\delta$ satisfy equation (3.19).

Now, substituting $\xi$ for $X$ and $Z$ into equation (3.17), using properties (2.1), (2.2), (2.4), (2.7), (2.9), (2.10) and (2.11), we achieve

$$\nu(U) = \left( -(\rho(\xi) + \mu(\xi)) + \frac{dr(\xi)}{r - n(n - 1)} \right) \eta(U).$$

From equations (3.19) and (3.20), it follows that

$$\nu(U) = \nu(\xi)\eta(U).$$

Similarly, substituting $\xi$ for $X$ and $U$ into equation (3.17), we obtain

$$\mu(Z) = \left( -(\rho(\xi) + \nu(\xi)) + \frac{dr(\xi)}{r - n(n - 1)} \right) \eta(Z).$$

Using equation (3.19) and the above equation, we achieve

$$\mu(Z) = \mu(\xi)\eta(Z).$$

Finally, substituting $\xi$ for $Z$ and $U$ into equation (3.17) yields

$$\rho(X) = -(\mu(\xi) + \nu(\xi))\eta(X) + \frac{dr(X)}{r - n(n - 1)}.$$

It follows from equation (3.19) that

$$\rho(X) = \left( \rho(\xi) - \frac{dr(\xi)}{r - n(n - 1)} \right) \eta(X) + \frac{dr(X)}{r - n(n - 1)}.$$

From equations (3.19), (3.21), (3.23) and (3.25), we thus obtain

$$\rho(X) + \mu(X) + \nu(X) = \frac{dr(X)}{r - n(n - 1)}.$$

Hence the following theorem is attained:

Theorem 3.5. If a Sasakian manifold is weakly $M$–projective Ricci symmetric, $r \neq n(n - 1)$, then

1. The associated 1-forms $\rho$, $\mu$, and $\nu$ are expressed as equations (3.25), (3.23) and (3.21), respectively.

2. The sum of the associated 1-forms is expressed as equation (3.26).

By proceeding in a similar manner as previous subsection, we achieve

$$[r - n(n - 1)] [\rho(X) + \mu(X) + \nu(X)] = dr(X).$$

In equation (3.27), if the scalar curvature $r$ equals $r = n(n - 1)$, then it is evident that equation (3.27) holds.

Hence we can state the following:
Theorem 3.6. There always exists a weakly $M$–projective Ricci symmetric Sasakian manifold whose scalar curvature $r$ equals $n(n-1)$.

Let us now assume that the $M$–projective Ricci tensor $W^*$ is cyclic parallel. Then, considering equation (1.4), we obtain

$$(3.28) \quad (\nabla_X W^*)(Z, U) + (\nabla_Z W^*)(U, X) + (\nabla_U W^*)(X, Z) = 0.$$ 

From equations (3.16) and (3.28), it follows that

$$(3.29) \quad w(X)W^*(Z, U) + w(Z)W^*(X, U) + w(U)W^*(Z, X) = 0,$$

where

$$(3.30) \quad w(X) = \rho(X) + \mu(X) + \nu(X).$$

Based on Walker’s Lemma [23], it must be either $w(X) = 0$ or $W^*(Z, U) = 0$. By virtue of Definition 3.2, since $W^*(Z, U) \neq 0$, it must be $w(X) = 0$. From equation (3.30), we achieve

$$(3.31) \quad \rho(X) + \mu(X) + \nu(X) = 0,$$

for all $X \in \chi(M)$. Considering these results, we can specify the following corollary.

Corollary 3.7. There is no weakly $M$–projective Ricci symmetric Sasakian manifold $(M^n, g)$ whose $M$–projective Ricci tensor $W^*$ is cyclic parallel unless $\rho + \mu + \nu$ equals zero everywhere.

Further, it is the result of equation (3.26) that a weakly $M$–projective Ricci symmetric Sasakian manifold $(M^n, g)$ whose $M$–projective Ricci tensor $W^*$ is cyclic parallel, is of constant scalar curvature. Therefore, we can state this result in the following theorem.

Theorem 3.8. If a weakly $M$–projective Ricci symmetric Sasakian manifold $(M^n, g)$ has a cyclic parallel $M$–projective Ricci tensor $W^*$, then such a manifold is of constant scalar curvature.

References


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