Study of P-curvature tensor and other related tensors

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Abstract. In this paper the relationships between W_2, P and other related tensors have been obtained and corresponding propositions are made. Further, the condition for P-curvature tensor to satisfy the Bianchi differential identity has been established.

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1 Introduction

The W_2 curvature tensor defined by [4] has been widely studied in differential geometry as well as in the space time of general relativity. [3] have studied it in P-Sasakian manifold; [5] studied it for Sasakian manifold. [7] have introduced the notion of weekly W_2-symmetric manifolds and studied their properties. [10] have studied this tensor in Kernmotsu manifolds, while [8] considered $N(k)-$quasi Einstein manifolds satisfying the conditions$R(\xi, X).W_2$ =0. Further [9] have studied Lorentzian Para-Sasakian manifold satisfying some conditions on W_2-curvature tensor. [1] have studied space times satisfying Einstein field equations with vanishing of W_2-curvature as well as existence of killing and conformal killing vector fields. Further, the vanishing and divergence of W_2-tensor have also been studied in perfect fluid space-times. The P-curvature tensor has been defined by breaking the W_2-curvature tensor in skew -symmetric part and some of its properties have been studied [4]. Further, W_2-curvature tensor was shown to extend Pirani formulation of gravitational waves to Einstein space ([6]). Consider an n-dimensional space V_n in which the tensors:

(1.1) $$C(X,Y,Z,T) = R(X,Y,Z,T) - (R/n(n-1))[g(X,T)g(Y,Z) - g(Y,T)g(X,Z)]$$

(1.2) $$\begin{aligned} L(X,Y,Z,T) =& R(X,Y,Z,T) - (1/n-2)[g(Y,Z)Ric(X,T) - g(X,Z)Ric(Y,T) + \\ & g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z)] \end{aligned}$$

(1.3)
$$\begin{aligned} V(X,Y,Z,T) =& R(X,Y,Z,T) - (1/n-2)[g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z) + \\ & g(Y,Z)Ric(X,T) - g(X,Z)Ric(Y,T)] \\ & + R/(n-1)(n-2)[g(X,T)g(Y,Z) - g(Y,T)g(X,Z)] \end{aligned}$$

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are known as concircular curvature tensor, conharmonic curvature tensor and conformal curvature tensor respectively. These tensors satisfy the symmetric and skew-symmetric as well as the cyclic properties possessed by the Riemann curvature tensor $R(X,Y,Z,T)$. The projective curvature tensor is given by:

$$W(X,Y,Z,T) = R(X,Y,Z,T) - (1/n-1)[g(X,T)Ric(Y,Z) - g(X,Z)Ric(Y,T)]. \tag{1.4}$$

It is seen that $W(X,Y,Z,T) = -W(X,Y,T,Z) but W(X,Y,Z,T) \neq -W(Y,X,Z,T)$. Further the tensor $W(X,Y,Z,T)$ satisfies only the following cyclic property:

$$W(X,Y,Z,T) + W(X,Z,T,Y) + W(X,T,Y,Z) = 0. \tag{1.5}$$

The W_2 - curvature tensor is defined as [4]:

$$W_2(X,Y,Z,T) = R(X,Y,Z,T) - (1/n-1)[g(Y,Z)Ric(X,T) - g(X,Z)Ric(Y,T)]. \tag{1.6}$$

It is seen that $W_2(X,Y,Z,T) = -W_2(Y,X,Z,T)$ but $W_2(X,Y,Z,T) \neq W_2(X,Y,T,Z)$. Further the tensor $W_2(X,Y,Z,T)$ satisfies only the following cyclic property:

$$W_2(X,Y,Z,T) + W_2(Y,Z,X,T) + W_2(Z,X,Y,T) = 0. \tag{1.7}$$

The tensor $W_7(X,Y,Z,T)$ is defined as [6]:

(1.8)

$$W_7(X,Y,Z,T) = R(X,Y,Z,T) - (1/n-1)[g(X,T)Ric(Y,Z) - g(Y,Z)Ric(X,T)].$$

It is seen that $W_7(X,Y,Z,T) \neq -W_7(Y,X,Z,T)$and $W_7(X,Y,Z,T) \neq W_7(X,Y,T,Z)$. Further none of the cyclic property is satisfied.

$$W_7(X,Y,Z,T) + W_7(X,Z,T,Y) + W_7(X,T,Y,Z) \neq 0$$

and

$$W_7(X,Y,Z,T) + W_7(Y,Z,X,T) + W_7(Z,X,Y,T) \neq 0.$$

From equations (1.1) to (1.8) it is seen that for an empty gravitational field characterized by $Ric(X,Y) = 0$, these six fourth rank tensors are identical. In the space V_n following relationships exist between these tensors.

$$V(X,Y,Z,T) - L(X.Y.Z.T) = (n/n-2)[R(X,Y,Z,T) - C(X,Y,Z,T)], \tag{1.9a}$$

which in V_4 reduces to:

$$V(X,Y,Z,T) - L(X.Y.Z.T) = 2[R(X,Y,Z,T) - C(X,Y,Z,T)]. \tag{1.9b}$$

$$W_2(X,Y,Z,T) + W_7(X,Y,Z,T) - W(X,Y,Z,T) = R(X,Y,Z,T). \tag{1.10}$$

From the properties of these tensors, we have the following proposition.

Proposition 1.1. *Curvature tensors having (skew) symmetric properties are the only ones that satisfy the cyclic properties.*

2 The W_2-curvature tensor

The Bianchi differential identity is given by:

$$(\nabla_u R)(X,Y,Z,T) + (\nabla_Z R)(X,Y,T,U) + (\nabla_T R)(X,Y,U,Z) = 0. \tag{2.1}$$

If Ricci tensore R(X,Y) is of Codazzi type, then by [2]

$$(\nabla_x Ric)(Y,Z) = (\nabla_y Ric)(X,Z) = (\nabla_z Ric)(X,Y). \tag{2.2}$$

Using (2.2) for W_2-curvature tensor in V_4, it was found [1]

$$\begin{aligned}(\nabla_x W_2)(Y,Z,T,U)+&(\nabla_y W_2)(Z,X,T,U) + (\nabla_z W_2)(X,Y,T,U) = \\ &(1/3)[g(Y,T)(\nabla_x Ric)(Y,Z) - g(Z,T)(\nabla_x Ric)(Y,U)+ \\ &g(Z,T)(\nabla_y Ric)(X,U) - g(X,T)(\nabla_y Ric)(Z,U)+ \\ &g(X,T)(\nabla_z Ric)(Y,U) - g(Y,T)(\nabla_z Ric)(X,U)] = 0.\end{aligned} \tag{2.3}$$

Hence, W_2-curvature tensor satisfies Bianchi type differential identity. Conversely, starting with equation (2.3) they found equation (2.2). Contracting $W_2(X,Y,Z,T)$, it was found that ([4]):

$$W_2(X,Y) = (n/n-1)[Ric(X,Y) - (R/n)g(X,Y)], \tag{2.4}$$

which vanishes in the Einstein space. Further, the scalar in variant W_2 was found to be identically equal to zero. Thus, we have the following proposition.

Proposition 2.1. *W_2-curvature tensor satisfies Bianchi differential identity if and only if Ricci tensor is of Codazzi type and on contraction W_2-curvature tensor vanishes in an Einstein space with scalar invariant W_2 being identically equal to zero.*

Note: The $W_7(X,Y,Z,T)$ curvature tensor in V_4 on contraction gives:

$$W_7(X,Y) = (2/3)[Ric(X,Y) + (R/2)g(X,Y)].$$

Hence $W_7(X,Y)$ does not vanish in the Einstein space.

3 The P-curvature tensor

Breaking W_2- curvature tensor in skew-symmetric parts in Z,T, the P-curvature tensor has been defined [4] as :

$$\begin{aligned}P(X,Y,Z,T) =&(1/2)[W_2(X,Y,Z,T)W_2(X,Y,T,Z)] = R(X,Y,Z,T)- \\ &1/2(n-1)[g(Y,Z)Ric(X,T) - g(X,Z)Ric(Y,T)+ \\ &g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z)].\end{aligned} \tag{3.1}$$

Using (1.3) and (3.1), we get

$$\begin{aligned}P(X,Y,Z,T) =&n/2(n-1)R(X,Y,Z,T) + (n-2)/2(n-1)V(X,Y,Z,T)- \\ &R/(n-1)[g(X,T)g(Y,Z) - (Y,T)g(X,Z)].\end{aligned} \tag{3.2}$$

Using (1.2) and (3.1), we get

(3.3) $P(X, Y, Z, T) = n/(2(n-1)R(X, Y, Z, T) + (n-2)/2(n-1)L(X, Y, Z, T).$

For an electromagnetic field (or more generally in the case of space with vanishing scalar curvature in V_4 the equations (3.2) and (3.3) respectively become:

$$3P(X, Y, X, T) = 2R(X, Y, Z, T) + V(X, Y, Z, T),$$

$$3P(X, Y, Z, T) = 2R(X, Y, Z, T) + L(X, Y, Z, T);$$

$$P(X, Y, Z, T) = -P(Y, X, Z, T),$$
$$P(X, Y, Z, T) = -P(X, Y, T, Z),$$
$$P(X, Y, Z, T) = P(Z, T, X, Y).$$

Further, both the cyclic properties are satisfied:

(3.7a) $$P(X, Y, Z, T) + P(Y, Z, X, T) + P(Z, X, Y, T) = 0$$

and

(3.7b) $$P(X, Y, Z, T) + P(X, Z, T, Y) + P(X, T, Y, Z) = 0.$$

Thus, it is observed that $P(X, Y, Z, T)$possesses all skew-symmetric and symmetric properties as well as both cyclic properties of $R(X, Y, Z, T)$.

3.1 Bianchi identity for the P(X,Y,Z,T) tensor

The Bianchi differential identity is given by (2.1). Consider V_4 to be the 4-dimensional space time of general relativity, then the equation (3.1)becomes:

(3.8)
$$P(X, Y, Z, T) = R(X.Y.Z, T) - (1/6)[g(Y, Z)Ric(X, T) - g(X, Z)Ric(Y, T)$$
$$+g(X, T)Ric(Y, Z) - g(Y, T)Ric(X, Z)].$$

In order to check if P-curvature tensor satisfies Bianchi differential identity, we compute:

(3.9)
$$\nabla_x P(Y, Z, T, U) + \nabla_y P(Z, X, T, U) + \nabla_z P(X, Y, T, U) = (\nabla_x R)(Y, Z, T, U)-$$
$$(1/6)[g(Z, T)(\nabla_x Ric)(Y, U) - g(Y, T)(\nabla_x Ric)(Z, U) + g(Y, U)(\nabla_x Ric)(Z, T)-$$
$$g(Z, U)(\nabla_x Ric)(Y, T)] + (\nabla_y R)(Z, X, T, U) - (1/6)[g(X, T)(\nabla_y Ric)(Z, U)-$$
$$g(Z, T)(\nabla_y Ric)(X, U) + g(Z, U)(\nabla_y Ric)(X, T) - g(X, U)(\nabla_y Ric)(Z, T)]+$$
$$(\nabla_z R)(X, Y, T, U) - 1/6)[g(Y, T)(\nabla_z Ric)(X, U) - g(X, T)(\nabla_z Ric)(Y, U)+$$
$$g(X, U)(\nabla_z Ric)(Y, T) - g(Y, U)(\nabla_z Ric)(X, T)].$$

Using the equation (2.1), the equation (3.9) reduces to:

(3.10)
$$\nabla_x P(Y, Z, T, U) + \nabla_y P(Z, X, T, U) + \nabla_z P(X, Y, T, U) = -(1/6)[g(X, T)\{(\nabla_y Ric)(Z, U)-$$
$$(\nabla_z Ric)(Y, Z)\} + g(Y, T)\{(\nabla_z Ric)(X, U) - (\nabla_x Ric)(Z, U)\} + g(Z, T)\{(\nabla_x Ric)(Y, U)-$$
$$(\nabla_y Ric)(X, U)\} + g(Y, U)\{\nabla_x Ric)(Z, T) - (\nabla_z Ric)(X, T)\} + g(Z, U)\{(\nabla_y Ric)(X, T)-$$
$$(\nabla_x Ric)(Y, T)\} + g(X, U)\{(\nabla_z Ric)(Y, T) - (\nabla_y Ric)(Z, T)\}].$$

If the Ricci tensor is of Coddazi type, then using (2.2) the right hand side of equation (3.10) becomes zero and we have:

$$\nabla_x P(Y,Z,T,U) + \nabla_y P(Z,X,T,U) + \nabla_z P(X,Y,T,U) = 0. \tag{3.11}$$

Hence, $P(X,Y,Z,T)$satisfies Bianchi differential identity. Conversely, if P-curvature tensor satisfied Bianchi differential identity, then by (3.9) and (2.1), we have:

$$\begin{aligned} &g(X,T)\{(\nabla_y Ric)(Z,U) - (\nabla_z Ric)(Y,Z)\} + g(Y,T)\{(\nabla_z Ric)(X,U) - \\ &(\nabla_x Ric)(Z,U)\} + g(Z,T)\{(\nabla_x Ric)(Y,U) - (\nabla_y Ric)(X,U)\} + \\ &g(Y,U)\{\nabla_x Ric)(Z,T) - (\nabla_z Ric)(X,T)\} + g(Z,U)\{(\nabla_y Ric)(X,T) - \\ &(\nabla_x Ric)(Y,T)\} + g(X,U)\{(\nabla_z Ric)(Y,T) - (\nabla_y Ric)(Z,T)\} = 0. \end{aligned} \tag{3.12}$$

For equation (3.12) to hold, equation (2.2) must be satisfied. Thus, we have the following theorem.

Theorem 3.1. *In V_4 the P-curvature tensor satisfies Bianchi Differential identity if and only if the Ricci tensor is of Codazzi type.*

4 Conclusions

The geometrical and physical properties of W_2-curvature tensor have been fairly widely studied. The P-curvature tensor has been defined by breaking W_2-curvature tensor in skew- symmetric part in Z, T. Further, the P-curvature tensor satisfies the skew-symmetric and symmetric, as well as cyclic properties that are satisfied by the Riemann curvature tensor. Therefore, the tensor $P(X,Y,Z,T)$ can be useful for studying the cosmological models. Also, by checking the consequences of the divergence of the contracted part $P(X,Y)$, the possible applications in the Einstein field equations as well as in perfect fluid space-time can be explored.

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