Normalized null hypersurfaces in Lorentzian manifolds admitting conformal vector fields

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Abstract. In this paper, we study the geometry of null hypersurface in Lorentzian manifolds furnished with a conformal vector field W with special attention paid to conformally stationary spacetime. We prove that an Einstein null hypersurface in Lorentzian manifolds of quasi-constant curvature for which the closed conformal rigging vector field is a curvature vector field, is locally a product of null curves, Euclidean spaces, and spheres.

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1 Introduction

A submanifold in a semi-Riemannian manifold is null if the induced metric tensor is degenerate on it. Null hypersurfaces are particularly important cases of such objects due to their application in physics and especially in general relativity. For example, the Schwarzschild black hole horizon is a lightlike hypersurface (submanifold of codimension 1) of the first non-trivial solution of Einstein's equations, found by the astrophysicist Karl Schwarzschild at the end of 1915.

The main difference between lightlike and non-degenerate hypersurfaces arises due to the absence of natural projector on the former. This prevents inducing as usual geometric objects (linear connection, second fundamental form, Gauss, Codazzi and Ricci equations, etc) on lightlike hypersurfaces.

A vector field W on $\overline{M}$ is said to be conformal if

$$\mathcal{L}_W \overline{g} = 2\Psi \overline{g}, \tag{1.1}$$

for some function $\Psi \in C^\infty(\overline{M})$, where $\mathcal{L}$ stands for the Lie derivative of the Lorentz metric of $\overline{M}$. The function Ψ is called the conformal factor of W. In particular, W is Killing if and only if $\Psi = 0$. A spacetime $(\overline{M}, \overline{g})$ admitting a timelike Killing vector field is called stationary. As it is well-known, if a spacetime has a timelike conformal vector field, then it is globally conformal to a stationary spacetime. For this reason, such spacetimes are called conformally stationary (CS) spacetimes [2]. When Ψ is constant, we say that W is homothetic.

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The class of conformally stationary spacetimes includes the family of generalized Robertson-Walker spacetimes. Recently in [12], Gutiérrez.M and Olea.B studied hypersurfaces in Generalized Robertson-Walker spacetimes and they showed that a totally umbilic null hypersurface gives rise to a local decomposition of the fibre as a twisted product and viceversa, providing a deep insight of twisted decompositions in Lorentzian geometry.

B.Y.Chen and K.Yano[5] introduced the notion of a Riemannian manifold of quasi-constant curvature as a Riemannian manifold $(\overline{M}, \overline{g})$ endowed with the curvature tensor $\overline{R}$ satisfying the following equation:

$$\begin{aligned} \overline{g}(\overline{R}(X,Y)Z,T) &= \alpha\{\overline{g}(Y,Z)\overline{g}(X,T) - \overline{g}(X,Z)\overline{g}(Y,T)\} \\ &+ \beta\{\overline{g}(X,T)\theta(Y)\theta(Z) - \overline{g}(X,Z)\theta(Y)\theta(T) \\ &+ \overline{g}(Y,Z)\theta(X)\theta(T) - \overline{g}(Y,T)\theta(X)\theta(Z)\}, \end{aligned} \tag{1.2}$$

for any vector fields X, Y, Z and W of $\Gamma(T\overline{M})$, where α and β are smooth functions and

$$\theta(X) = \overline{g}(X, K), \tag{1.3}$$

is $\overline{g}-$dual to a non vanishing smooth unit vector field K called the curvature vector field of $\overline{M}$.

The classification of Einstein null hypersurfaces in semi-Riemannian manifolds was studied by many authors, see for example [13],[6],[7]. In [13], their main results focused on the geometry of Einstein lightlike hypersurfaces M of a Lorentzian space form $\overline{M}(\overline{k})$ of constant curvature $\overline{k}$, whose shape operator is quasi-conformal to the shape operator of its screen distribution. They proved a characterization theorem for screen quasi-Einstein null hypersurfaces of a Lorentzian space form (see [13, Theorem 5.8]).

The objective of this paper is to contribute to better understand the geometry of null hypersurfaces in Lorentzian manifold furnished with conformal vector fields. We extend the above just mentioned characterization theorem in Lorentzian manifolds furnished with closed conformal rigging vector field ζ(see section 3, Theorem 3.11). After some technical results, we prove the following main Theorem.

Theorem 1.1. *Let (M, ζ) be an Einstein normalized null hypersurface in a conformally stationary spacetime $(\overline{M}^{n+2}, \overline{g})$ of quasi-constant sectional curvature such that ζ is closed homothetic with homothetic factor Ψ. Suppose that in addition to the eigenfunction 0, $\overset{*}{A}_\xi$ has exactly two eigenfunctions $\overset{\star}{k}_1$ and $\overset{\star}{k}_2$, and that the curvature vector field K of $\overline{M}$ is pointwise transverse to M. Then M is locally isometric to $L \times M_{\overset{\star}{k}_1} \times M_{\overset{\star}{k}_2}$, with $M_{\overset{\star}{k}_1} \times M_{\overset{\star}{k}_2}$ either an open part of*

$$\mathbb{R}^l \times \mathbb{S}^{n-l}\Big(\sqrt{\frac{|\,\kappa\,|}{2(n-1)l}\frac{n-l-1}{|\,\Psi\,|}}\Big) \quad or \quad \mathbb{R}^{n-l} \times \mathbb{S}^{l}\Big(\sqrt{\frac{|\,\kappa\,|}{2(n-1)(n-l)}\frac{l-1}{|\,\Psi\,|}}\Big),$$

where $l \in \{1..., n,\}$ is such that $\iota_1 = ... = \iota_l = \overset{\star}{k}_1$, $\iota_{l+1} = ... = \iota_n = \overset{\star}{k}_2$, L is a null curve, and $\kappa = \frac{1}{2}\overline{g}(\zeta, \zeta)$.

2 Preliminaries

Let (M, g) be a null hypersurface in a conformally stationary spacetime $(\overline{M}^{n+2}, \overline{g})$, $(n \geq 1)$ i.e a hypersurface for which the induced metric tensor $g = \overline{g}_{|_M}$ is degenerate on it. A *screen distribution* on M, is a complementary bundle of $TM^\perp$ in TM. It is then a rank n non-degenerate distribution over M. From [4], it is known that for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, such that for any non-zero section ξ of $TM^\perp$ on a coordinate neighborhood $\mathscr{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathscr{U}$ satisfying

$$\overline{g}(N, \xi) = 1, \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \qquad \forall\, W \in \mathscr{S}(N)|_{\mathscr{U}}) \tag{2.1}$$

where $\mathscr{S}(N)$ denotes the fixed screen distribution.

Then $T\overline{M}$ admits the splitting:

$$T\overline{M}|_M = TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus \mathscr{S}(N). \tag{2.2}$$

We call $tr(TM)$ a *(null) transverse vector bundle* along M. [8] A rigging for M is a vector field ζ defined on some open set containing M such that $\zeta_p \notin T_pM$ for each $p \in M$.

Given a rigging ζ for M, let $\overline{\eta}$ denote the 1-form $\overline{g}$-metrically equivalent to ζ, i.e. $\overline{\eta} = \overline{g}(\zeta,\ .)$. Take $\eta = i^*\overline{\eta}$, being $i : M \hookrightarrow \overline{M}$ the canonical inclusion. Next, consider the tensors

$$\breve{g} = \overline{g} + \overline{\eta} \otimes \overline{\eta} \quad \text{and} \quad \widetilde{g} = i^\star \breve{g}. \tag{2.3}$$

It is easy to show that $\widetilde{g}$ defines a Riemannian metric on the (whole) hypersurface M. The *rigged vector field* of ζ is the $\widetilde{g}$-metrically equivalent vector field to the 1-form ω and it is denoted by ξ. In fact the rigged vector field ξ is the unique lightlike vector field in M such that $\overline{g}(\zeta, \xi) = 1$. Moreover, ξ is $\widetilde{g}$-unitary. To a rigging ζ for M is associated the screen distribution $\mathscr{S}(\zeta)$ given by $\mathscr{S}(\zeta) = TM \cap \zeta^\perp$. It is the $\widetilde{g}$-orthogonal subspace to ξ and the corresponding null transverse vector field on M is

$$N = \zeta - \frac{1}{2}\overline{g}(\zeta, \zeta)\xi. \tag{2.4}$$

A null hypersurface M equipped with a rigging ζ is said to be normalized and is denoted (M, ζ) (the latter is called a normalization of the null hypersurface). We say that ζ is a *null rigging* for M if the restriction of ζ to the null hypersurface M is a null vector field. The screen distribution $\mathscr{S}(\zeta) = \ker \omega$ is integrable whenever ω is closed, in particular if the rigging is closed. Throughout, the ambient Lorentzian metric $\overline{g}$ will also be denoted $\langle\ ,\ \rangle$.

The Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \qquad \overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.5}$$

$$\nabla_X PY = \overset{\star}{\nabla}_X PY + C(X, PY)\xi, \qquad \nabla_X \xi = -\overset{\star}{A}_\xi X - \tau(X)\xi, \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $(\overline{M}, \overline{g})$, ∇ denotes the connection on M induced from $\overline{\nabla}$ through the projection along the null transverse vector field N and $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $\mathscr{S}(\zeta)$ induced from ∇ through the projection morphism P of $\Gamma(TM)$ onto $\Gamma\big(\mathscr{S}(\zeta)\big)$ with respect to the decomposition (2.6). Now the $(0,2)$ tensors B and C are the second fundamental forms on TM and $\mathscr{S}(\zeta)$ respectively, A_N and $\overset{\star}{A}_\xi$ are the shape operators on TM with respect to the rigging ζ and the rigged vector field ξ respectively and τ a 1-form on TM defined by

$$\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi). \tag{2.7}$$

A null hypersurface $(M, g, S(TM))$ is said to be totally umbilic (resp. totally geodesic) if there exists a smooth function c on M such that at each $p \in M$ and for all $u, v \in T_pM$, $B(p)(u,v) = c(p)g(u,v)$ (resp. B^N vanishes identically on M).

The (non normalized) null mean curvature is $\mathbf{H}_\xi = tr(\overset{\star}{A}_\xi)$. Let denote by $\overline{R}$ and R the Riemannian curvature tensors of $\overline{\nabla}$ and ∇, respectively. Then the following are the Gauss-Codazzi equations [4].

$$\begin{aligned} \Big\langle \overline{R}(X,Y)Z, \xi \Big\rangle &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad + \tau^N(X)B(Y,Z) - \tau^N(Y)B(X,Z), \end{aligned} \tag{2.8}$$

$$\begin{aligned} \Big\langle \overline{R}(X,Y)Z, PW \Big\rangle &= \Big\langle R(X,Y)Z, PW \Big\rangle + B(X,Z)C^N(Y,PW) \\ &\quad - B(Y,Z)C^N(X,PW), \end{aligned} \tag{2.9}$$

$$\begin{aligned} \Big\langle \overline{R}(X,Y)\xi, N \Big\rangle &= \Big\langle R(X,Y)\xi, N \Big\rangle = C^N(Y,X) \\ &\quad - C^N(X,Y) \\ &\quad - 2d\tau^N(X,Y), \quad \forall X,Y,Z,W \in \Gamma(TM|_{\mathscr{U}}). \end{aligned} \tag{2.10}$$

$$\begin{aligned} \Big\langle \overline{R}(X,Y)PZ, N \Big\rangle &= \Big\langle (\nabla_X A_N)Y, PZ \Big\rangle - \Big\langle (\nabla_Y A_N)X, PZ \Big\rangle \\ &\quad + \tau^N(Y)\Big\langle A_N X, PZ \Big\rangle - \tau^N(X)\Big\langle A_N Y, PZ \Big\rangle. \end{aligned} \tag{2.11}$$

for every X, Y and Z in $\Gamma(TM)$.

Lemma 2.1. *For every $X, Y, Z \in \Gamma(TM)$ we have*

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = g\Big((\nabla_X \overset{*}{A}_\xi)Y, Z\Big) - g\Big((\nabla_Y \overset{*}{A}_\xi)X, Z\Big). \tag{2.12}$$

Proposition 2.2. *[3]Let $(\overline{M}(\overline{k}), \overline{g})$ be a semi-Riemannian manifold of constant curvature $\overline{k}$ and M a normalized null hypersurface of $\overline{M}$. For any $X, Y, Z \in \Gamma(TM)$, we have*

(a) $R(X,Y)Z = \overline{k}\{g(Y,Z)X - g(X,Z)Y\} - B(X,Z)A_N Y + B(Y,Z)A_N X.$

(b) $\Big(\nabla_X B\Big)(Y,Z) - \Big(\nabla_Y B\Big)(X,Z) = B(X,Z)\tau(Y) - B(Y,Z)\tau(X).$

(c) $\Big(\nabla_X A_N\Big)Y - \Big(\nabla_Y A_N\Big)X = \overline{k}\{\eta(X)Y - \eta(Y)X\} - \tau(X)A_N Y - \tau(Y)A_N X.$

(d) $\left(\nabla_X \overset{\star}{A}_\xi\right)Y - \left(\nabla_Y \overset{\star}{A}_\xi\right)X = \tau(Y)\overset{\star}{A}_\xi X - \tau(X)\overset{\star}{A}_\xi Y - 2d\tau(X,Y)\xi.$

(e) $\overset{\star}{R}(X,Y)Z = \overline{k}\{g(Y,Z)X - g(X,Z)Y\} - B(X,Z)A_N Y + B(Y,Z)A_N X + C(Y,PZ)\overset{\star}{A}_\xi X - C(X,PZ)\overset{\star}{A}_\xi Y.$

3 Null hypersurface in Lorentzian manifold admitting a conformal vector field

Throughout, W denotes a fixed globally defined conformal vector field in a conformally stationary spacetime $(\overline{M},\overline{g})$.

Let (M,ζ) be a null hypersurface of $\overline{M}$. The global vector field W has the following pointwise decomposition along M

$$W = W_{\mathscr{S}} + \kappa\xi + \varrho N, \tag{3.1}$$

where ϱ and κ are smooth functions on M, and $W_{\mathscr{S}} \in \Gamma(\mathscr{S}(\zeta))$.

By Koszul's formula, it holds

$$2g(\overline{\nabla}_X W, Y) = (L_W)g(X,Y) + d\alpha(X,Y), \quad X.Y \in \Gamma(T\overline{M}), \tag{3.2}$$

where α is the $\overline{g}-$dual to W. Define a skew symmetric tensor $\overline{\phi}$ of type $(1,1)$ on $\overline{M}$ by

$$d\alpha(X,Y) = 2\overline{g}(\overline{\phi}X,Y), \quad X,Y \in \Gamma(T\overline{M}), \tag{3.3}$$

where $d\alpha(X,Y) = X\cdot\alpha(Y) - Y\cdot\alpha(X) - \alpha([X,Y])$. Then using Eqs.(1.1)-(3.2), we immediately get

$$\overline{\nabla}_X W = \Psi X + \overline{\phi}X, \quad X \in \Gamma(T\overline{M}). \tag{3.4}$$

From (3.4), the following holds.

Proposition 3.1. *Let M be a normalized null hypersurface in Lorentzian manifold endowed with a conformal vector W with conformal factor Ψ. Then for all $X \in \mathfrak{X}(M)$,*

$$\begin{cases} \overset{\star}{\nabla}_X W_{\mathscr{S}} = \kappa\overset{\star}{A}_\xi X + \varrho A_N X + \Psi PX + P(\phi(X)), & (3.5)\\ X\cdot\kappa = \kappa\tau(X) - C(X,W_{\mathscr{S}}) + \Psi\eta(X) + \overline{g}(\phi X, N), & (3.6)\\ X\cdot\varrho = -\varrho\tau(X) - B(X,W_{\mathscr{S}}) + \overline{g}(\overline{\phi}X,\xi), & (3.7)\end{cases}$$

where $P:\Gamma(TM)\to\Gamma(\mathscr{S}(\zeta))$, and $\overline{P}:\Gamma(T\overline{M})\to\Gamma(TM)$ are the natural projections related to the decomposition (2.2)*, and $\phi = \overline{P}\circ\overline{\phi}$.*

Proof. Setting $\overline{\phi}X = \phi X + \overline{g}(\overline{\phi}X,\xi)N$, where $\phi X \in \Gamma(TM)$ for every $X \in \Gamma(TM)$. By straightforward computation, we have,

$$\begin{aligned} &\Psi PX + P(\phi X) + (\overline{g}(\phi X,N) + \Psi\eta(X))\xi + \overline{g}(\overline{\phi}X,\xi)N = \Psi X + \overline{\phi}(X) = \overline{\nabla}_X W\\ = \ &\overset{\star}{\nabla}_X W_{\mathscr{S}} - \overline{g}(W,N)\overset{\star}{A}_\xi X - \overline{g}(W,\xi)A_N X\\ &+\Big(C(X,W_{\mathscr{S}}) + X\cdot\overline{g}(W,N) - \overline{g}(W,N)\tau(X)\Big)\xi\\ &+\Big(X\cdot\overline{g}(W,\xi) + \overline{g}(W,\xi)\tau(X) + B(X,W_{\mathscr{S}})\Big)N. \end{aligned} \tag{3.8}$$

The result follows matching the tangential, radical and transverse parts of (3.8). □

From the above Proposition, the following lemma holds.

Lemma 3.2. *Let M be a normalized null hypersurface of Lorentzian manifold endowed with a conformal vector W with conformal factor Ψ. If τ vanishes along the screen, then κ and ϱ satisfy the following:*

$$\overset{*}{\nabla}\kappa = -A_N W_{\mathscr{S}} - P(\overline{P}(\overline{\phi}(N))), \text{ and } \overset{*}{\nabla}\varrho = -\overset{*}{A}_\xi W_{\mathscr{S}} - P(\phi(\xi))\xi. \tag{3.9}$$

Proof. From equation (3.6) and (3.7), we have for every $X \in \mathscr{S}(\zeta))$,

$$\begin{cases} X\cdot\kappa = -g(A_N W_{\mathscr{S}}, X) - \overline{g}(\overline{\phi}(N), X) \\ X\cdot\varrho = -g(\overset{*}{A}_\xi W_{\mathscr{S}}, X) - \overline{g}(\phi(\xi), X) \end{cases} \Rightarrow \begin{cases} g(\overset{*}{\nabla}\kappa + A_N W_{\mathscr{S}} - \overline{P}(\overline{\phi}(N)), X) = 0, \\ g(\overset{*}{\nabla}\varrho + \overset{*}{A}_\xi W_{\mathscr{S}} - P(\phi(\xi)), X) = 0. \end{cases} \tag{3.10}$$

The claim follows using the fact that g is Riemannian along the screen. □

Now, we deduce the following from(3.9).

Theorem 3.3. *Let (M,ζ) be a normalized null hypersurface of Lorentzian manifold $(\overline{M},\overline{g})$ endowed with a conformal vector field W with conformal factor Ψ. If $\tau = 0$, then we have the following.*

$$\begin{aligned} \Delta^\star\varrho &= -W_{\mathscr{S}}\cdot\mathbf{H}_\xi + \overline{R}ic(W_{\mathscr{S}},\xi) - \kappa tr(\overset{\star}{A}{}^2_\xi) - \varrho tr(A_N\circ\overset{\star}{A}_\xi) + \overline{g}\Big(\overset{\star}{A}_\xi W_{\mathscr{S}}, A_N\xi\Big) \\ &- n\mathbf{H}_\xi\Psi + div^{\overset{\star}{\nabla}}(P(\phi(\xi))). \end{aligned} \tag{3.11}$$

Moreover, using $W = \zeta$ as a rigging for M leads to

$$-\kappa tr(\overset{\star}{A}{}^2_\xi) - \varrho tr(A_N\circ\overset{\star}{A}_\xi) - n\mathbf{H}_\xi\Psi + div^{\overset{\star}{\nabla}}(P(\phi(\xi))) = 0, \tag{3.12}$$

$$2div^{\overset{\star}{\nabla}}(P(\phi(\xi))) = div^{\overset{\star}{\nabla}}(A_N\xi) = \xi\cdot\mathbf{H}_N - \overline{R}ic(\xi,N) + \overline{g}(\overline{R}(\xi,N)\xi,N), \tag{3.13}$$

where $\mathbf{H}_\xi$ (resp. $\mathbf{H}_N$) is the (non normalized) null mean curvature of (M,ζ)(resp. the trace of A_N).

Proof. $-div(\overset{\star}{\nabla}\varrho) = -\Delta^\star\varrho = div^{\overset{\star}{\nabla}}(\overset{\star}{A}_\xi W_{\mathscr{S}}) + div^{\overset{\star}{\nabla}}(P(\phi(\xi)))$. Since $\overset{\star}{A}_\xi$ is diagonalizable, from the quasi-orthonormal frame field $\{\overset{\star}{E}_0 = \xi, \overset{\star}{E}_1, ..., \overset{\star}{E}_n\}$, we have,

$$\begin{aligned} div^{\overset{\star}{\nabla}}(\overset{\star}{A}_\xi W_{\mathscr{S}}) &= \sum_{i=1}^n \overline{g}\Big(\overset{\star}{\nabla}_{\overset{\star}{E}_i}(\overset{\star}{A}_\xi W_{\mathscr{S}}), \overset{\star}{E}_i\Big) \\ &\overset{(2.8)}{=} \sum_{i=1}^n \overline{g}\Big((\overset{\star}{\nabla}_{\overset{\star}{E}_i}\overset{\star}{A}_\xi)W_{\mathscr{S}}, \overset{\star}{E}_i\Big) + \sum_{i=1}^n \overline{g}\Big(\overset{\star}{A}_\xi(\varrho A_N\overset{\star}{E}_i + \kappa\overset{\star}{A}_\xi\overset{\star}{E}_i + \Psi\overset{\star}{E}_i), \overset{\star}{E}_i\Big) \\ &= \sum_{i=1}^n \overline{g}\Big((\overset{\star}{\nabla}_{\overset{\star}{E}_i}\overset{\star}{A}_\xi)W_{\mathscr{S}} + C(\overset{\star}{\nabla}_{\overset{\star}{E}_i}\overset{\star}{A}_\xi, W_{\mathscr{S}})\xi, \overset{\star}{E}_i\Big) + \varrho tr(\overset{\star}{A}_\xi\circ A_N) \\ &\quad + \kappa tr(\overset{\star}{A}{}^2_\xi) + n\Psi\mathbf{H}_\xi \\ &= \sum_{i=1}^n \overline{g}\Big((\nabla_{\overset{\star}{E}_i}\overset{\star}{A}_\xi)W_{\mathscr{S}}, \overset{\star}{E}_i\Big) + \varrho tr(\overset{\star}{A}_\xi\circ A_N) + \kappa tr(\overset{\star}{A}{}^2_\xi) + n\Psi\mathbf{H}_\xi. \end{aligned}$$

Now, Gauss-Codazzi equations lead to

$$\begin{aligned}
-\overset{\star}{\Delta}\varrho &= \sum_{i=1}^{n}\overline{g}\Big((\nabla_{W_{\mathscr{S}}}A_N)\overset{\star}{E}_i,\overset{\star}{E}_i\Big)+\sum_{i=1}^{n}\overline{g}\Big(\overline{R}(\overset{\star}{E}_i,W_{\mathscr{S}})\overset{\star}{E}_i,N\Big)+\varrho tr(\overset{\star}{A}_\xi\circ A_N)\\
&\quad +\kappa tr(\overset{\star}{A}{}^2_\xi)+n\Psi\mathbf{H}_\xi\\
&= \sum_{i=1}^{n}\overline{g}\Big((\nabla_{W_{\mathscr{S}}}\overset{\star}{A}_\xi)\overset{\star}{E}_i,\overset{\star}{E}_i\Big)-\overline{R}ic(W_{\mathscr{S}},\xi)+\overline{g}(\overline{R}(\xi,W_{\mathscr{S}})\xi,N)+\overline{g}(\overline{R}(\xi,\xi)W_{\mathscr{S}},N)\\
&\quad +\varrho tr(\overset{\star}{A}_\xi\circ A_N)+\kappa tr(\overset{\star}{A}{}^2_\xi)+n\Psi\mathbf{H}_\xi\\
&= W_{\mathscr{S}}\cdot\mathbf{H}_\xi-\overline{R}ic(W_{\mathscr{S}},\xi)+\kappa tr(\overset{\star}{A}{}^2_\xi)+\varrho tr(A_N\circ\overset{\star}{A}_\xi)-\overline{g}\Big(\overset{\star}{A}_\xi W_{\mathscr{S}},A_N\xi\Big)+n\Psi\mathbf{H}_\xi.
\end{aligned}$$

Setting $W=\zeta$, it follows from (2.4) that $\varrho=1$, $\kappa=\frac{1}{2}\overline{g}(\zeta,\zeta)$, and we get for every $X\in\mathscr{S}(\zeta)$,

$$2\overline{g}(P(\phi(\xi)),X)=2\overline{g}(\phi(\xi),X)=d\alpha(\xi,X)=d\eta(\xi,X)=\overline{g}(A_N\xi,X). \tag{3.14}$$

That is $P(\phi(\xi))=\frac{1}{2}A_N\xi$. From this, we have

$$\begin{aligned}
div^{\overset{\star}{\nabla}}(A_N\xi) &= \sum_{i=1}^{n}\overline{g}\Big(\overset{\star}{\nabla}_{\overset{\star}{E}_i}(A_N\xi),\overset{\star}{E}_i\Big)\\
&= \sum_{i=1}^{n}\overline{g}\Big((\overset{\star}{\nabla}_{\overset{\star}{E}_i}A_N)\xi,\overset{\star}{E}_i\Big)\\
&= \sum_{i=1}^{n}\overline{g}\Big((\overset{\star}{\nabla}_{\overset{\star}{E}_i}A_N)\xi+C(\overset{\star}{\nabla}_{\overset{\star}{E}_i}A_N,\xi)\xi,\overset{\star}{E}_i\Big)\\
&= \sum_{i=1}^{n}\overline{g}\Big((\nabla_{\overset{\star}{E}_i}A_N)\xi,\overset{\star}{E}_i\Big).
\end{aligned}$$

Gauss-Codazzi equations leads to

$$\begin{aligned}
div^{\overset{\star}{\nabla}}(A_N\xi) &= \sum_{i=1}^{n}\overline{g}\Big((\nabla_\xi A_N)\overset{\star}{E}_i,\overset{\star}{E}_i\Big)+\sum_{i=1}^{n}\overline{g}\Big(\overline{R}(\overset{\star}{E}_i,\xi)\overset{\star}{E}_i,N\Big)\\
&= \sum_{i=1}^{n}\overline{g}\Big((\nabla_\xi A_N)\overset{\star}{E}_i,\overset{\star}{E}_i\Big)-\overline{R}ic(\xi,N)\\
&\quad +\overline{g}(\overline{R}(N,N)\xi,\xi)+\overline{g}(\overline{R}(\xi,N)\xi,N)\\
&= \xi\cdot\mathbf{H}_N-\overline{R}ic(\xi,N)+\overline{g}(\overline{R}(\xi,N)\xi,N).
\end{aligned} \tag{3.15}$$

□

Definition 3.1. [13] A normalized null hypersurface (M,ζ) of a semi-Riemannian manifold is locally screen quasi-conformal if the shape operators A_N and $\overset{\star}{A}_\xi$ of M and $\mathscr{S}(\zeta)$ satisfy

$$A_N=a\overset{\star}{A}_\xi+bP$$

in $\Gamma(TM)$ for some functions a and b. For $b=0$, M is (locally) screen conformal.

Theorem 3.4. *Let $(\overline{M}(\overline{k}), \overline{g})$ be a Lorentzian manifold of constant sectional curvature $\overline{k}$ and (M, ζ) a screen integrable normalized null hypersurface of $\overline{M}$ such that ζ is a conformal vector field with conformal factor Ψ. Then,*

(a) M is screen quasi-conformal along the screen.

(b) M admits an induced symmetric Ricci tensor Ric on $\mathscr{S}(\zeta)$, and $\forall X, Y \in \mathscr{S}(\zeta)$,

$$Ric(X,Y) = n\overline{k}g(X,Y) + B(X,Y)\Big(n\frac{\overline{g}(\zeta,\zeta)}{2}\mathbf{H}_\xi + (1-n)\Psi\Big) + \frac{\overline{g}(\zeta,\zeta)}{2}\overline{g}(\overset{\star}{A}{}_\xi^2 X, Y). \tag{3.16}$$

(c) $\forall X, Y \in \mathscr{S}(\zeta)$, the ricci tensor $\overset{\circ}{R}ic$ of $\overset{\circ}{M}$ is given by

$$\begin{aligned} \overset{\circ}{R}ic(X,Y) &= \overline{k}(n-1)g(X,Y) + g(\overset{\star}{A}_\xi X, Y)\Big((2-n)\Psi - \overline{g}(\zeta,\zeta)\mathbf{H}_\xi\Big) \\ &\quad -\Psi n \mathbf{H}_\xi g(X,Y) + \overline{g}(\zeta,\zeta)\overline{g}(\overset{\star}{A}{}_\xi^2 X, Y). \end{aligned} \tag{3.17}$$

(d) If M is totally umbilic, then

$$\begin{aligned} R(X,Y)Z &= \{\overline{k} - c(c\kappa + \Psi)\}\Big[g(Y,Z)X - g(X,Z)Y\Big] \\ \overset{\circ}{R}(X,Y)X &= \{\overline{k} - 2c(c\kappa + \Psi)\}\Big[g(Y,Z)X - g(X,Z)Y\Big] \forall X, Y \in \mathscr{S}(\zeta). \end{aligned}$$

Moreover, it has constant sectional curvature $\overline{k}$ and this is also the case for the screen foliation whenever M is totally geodesic.

Proof. (a) Being M screen integrable, from (3.3) is it easy to see that for all $X, Y \in \mathscr{S}(\zeta)$, $0 = d\overline{\alpha}(X,Y) = d\eta(X,Y) = 2\overline{g}(\overline{\phi}(X), Y) = 2\overline{g}(P(\phi(X)), Y) = 0$. That is $P(\phi(X)) = 0$ along the screen. As $W = \zeta$, then it follows from (2.4) that $\varrho = 1$, $\kappa = \frac{1}{2}\overline{g}(\zeta,\zeta)$. Using this together with the fact that $P(\phi(X) = 0$, (3.5) leads to

$$\frac{1}{2}\overline{g}(\zeta,\zeta)\overset{\star}{A}_\xi X + A_N X + \Psi PX = 0, \tag{3.18}$$

for every $X \in \mathscr{S}(\zeta)$. That is M is screen quasi-conformal along the screen.

(b) From (1.1), and by straightforward computation, we have for all $X, Y \in \Gamma(TM)$,

$$\begin{aligned} 2\Psi\overline{g}(X,Y) &= -\overline{g}(\zeta,\zeta)\overline{g}(\overset{\star}{A}_\xi X, Y) - \overline{g}(A_N X, Y) \\ &\quad -\overline{g}(X, A_N Y) + \tau(X)\eta(Y) + \tau(Y)\eta(X). \end{aligned} \tag{3.19}$$

Setting $Y := \overset{\star}{A}_\xi Y$ in (3.19),leads to

$$\begin{aligned} \overline{g}(A_N X, \overset{\star}{A}_\xi Y) &= -\overline{g}(\zeta,\zeta)\overline{g}(\overset{\star}{A}{}_\xi^2 X, Y) - \overline{g}(A_N PX, \overset{\star}{A}_\xi PY) \\ &\quad +\tau(X)\eta(Y) + \tau(Y)\eta(X) - 2\Psi B(X,Y). \end{aligned} \tag{3.20}$$

Thus, for $X, Y \in \mathscr{S}(\zeta)$, we get

$$\overline{g}(A_N X, Y) = -\frac{\overline{g}(\zeta,\zeta)}{2} B(X,Y) - \Psi \overline{g}(X,Y), \tag{3.21}$$

$$\overline{g}(A_N X, \overset{\star}{A}_\xi Y) = -\frac{\overline{g}(\zeta,\zeta)}{2} \overline{g}(\overset{\star}{A}{}^2_\xi X, Y) - \Psi B(X,Y) = \overline{g}(A_N Y, \overset{\star}{A}_\xi X). \tag{3.22}$$

The Ricci tensors of M and $\overline{M}$ are related by (see [14], p.69)

$$R(X,Y) = \overline{R}ic(X,Y) + nB(X,Y)\mathbf{H}_N - \overline{g}(A_N X, \overset{\star}{A}_\xi Y) - \overline{g}(R(\xi, Y)X, N), \tag{3.23}$$

for all $X, Y \in \Gamma(TM)$. Furthermore, since $\overline{M}$ has constant sectional curvature $\overline{k}$, we have

$$\overline{g}(R(\xi, Y)X, N) = \overline{g}(\overline{R}(\xi, Y)X, N) = \overline{k} g(X,Y). \tag{3.24}$$

Then using Eqs.(3.22)-(3.23), we immediately get for all $X, Y \in \mathscr{S}(\zeta)$,

$$R(X,Y) - R(Y,X) = \overline{g}(A_N Y, \overset{\star}{A}_\xi X) - \overline{g}(A_N X, \overset{\star}{A}_\xi Y) = 0. \tag{3.25}$$

Hence, Ric is symmetric along the screen. Now as $\overline{R}ic(X,Y) = \overline{k}(n+1)g(X,Y)$ for any $X, Y \in \Gamma(TM)$, it follows from Eq.(3.23) that the induced Ricci in a conformally stationary spacetime of constant curvature $\overline{k}$ satisfies

$$Ric(X,Y) = n\Big(\overline{k} g(X,Y) + B(X,Y)\mathbf{H}_N\Big) - g(A_N X, \overset{\star}{A}_\xi Y), \tag{3.26}$$

for all $X, Y \in \Gamma(TM)$. It is straightforward to see that

$$\mathbf{H}_N = -\frac{\overline{g}(\zeta,\zeta)}{2}\mathbf{H}_\xi - \frac{1}{n} div(\zeta) = -\frac{\overline{g}(\zeta,\zeta)}{2}\mathbf{H}_\xi - \Psi. \tag{3.27}$$

Hence, item (b) follows from Eqs.(3.26) and (3.27).

(c) Let $\overset{\circ}{R}$ the Riemannian curvature of a screen leaf $\overset{\circ}{M}$. By straightforward computation, we have

$$\begin{aligned} R(X,Y)Z &= \overset{\circ}{R}(X,Y)Z - C(Y,Z)\overset{\star}{A}_\xi X + C(X,Z)\overset{\star}{A}_\xi Y \\ &\quad + \Big[C(X, \overset{\star}{\nabla}_Y Z) + X\cdot C(Y,Z) - C(Y, \overset{\star}{\nabla}_X Z) \\ &\quad - Y\cdot C(X,Z) - C([X,Y],Z)\Big]\xi, \end{aligned} \tag{3.28}$$

for all $X, Y, Z \in \mathscr{S}(\zeta)$.

Thus, from Eqs.(2.5)-(3.28), we have

$$\begin{aligned} \overline{g}(\overset{\circ}{R}(X,Y)Z, W) &= \overline{g}(\overline{R}(X,Y)Z, W) + B(Y,Z)C(X,W) - B(X,Z)C(Y,W) \\ &\quad + C(Y,Z)B(X,W) - C(X,Z)B(Y,W). \end{aligned} \tag{3.29}$$

Now, from (3.29), the Ricci curvature $\overset{\circ}{Ric}$ of $\overset{\circ}{M}$ is given by

$$\begin{aligned} \overset{\circ}{Ric}(X,Y) &= \overline{k}(n-1)g(X,Y) + n\Big(B(X,Y)\mathbf{H}_N + g(A_N X,Y)\mathbf{H}_\xi\Big) \\ &\quad -g(\overset{\star}{A}_\xi Y, A_N X) - g(\overset{\star}{A}_\xi X, A_N Y). \end{aligned} \tag{3.30}$$

Hence, item (c) follows using Eqs.(3.22)-(3.21) in (3.30).

(d). Being M totally umbilic, by definition, there exist c such that $B(X,Y) = cg(X,Y)$, and from item (a) of Proposition 3.4, we have $A_N X = -\frac{\overline{g}(\zeta,\zeta)}{2}\overset{\star}{A}_\xi X - \Psi X = -(c\kappa+\Psi)X$. That is $\mathscr{S}(\zeta)$ is totally umbilic. Hence, Theorem 3.8 together with Eqs.(3.26)-(3.30) give immediately for every $X,Y \in \mathscr{S}(\zeta)$,

$$\begin{aligned} Ric(X,Y) &= \Big\{n\overline{k} - c(c\kappa+\Psi)(n-1)\Big\}g(X,Y), \\ \overset{\circ}{R}(X,Y) &= \Big\{\overline{k}(n-1) - 2c(c\kappa+\Psi)(n-1)\Big\}g(X,Y). \end{aligned}$$

Also items (a) and (e) of Proposition 2.2, together with Theorem (3.8), give:

$$\begin{aligned} R(X,Y)Z &= \{\overline{k} - c(c\kappa+\Psi)\}\Big[g(Y,Z)X - g(X,Z)Y\Big], \\ \overset{\circ}{R}(X,Y)X &= \{\overline{k} - 2c(c\kappa+\Psi)\}\Big[g(Y,Z)X - g(X,Z)Y\Big], \forall X,Y \in \mathscr{S}(\zeta). \end{aligned}$$

□

Suppose that $\overset{\circ}{M}$ is an Einstein manifold, then there exist a constant $\overset{\circ}{\Lambda}$ such that $\overset{\circ}{Ric}(X,Y) = \overset{\circ}{\Lambda} g(X,Y)$, for all $X,Y \in \mathscr{S}(\zeta)$. Hence, for all $X,Y \in \mathscr{S}(\zeta)$,

$$\begin{aligned} \overset{\circ}{\Lambda} g(X,Y) &= \overline{k}(n-1)g(X,Y) + g(\overset{\star}{A}_\xi X,Y)\Big((2-n)\Psi - \overline{g}(\zeta,\zeta)\mathbf{H}_\xi\Big) \\ &\quad -\Psi n\mathbf{H}_\xi g(X,Y) + \overline{g}(\zeta,\zeta)\overline{g}(\overset{\star}{A}{}^2_\xi X,Y). \end{aligned} \tag{3.31}$$

Setting $X = Y = \overset{\star}{E}_i$ in (3.31), it follows that the screen principal curvature, say $\overset{\star}{\gamma}_i$, is a solution of

$$y^2 + \Big((2-n)\frac{\Psi}{\overline{g}(\zeta,\zeta)} - n\mathbf{H}_\xi\Big)y + \frac{1}{\overline{g}(\zeta,\zeta)}\Big(\overline{k}(n-1) - \overset{\circ}{\Lambda} - \Psi n\mathbf{H}_\xi\Big) = 0. \tag{3.32}$$

The equation (3.32) has at most two distinct solutions which are real-valued functions on M. Assume there exists $l \in \{1...,n,\}$ such that $\iota_1 = ... = \iota_l = \overset{\star}{\gamma}_1$, and $\iota_{l+1} = ... = \iota_m = \overset{\star}{\gamma}_2$.

Thus, the following proposition holds.

Proposition 3.5. *Let $(\overline{M}(\overline{k}),\overline{g})$ be conformally stationary spacetime of constant sectional curvature $\overline{k}$, and (M,ζ) be a screen integrable normalized null hypersurface of $\overline{M}$ all of whose leaves are Einstein, and ζ a conformal timelike vector in $\overline{M}$ with* *conformal factor* Ψ*. Then* M *has at most two distinct screen principal curvatures* $\overset{\star}{\gamma}_1$*, and* $\overset{\star}{\gamma}_2$ *satisfying*

$$\overset{\star}{\gamma}_1 + \overset{\star}{\gamma}_2 = \Big((n-2)\frac{\Psi}{\overline{g}(\zeta,\zeta)} + n\mathbf{H}_\xi\Big), \tag{3.33}$$

$$\overset{\star}{\gamma}_1 \overset{\star}{\gamma}_2 = \frac{1}{\overline{g}(\zeta,\zeta)}\Big(\overline{k}(n-1) - \overset{\circ}{\Lambda} - \Psi n\mathbf{H}_\xi\Big). \tag{3.34}$$

A closed conformal vector field (say ζ) has the outstanding property that $\overline{\nabla}_X\zeta = \Psi X$ for all $X \in \Gamma(T\overline{M})$. If in addition it is timelike then it can act as a rigging field for any null hypersurface in a Lorentzian space. Since $\zeta = \frac{\overline{g}(\zeta,\zeta,)}{2}\xi + N$, the associated screen distribution $\mathscr{S}(\zeta)$ is integrable. Also the following is straightforward

$$\begin{cases} \frac{1}{2}\overline{g}(\zeta,\zeta)\overset{\star}{A}_\xi X + A_N X + \Psi P X = 0, & (3.35)\\ X \cdot (\frac{1}{2}\overline{g}(\zeta,\zeta)) = \Psi\eta(X), & (3.36)\\ \tau(X) = 0. & (3.37)\end{cases}$$

Now, let $\mathbb{I}$ be the second fundamental form of any leaf $\overset{\circ}{M}$, from eqs. (3.35)-(3.37), it is easy to see that

$$\mathbb{I}(X,Y) = C(X,Y)\xi + B(X,Y)N = B(X,Y)(N - \frac{\overline{g}(\zeta,\zeta)}{2}\xi) - \Psi\overline{g}(X,Y)\xi, \tag{3.38}$$

for every $X, Y \in \Gamma(TM)$. In particular, M is totally umibilic in $\overline{M}$ if and only if each leaf $\overset{\circ}{M}$ of $\mathscr{S}(\zeta)$ is a totally umbilic codimention 2 submanifold of $\overline{M}$. The transversal mean curvature vector field of a leaf $\overset{\circ}{M}$ is $\mathbf{H} = tr(A_N)\xi + tr(\overset{\star}{A}_\xi)N = \mathbf{H}_N\xi + \mathbf{H}_\xi N$.

Taking trace in (3.38), we get

$$\overline{g}(\mathbf{H},\mathbf{H}) = -\mathbf{H}_\xi\Big(\overline{g}(\zeta,\zeta)\mathbf{H}_\xi - 2\Psi\Big). \tag{3.39}$$

A codimension-two spacelike submanifold of a Lorentzian manifold is called a future Marginally Outer Trapped Submanifold (MOTS) if its mean curvature vector is lightlike or zero. The notion of a MOTS was introduced in general relativity to study spacetime singularities and black holes [9],[11].

From (3.39), the following holds.

Theorem 3.6. *Let* $(\overline{M},\overline{g})$ *be a conformally stationary Lorentzian manifold which admits a closed conformal timelike vector field* ζ *with conformal factor* Ψ*. Then, the intersection of a generic leaf of the integrable distribution* $\zeta^\perp$ *with any normalized null hypersurface without mimimal point* (M,ζ) *in* $(\overline{M},\overline{g})$ *is marginally outer trapped (codimension 2) surface if and only if* $\mathbf{H}_\xi = -\frac{2\Psi}{\overline{g}(\zeta,\zeta)}$.

We have also the following.

Theorem 3.7. *Let* $(\overline{M}(\overline{k}),\overline{g})$ *be a Lorentzian manifold of constant sectional curvature* $\overline{k}$ *and* (M,ζ) *be a normalized null hypersurface of* $\overline{M}$ *endowed with a closed conformal rigging vector field* ζ*, with conformal factor* Ψ*. Then, the trace* $\mathbf{H}_N$ *of* A_N *is constant along the* $\xi-$*orbits if and only if* $\overline{k} = 0$.

Proof. We have $A_N\xi = 0$ as ζ is closed and conformal. Together with (3.15), we have

$$\xi \cdot \mathbf{H}_N - \overline{R}ic(\xi, N) + \overline{g}(\overline{R}(\xi, N)\xi, N) = 0.$$

Being $\overline{M}$ of constant curvature $\overline{k}$, it is easy to see that

$$\begin{aligned} -\overline{R}ic(\xi, N) + \overline{g}(\overline{R}(\xi, N)\xi, N) &= -\sum_{i=1}^{n} \overline{g}(\overline{R}(\overset{\star}{E}_i, \xi)N, \overset{\star}{E}_i) - \overline{g}(R(\xi, \xi)N, N) \\ &= n\overline{k}. \end{aligned} \tag{3.40}$$

From this, the claim follows. □

Let's recall the following known results.

Theorem 3.8. *[10] Let (M, ζ) be a lightlike hypersurface of a semi-Riemannian space form $(\overline{M}(\overline{k}), \overline{g})$ such that $\mathscr{S}(\zeta)$ is totally umbilic. Then $C = 0$ or $B = 0$. Moreover*

(1) $C = 0$ implies $\mathscr{S}(\zeta)$ is totally geodesic and $\overline{k} = 0$.

(2) $B = 0$ implies that M is totally geodesic immersed in $\overline{M}(\overline{k})$ and the induced connection ∇ is metric.

Setting $\frac{1}{2}\overline{g}(\zeta, \zeta) = \kappa$. From (3.18), we easily get for all $Y \in \mathscr{S}(\zeta)$, and $Z \in \Gamma(TM)$,

$$C(Y, PZ) = -\kappa B(Y, Z) - \Psi\overline{g}(Y, Z)\ . \tag{3.41}$$

Differentiating (3.41) gives

$$(\nabla_X C)(Y, PZ) = -\kappa(\nabla_X B)(Y, PZ) - (X \cdot \kappa)B(Y, Z) - X \cdot \overline{g}(Y, Z). \tag{3.42}$$

If $\overline{M}$ is of constant curvature $\overline{k}$, then using (3.42) and (2.8) into (2.11), leads to

$$\begin{aligned} \overline{k}\{\overline{g}(Y, Z)\eta(X) - \overline{g}(X, Z)\eta(Y)\} &= -\kappa\Big((\nabla_X B)(Y, PZ) - (\nabla_Y B)(X, PZ)\Big) \\ &+ (Y \cdot \kappa)(B(X, Z) - (X \cdot \kappa)B(Y, Z) \\ &+ \kappa\Big(B(X, Z))\tau(Y) - B(Y, Z)\tau(X)\Big) \\ &+ (Y \cdot \Psi)(g(X, Z) - (X \cdot \Psi)g(Y, Z)) \\ &+ \Psi\Big(g(X, Z)\tau(Y) - g(Y, Z)\tau(X)\Big) \\ &= B(X, Z)(Y \cdot \kappa + 2\kappa\tau(Y)) - B(Y, Z)(X \cdot \kappa + 2\kappa\tau(X)) \\ &+ (Y \cdot \Psi)(g(X, Z) - (X \cdot \Psi)g(Y, Z)) \\ &+ \Psi\Big(g(X, Z)\tau(Y) - g(Y, Z)\tau(X)\Big). \end{aligned} \tag{3.43}$$

Replacing X by ξ in (3.43), we get

$$\overline{k}\overline{g}(Y, Z) = -B(Y, Z)(\xi \cdot \kappa + 2\kappa\tau(\xi)) - \xi \cdot \Psi g(Y, Z) - g(Y, Z)\Psi\tau(\xi). \tag{3.44}$$

From this we have the following proposition.

Proposition 3.9. *Let (M,ζ) be a screen integrable totally umbilic null hypersurface of a Lorentzian manifold $(\overline{M}(\overline{k}),\overline{g})$ of constant sectional curvature $\overline{k}$ furnished with a conformal rigging vector field ζ. i.e., $L_\zeta\overline{g}=2\Psi\overline{g}$. Then,*

$$\overline{k}+c(\xi\cdot\kappa)+\xi\cdot\Psi=0.$$

Proof. For a conformal rigging, $\tau(\xi)=0$. The claim follows using this in (3.44) together with the fact that M is totally umbilic. □

Replacing X by ξ in (3.43), and using the fact that $\tau(X)=0$ when ζ is closed conformal, we get

$$\overline{k}\overline{g}(Y,Z)=(\xi.\kappa)B(Y,Z)+(\xi\cdot\Psi)g(Y,Z). \tag{3.45}$$

Thus, we have the following classification theorem.

Theorem 3.10. *Let (M,ζ) be a normalized null hypersurfaces in a Lorentzian manifols $(\overline{M}(\overline{k}),\overline{g})$ of constant curvature $\overline{k}$ furnished with a closed conformal vector field ζ with conformal factor Ψ.*

(a) If $\Psi=0$(i.e ζ is a Killing field) then $\overline{k}=0$.

(b) If $\Psi\neq 0$, then, either $\Psi-\frac{\xi\cdot\Psi}{\Psi}\kappa=0$ and $\overline{k}=0$ or $\overline{k}=\xi\cdot\Psi$.

Proof. Differentiating $2\kappa=\overline{g}(\zeta,\zeta)$ with respect to ξ, we have $\xi\cdot\kappa=\Psi$. Then, if $\Psi=\xi\cdot\kappa=0$, from (3.45), it is immediate that $\overline{k}=0$. Now assume that $\Psi\neq 0$. From (3.45) and (3.41) we get

$$B(X,Y)=\frac{\overline{k}-\xi\cdot\Psi}{\xi\cdot\kappa}g(X,Y) \text{ and } C(X,Y)=-\Big[\frac{\overline{k}-\xi\cdot\Psi}{\xi\cdot\kappa}\kappa+\Psi\Big]g(X.Y), \tag{3.46}$$

which means that both $\mathscr{S}(\zeta)$ and M are totally umbilic. Applying Theorem 3.8, we get $B=0$ or $C=0$. We distinguish the two cases $C=0$ and $C\neq 0$. For $C=0$ item (1) in Theorem 3.8 asserts that $\overline{k}=0$ and $\Psi-\frac{\xi\cdot\Psi}{\Psi}\kappa=0$. If $C\neq 0$ then necessarily $B=0$ and we have $\overline{k}=\xi\cdot\Psi$. □

Now let suppose the null hypersurface (M^{n+1},g) Einstein's, and ζ closed homothetic. By (3.18), M is screen quasi-conformal with quasi-conformal adapted pair $(-\frac{1}{2}\overline{g}(\zeta,\zeta),-\Psi)$. Hence by [13, Theorem 5.7], (M,ζ) is null screen isoparametric with at most two distinct screen principal curvatures $\overset{\star}{k}_1$ and $\overset{\star}{k}_2$ satisfying

$$\overset{\star}{k}_1+\overset{\star}{k}_2 = (n-1)\frac{2\Psi}{\overline{g}(\zeta,\zeta)}+n\mathbf{H}_\xi, \tag{3.47}$$

$$\overset{\star}{k}_1\overset{\star}{k}_2 = \frac{2}{\overline{g}(\zeta,\zeta)}(n\overline{k}-\Lambda). \tag{3.48}$$

Assume there exists $l\in\{1,...,n,\}$ such that $\iota_1=...=\iota_l=\overset{\star}{k}_1$, and

$\iota_{l+1}=...=\iota_m=\overset{\star}{k}_2$.

Hence, $n\mathbf{H}_\xi=l\overset{\star}{k}_1+(n-l)\overset{\star}{k}_2$, then from (3.47), we have

$$(l-1)\overset{\star}{k}_1+(n-l-1)\overset{\star}{k}_2=(1-n)\frac{\Psi}{\kappa}. \tag{3.49}$$

Setting $X=W=\overset{\star}{E}_i$, and $Y=Z=\overset{\star}{E}_j$ in (3.29), we have

$$\begin{aligned} g(\overset{\circ}{R}(\overset{\star}{E}_i,\overset{\star}{E}_j)\overset{\star}{E}_j,\overset{\star}{E}_i) &= \overline{k}+\overset{\star}{k}_j\, g(A_N\overset{\star}{E}_i,\overset{\star}{E}_i)+\overset{\star}{k}_i\, g(A_N\overset{\star}{E}_j,\overset{\star}{E}_j)\\ &\overset{(3.21)}{=} \overline{k}-\overset{\star}{k}_j\left(\frac{\overline{g}(\zeta,\zeta)}{2}\overset{\star}{k}_i+\Psi\right)-\overset{\star}{k}_i\left(\overline{g}(\zeta,\zeta)\overset{\star}{k}_j+\Psi\right)\\ &= \overline{k}-\overline{g}(\zeta,\zeta)\overset{\star}{k}_i\overset{\star}{k}_j-\Psi(\overset{\star}{k}_i+\overset{\star}{k}_j), \end{aligned} \tag{3.50}$$

for all $1\leq (i,j)\leq n$. Thus, from (3.50), the sectional curvature $\overset{\star}{c}$ of any leaf $\overset{\circ}{M}$ is given by

$$\overset{\star}{c}(\overset{\star}{E}_i,\overset{\star}{E}_j)=\frac{g(\overset{\circ}{R}(\overset{\star}{E}_i,\overset{\star}{E}_j)\overset{\star}{E}_j,\overset{\star}{E}_i)}{g(\overset{\star}{E}_i,\overset{\star}{E}_i)g(\overset{\star}{E}_j,\overset{\star}{E}_j)-g^2(\overset{\star}{E}_i,\overset{\star}{E}_j)}=\overline{k}-\overline{g}(\zeta,\zeta)\overset{\star}{k}_i\overset{\star}{k}_j-\Psi(\overset{\star}{k}_i+\overset{\star}{k}_j). \tag{3.51}$$

Hence

$$\begin{aligned} \overset{\star}{c}_1 &= \overline{k}-\overline{g}(\zeta,\zeta)\overset{\star}{k}{}_1^2-2\Psi\overset{\star}{k}_1=\overline{k}-2(\kappa\overset{\star}{k}{}_1^2-\Psi\overset{\star}{k}_1),\\ \overset{\star}{c}_2 &= \overline{k}-\overline{g}(\zeta,\zeta)\overset{\star}{k}{}_2^2-2\Psi\overset{\star}{k}_2=\overline{k}-2(\kappa\overset{\star}{k}{}_2^2-\Psi\overset{\star}{k}_2). \end{aligned} \tag{3.52}$$

If $\overset{\star}{A}_\xi$ has exactly two distinct screen principal curvatures $\overset{\star}{k}_1$ and $\overset{\star}{k}_2$, then by [13, Theorem 5.8], (M,ζ) is locally diffeomorphic to a product $M=L\times M_{\overset{\star}{k}_1}\times M_{\overset{\star}{k}_2}$, where L is a null geodesic, and $M_{\overset{\star}{k}_1}$ and $M_{\overset{\star}{k}_2}$ are two Riemannian manifolds.

As ζ is homothetic, $\overset{\star}{c}_i$ $(i=1,2)$ are consatant along $\mathscr{S}(\zeta)$. Now, we assert that $\overline{k}=0$.

Indeed, $0=\overline{R}(\xi,N)\zeta=\overline{k}(\overline{g}(N,\zeta)\xi-\overline{g}(\xi,\zeta)N)=\overline{k}(\frac{1}{2}\kappa\xi-N)$, that is $\overline{k}=0$. Therefore, by [13, Corollary 4.6] we know that one of the screen principal curvature is zero. This together with equation (3.49) and (3.52) give either

$$\Big[\overset{\star}{k}_1=0,\ \ \overset{\star}{c}_1=0\ ,\overset{\star}{k}_2=\frac{n-1}{(n-l-1)}\frac{\Psi}{\mid\kappa\mid},\ \ \overset{\star}{c}_2=\Big(\sqrt{\frac{2(n-1)l}{\mid\kappa\mid}}\frac{\Psi}{(n-l-1)}\Big)^2\Big]$$

or

$$\Big[\overset{\star}{k}_2=0,\ \ \overset{\star}{c}_2=0\ ,\overset{\star}{k}_1=\frac{n-1}{(l-1)}\frac{\Psi}{\mid\kappa\mid},\ \ \overset{\star}{c}_1=\Big(\sqrt{\frac{2(n-1)(n-l)}{\mid\kappa\mid}}\frac{\Psi}{(l-1)}\Big)^2\Big].$$

Also, using the fact that $\overset{\star}{k}_1$ and $\overset{\star}{k}_2$ are constant along the screen, then it is easy to show that $M_{\overset{\star}{k}_1}$ and $M_{\overset{\star}{k}_2}$ are totally geodesic submanifolds of $\overset{\circ}{M}$, and totally umbilic submanifolds of $\overline{M}$ of constant sectional curvature. Hence, as either $\overset{\star}{c}_2>0$ or $\overset{\star}{c}_1>0$, then by [1, Proposition 3.1], $M_{\overset{\star}{k}_1}\times M_{\overset{\star}{k}_2}$ is either an open part of

$$\mathbb{R}^l\times\mathbb{S}^{n-l}\Big(\sqrt{\frac{\mid\kappa\mid}{2(n-1)l}}\frac{n-l-1}{\mid\Psi\mid}\Big)\text{or }\ \mathbb{R}^{n-l}\times\mathbb{S}^{l}\Big(\sqrt{\frac{\mid\kappa\mid}{2(n-1)(n-l)}}\frac{l-1}{\mid\Psi\mid}\Big).$$

From this, the following Theorem holds.

Theorem 3.11. *Let (M,ζ) be an Einstein normalized null hypersurface in a conformally stationary spacetime $(\overline{M}^{n+2}(\overline{k}),\overline{g})$, $n\geq 2$, of constant sectional curvature $\overline{k}$ such that ζ is closed homothetic with non vanishing homothetic factor Ψ. Suppose that in addition to the eigenfunction 0, $\overset{*}{A}_\xi$ has exactly two eigenfunctions $\overset{\star}{k}_1$ and $\overset{\star}{k}_2$. Then M is locally isometric to $L\times M_{\overset{\star}{k}_1}\times M_{\overset{\star}{k}_2}$, with $M_{\overset{\star}{k}_1}\times M_{\overset{\star}{k}_2}$ either an open part of*

$$\mathbb{R}^l\times\mathbb{S}^{n-l}\Big(\sqrt{\frac{|\kappa|}{2(n-1)l}}\frac{n-l-1}{|\Psi|}\Big)\quad or\quad \mathbb{R}^{n-l}\times\mathbb{S}^{l}\Big(\sqrt{\frac{|\kappa|}{2(n-1)(n-l)}}\frac{l-1}{|\Psi|}\Big),$$

where $l\in\{1...,n,\}$ is such that $\iota_1=...=\iota_l=\overset{\star}{k}_1$, $\iota_{l+1}=...=\iota_n=\overset{\star}{k}_2$, L a null curve, and $\kappa=\frac{1}{2}\overline{g}(\zeta,\zeta)$.

3.1 Proof of Theorem 1.1

Proof. Take $\zeta=K$ to be a rigging vector field. Being $\overline{M}$ of quasi-constant curvature, replacing T by N in (1.2) together with the fact that $\theta(N)=\frac{1}{2}\overline{g}(\zeta,\zeta)$, we have that

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z,N) &= \alpha\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\}\\ &+ \beta\{\eta(Y)\eta(Z)\eta(X)-\overline{g}(X,Z)\eta(Y)(\frac{1}{2}\overline{g}(\zeta,\zeta))\\ &\quad +\overline{g}(Y,Z)\eta(X)(\frac{1}{2}\overline{g}(\zeta,\zeta))-\eta(Y)\eta(X)\eta(Z)\}\\ &= \alpha\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\}\\ &+ \frac{1}{2}\overline{g}(\zeta,\zeta)\beta\{\overline{g}(Y,Z)\eta(X)-\overline{g}(X,Z)\eta(Y)\}.\end{aligned}\tag{3.53}$$

Replace X by ξ in (3.53) and use the fact that $\eta(\xi)=1$, to get

$$\overline{g}(\overline{R}(\xi,Y)Z,N) = (\alpha+\beta(\frac{1}{2}\overline{g}(\zeta,\zeta)))g(Y,Z).\tag{3.54}$$

Also, replace T by ξ in (1.2) and use (2.8), together with the fact that $\theta(\xi)=\eta(\xi)=1$, to get

$$(\nabla_X B)(Y,Z)-(\nabla_Y B)(X,Z)=\beta\{\overline{g}(Y,Z)\eta(X)-\overline{g}(X,Z)\eta(Y)\}.\tag{3.55}$$

As $\tau=0$, replacing T by ξ to (1.2) and using Eqs.(3.55)-(2.8)-(2.11), together with the fact $A_NX=-\frac{1}{2}\overline{g}(\zeta,\zeta)\overset{\star}{A}_\xi X-\Psi PX$, we have

$$\begin{aligned}\beta\{\overline{g}(Y,Z)\eta(X)-\overline{g}(X,Z)\eta(Y)\} &= (\nabla_X B)(Y,Z)-(\nabla_Y B)(X,Z)\\ &= \frac{2}{\overline{g}(\zeta,\zeta)}\{(\nabla_Y C)(X,Z)-(\nabla_X C)(Y,Z)\}\\ &= -\frac{2}{\overline{g}(\zeta,\zeta)}\overline{g}(R(X,Y)Z,N).\end{aligned}\tag{3.56}$$

Finally, use the substitution $X:=\xi$ in (3.56), and obtain

$$\overline{g}(R(\xi,Y)Z,N)=-\frac{\overline{g}(\zeta,\zeta)}{2}\beta\{\overline{g}(Y,Z)\}.\tag{3.57}$$

Comparing (3.54) with (3.57), we have that $\alpha = -\beta\overline{g}(\zeta,\zeta)$.

As ζ is homothetic, replacing $X = \xi$, $Y = T = N$, $\zeta = Z$ to (1.2), we have

$$
\begin{aligned}
(3.58)\quad 0 = \overline{g}(\overline{R}(\xi,N)\zeta,N) &= \alpha\{\overline{g}(N,\zeta)\overline{g}(\xi,N) - \overline{g}(\xi,\zeta)\overline{g}(N,N)\}\\
&+ \beta\{\overline{g}(\xi,N)\theta(N)\theta(\zeta) - \overline{g}(\xi,\zeta)\theta(N)\theta(N)\\
&+ \overline{g}(N,\zeta)\theta(\xi)\theta(N) - \overline{g}(N,N)\theta(\xi)\theta(\zeta)\}\\
&= \alpha\frac{\overline{g}(\zeta,\zeta)}{2} + \beta\{\frac{\overline{g}(\zeta,\zeta)^2}{2} - \frac{\overline{g}(\zeta,\zeta)^3}{4} + \frac{\overline{g}(\zeta,\zeta)^2}{4}\}\\
(3.59)\quad &= \beta\{-\frac{\overline{g}(\zeta,\zeta)^3}{4} + \frac{\overline{g}(\zeta,\zeta)^2}{4}\}.
\end{aligned}
$$

That is $\beta = 0$ and consequently $\alpha = 0$. Hence $\overline{M}$ is flat along the null hypersurface. Therefore proceeding as in the proof of Theorem 3.11 the claim follows. Indeed, the proof of Theorem 3.11 still true if the curvature vector field of $\overline{M}$ is null just along the null hypersurface. □

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