Twisted products Berwald metrics of polar type

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Abstract. Let \((\mathcal{M}, F)\) be an \((n-1)\)-dimensional compact Finslerian manifold with \(n > 1\). Let consider a Finslerian metric on \(\mathcal{M}\) of the form 
\[
F(x, y) = \sqrt{(y_1^2 + f(x_1, x_2, \ldots, x_n)^2 F_2(x_2, \ldots, x_n, y_2, \ldots, y_n)}
\]
where \(f\) is a positive function on \(\mathcal{M}\) and 
\((x_1, \ldots, x_n, y_1, \ldots, y_n)\) is a local coordinate of a point \((x, y)\) in the tangent bundle of \(\mathcal{M}\). In this paper, we express the geometry of \((\mathcal{M}, F)\) in term of \(f\) and the geometry of \((\mathcal{M}, F)\). Curvatures are calculated in the Berwald case. An example of a twisted product Berwald metric is given for \(n = 3\).

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1 Introduction

Twisted product metrics are natural extensions of warped product metrics [1]. These both kinds of metrics play a major role in Differential Geometry as well as in General Relativity. For examples, the warped product metrics are used to construct Riemannian metrics with negative curvature [3] and the twisted product metrics are applied in theory of projective mappings. In general relativity, many basic solutions of the Einstein equation are warped products metrics[6]. In Finslerian realm the twisted product of manifolds was studied for the first time by Kozma, Peter and Shimada [5]. Later, in 2013, Peyghan, Tayebi and Nourmohammadi Far [7] studied locally dually flat twisted product Finsler manifold.

Let \((\mathcal{M}, F)\) and \((\mathcal{M}, \overline{F})\) be two Finslerian manifolds. Consider \(f_i : \mathcal{M} \times \mathcal{M} \rightarrow (0, \infty)\) with \(i = 1, 2\) two \(C^\infty\) maps. Then, on the product manifold \(\mathcal{M} \times \mathcal{M}\), one can define the Finslerian metric
\[
F(x_1, x_2, y_1, y_2) = \sqrt{f_1(x_1, x_2)F^2(x_1, y_1) + f_2(x_1, x_2)\overline{F}^2(x_2, y_2)}
\]
for any \((x_1, x_2) \in \mathcal{M} \times \mathcal{M}\) and \((y_1, y_2) \in \mathcal{M} \times \mathcal{M}\) where \(\mathcal{M} \equiv T\mathcal{M} \setminus \{0\}\). The couple \((f_1, \mathcal{M} \times \mathcal{M}, F)\) is called a doubly twisted product Finslerian manifold. If \(f_1(x_1, x_2) = 1\) then the doubly twisted product manifold \(f_1, \mathcal{M} \times \mathcal{M}\) is called a twisted product manifold [7] and is denoted by \(\mathcal{M} \times \mathcal{M}\).
In this paper we study the geometry of the twisted product Finslerian manifold \((\tilde{M} \times _{f} M, F)\) when \(F\) is a Berwald metric and \(\tilde{M} \equiv (0, \infty)\). In particular, if the twisted function \(f(x_1, x_2) = f(x_1)\) then \((\tilde{M} \times _{f} M, F)\) becomes a warped product Berwald manifold of polar type.

This work is organised as follows. In Section 2, we give some basic notions on Finslerian manifolds. The Section 3 is devoted to study the Berwald curvatures. The Berwald Ricci and scalar curvatures are evaluated in natural coordinates. Finally, as an example, we show that the application \(F : T((0, \infty) \times U) \rightarrow \mathbb{R}\) defined by

\[
F(x^1, x^2, x^3; y^1, y^2, y^3) = \sqrt{(y^1)^2 + fe^{\rho(x^2, x^3)}(y^2)^{2p}(y^3)^{2q}},
\]

where \(\rho\) is a \(C^\infty\) function on \(U \subset \mathbb{R}^2\), and \(p\) and \(q\) are some real numbers, is a twisted product Berwald metric of polar type on \((0, \infty) \times U\).

## 2 Some basic notions on Finslerian manifolds

Let \(M\) be an \(n\)-dimensional manifold. We denote by \(T_x M\) the tangent space at \(x \in M\) and by \(TM := \bigcup_{x \in M} T_x M\) the tangent bundle of \(M\). Set \(\tilde{T}M = TM \setminus \{0\}\) and \(\pi : \tilde{T}M \longrightarrow M : \pi(x, y) \longrightarrow x\) the natural projection. Let \((x^1, ..., x^n)\) be a local coordinate on an open subset \(U\) of \(M\) and \((x^1, ..., x^n, y^1, ..., y^n)\) be the local coordinate on \(\pi^{-1}(U) \subset \tilde{T}M\). The local coordinate system \((x^i)_{i=1,...,n}\) produces the coordinate bases \(\{\frac{\partial}{\partial x^i}\}_{i=1,...,n}\) and \(\{dx^i\}_{i=1,...,n}\) respectively, for \(T^*M\) and cotangent bundle \(T^*M\). We use Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise will be noted.

**Definition 2.1.** Let \(M\) be an \(n\)-dimensional manifold. A function \(F : TM \longrightarrow [0, \infty)\) is called a Finslerian metric on \(M\) if:

(i) \(F\) is \(C^\infty\) on the entire slit tangent bundle \(\tilde{T}M\),

(ii) \(F\) is positively 1-homogeneous on the fibers of \(TM\), that is \(\forall c > 0, F(x, cy) = cF(x, y)\),

(iii) the Hessian matrix \((g_{ij}(x, y))_{1 \leq i, j \leq n}\) with elements

\[
g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}
\]

is positive definite at every point \((x, y)\) of \(\tilde{T}M\).

Consider the differential map \(\pi_*\) of the submersion \(\pi : \tilde{T}M \longrightarrow M\). The vertical subspace of \(TT\tilde{M}\) is defined by \(V := ker(\pi_*)\) and is locally spanned by the set \(\{F \frac{\partial}{\partial F^1}, 1 \leq i \leq n\}\), on each \(\pi^{-1}(U) \subset \tilde{T}M\).

An horizontal subspace \(\mathcal{H}\) of \(TT\tilde{M}\) is by definition any complementary to \(V\). The bundles \(\mathcal{H}\) and \(V\) give a smooth splitting

\[
TT\tilde{M} = \mathcal{H} \oplus V.
\]

An Ehresmann connection is a selection of a horizontal subspace \(\mathcal{H}\) of \(TT\tilde{M}\). It is known [4] that \(\mathcal{H}\) can be canonically defined from the geodesics equation.
Definition 2.2. Let \( \pi : \dot{T}M \rightarrow M \) be the restricted projection.

(1) An Ehresmann-Finsler connection of \( \pi \) is the subbundle \( \mathcal{H} \) of \( T\dot{T}M \) given by

\[
\mathcal{H} := \ker \theta,
\]

where \( \theta : T\dot{T}M \rightarrow \pi^*TM \) is the bundle morphism defined by

\[
\theta|_{(x,y)} = \frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N^j_i dx^j)
\]

with \( N^j_i(x,y) := \frac{\partial G^j(x,y)}{\partial y^i} \) for

\[
G^j(x,y) := \frac{1}{4} g^{ij}(x,y) \left[ \frac{\partial g_{ij}}{\partial x^k}(x,y) + \frac{\partial g_{jk}}{\partial x^i}(x,y) - \frac{\partial g_{ik}}{\partial x^j}(x,y) \right] y^j k.
\]

(2) The form \( \theta : T\dot{T}M \rightarrow \pi^*TM \) induces a linear map

\[
\theta|_{(x,y)} : T_{(x,y)}\dot{T}M \rightarrow T_x M,
\]

for each point \((x,y) \in \dot{T}M; \) where \( x = \pi(x,y) \).

The vertical lift of a section \( \xi \) of \( \pi^*TM \) is a unique section \( v(\xi) \) of \( T\dot{T}M \) such that for every \((x,y) \in \dot{T}M, \)

\[
\pi_v(\xi(x,y))|_{(x,y)} = 0|_{(x,y)} \text{ and } \theta(v(\xi)(x,y)) = \xi(x,y).
\]

(3) The differential projection \( \pi_* : T\dot{T}M \rightarrow \pi^*TM \) induces a linear map

\[
\pi_*|_{(x,y)} : T_{(x,y)}\dot{T}M \rightarrow T_x M,
\]

for each point \((x,y) \in \dot{T}M; \) where \( x = \pi(x,y) \).

The horizontal lift of a section \( \xi \) of \( \pi^*TM \) is a unique section \( h(\xi) \) of \( T\dot{T}M \) such that for every \((x,y) \in \dot{T}M, \)

\[
\pi_h(h(\xi)(x,y))|_{(x,y)} = \xi(x,y) \text{ and } \theta(h(\xi)(x,y)) = 0|_{(x,y)}.
\]

We have the following.

Definition 2.3. A Finslerian tensor field \( T \) of type \((q,0; p_1, p_2) \) on \( \dot{T}M \) is a \( C^\infty \) section of the tensor bundle

\[
\pi^*TM \otimes \cdots \otimes \pi^*TM \otimes T^*\dot{T}M \otimes \cdots \otimes T^*\dot{T}M \otimes \bigotimes_{p_1-\text{times}}^q \pi^*TM.
\]

Remark 2.4. In a local chart,

\[
T = T_{i_1 \cdots i_{k_1}}^{j_1 \cdots j_{k_2}} \partial_{k_1} \otimes \cdots \otimes \partial_{k_2} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_{k_2}} \otimes e^{i_1} \otimes \cdots \otimes e^{i_{p_1}}
\]

where \((\partial_{k_1} \otimes \cdots \otimes \partial_{k_2} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_{k_2}} \otimes e^{i_1} \otimes \cdots \otimes e^{i_{p_1}})_{k \in \{1,\ldots,n\}^k, i \in \{1,\ldots,n\}^{p_1}, j \in \{1,\ldots,n\}^{p_2}} \)

is a basis section of this tensor and, the \( \partial_{k_2} := \frac{\partial}{\partial x^{j_{k_2}}} \) as well as \( e^{i_j} \) are respectively the basis sections for \( \pi^*TM \) and \( T^*\dot{T}M \) dual of \( T\dot{T}M \).
Example 2.5.  (1) The Hessian matrix $g$, defined in (2.1), is of type $(0,0;2,0)$.

(2) The Ehresmann-Finsler form $\theta$ is of type $(1,0;0,1)$.

The following lemma defines the Chern connection on $\pi^*TM$.

Lemma 2.1.  [8] Let $(M,F)$ be a Finslerian manifold and $g$ its fundamental tensor. There exists a unique linear connection $\nabla$ on the vector bundle $\pi^*TM$ such that, for all $X,Y \in \chi(TM)$ and for every $\xi, \eta \in \Gamma(\pi^*TM)$, one has the following properties:

(i) $\nabla_X \pi_* Y - \nabla_Y \pi_* X = \pi_*(X,Y)$,

(ii) $X(g(\xi,\eta)) = g(\nabla_X \xi, \eta) + g(\xi, \nabla_Y \eta) + 2A(\theta(X),\xi,\eta)$

where $A := \frac{\partial g}{\partial y^i} dx^i \otimes dx^j \otimes dx^k$ is the Cartan tensor.

One has $\nabla \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \Gamma^i_{jk} \frac{\partial}{\partial x^i}$ where

$$\Gamma^i_{jk} := \frac{\partial^2 G^i}{\partial y^j \partial y^k}$$

which can be written as

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^j} \right)$$

with

$$\left\{ \frac{\partial}{\partial x^i} := \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j} = h\left( \frac{\partial}{\partial x^i} \right) \right\}_{i=1,\ldots,n}$$

Definition 2.6. Let $F$ be a Finslerian metric on an $n$-dimensional manifold $M$ and $x \in M$. $F$ is called a Berwald metric if, for a local coordinate $(x^i, y^j)_{i=1,\ldots,n}$ in $TM$, the Christoffel symbols $\Gamma^i_{jk}$ of the Chern connection are only functions of the point $x$ in $M$.

Example 2.7. All Riemannian metrics and all locally Minkowskian metrics are examples of Berwald metrics. In fact,

(1) for Riemannian metrics, $\Gamma^i_{jk} = \gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^j} \right)$. In particular, the functions $\Gamma^i_{jk}$ are independant of $y$.

(2) for locally Minkowskian metrics, in a neighborhood $V$ of a point $x \in M$, the functions $\Gamma^i_{jk}$ vanish identically. Hence, on $V$, $\Gamma^i_{jk}$ can depend at most on $x$.

3 Berwald Ricci and scalar curvatures

Let $(\mathcal{M}, F)$ be an $n-1$ dimensional Finslerian manifold and $f$ a positive $C^\infty$ function on $(0,\infty) \times \mathcal{M}$. One can show that $F : T((0,\infty) \times \mathcal{M}) \rightarrow (0,\infty)$, defined by

$$F(x^1, x^2, \ldots, x^n; y^1, y^2, \ldots, y^n) = \sqrt{(y^1)^2 + f(x^1, x^2, \ldots, x^n)F^2(x^2, \ldots, x^n; y^2, \ldots, y^n)},$$

(3.1)
is a Finslerian metric on $M \equiv (0, \infty) \times \overline{M}$. In particular, if $f(x^1, x^2, \ldots, x^n) = f(x^1)$ then $F$ is warped product Finslerian metric of cylindrical type. If $f(x^1, x^2, \ldots, x^n) = f(x^2, \ldots, x^n)$ then $fF^2$ can be treated as a conformal metric of $F^2$ whose conformal factor is $f$. In this last case, $F$ can be seen as a simple product Finslerian metric.

For the Finsler metric $F(x, y) = \sqrt{(y^1)^2 + f(x)F^2(\varpi, \overline{\varpi})}$ where $(\varpi) = (x^2, \ldots, x^n)$ is a local coordinate in $\overline{M}$ and $(\overline{\varpi}) = (y^2, \ldots, y^n)$ are vector components in $T_{\varpi}\overline{M}$, the fundamental tensor is

\begin{equation}
(g_{ij}(x, y)) = \begin{pmatrix}
1 & 0 \\
0 & f(x)(\overline{g}_{ij}(\varpi, \overline{\varpi}))
\end{pmatrix}
\end{equation}

where $\overline{g}$ is the fundamental tensor associated with $\overline{F}$.

The inverse $g^{-1}$ of $g$ is given by

\begin{equation}
(g^{ij}(x, y)) = \begin{pmatrix}
1 & 0 \\
0 & f^{-1}(x)(\overline{g}^{ij}(\varpi, \overline{\varpi}))
\end{pmatrix}.
\end{equation}

**Definition 3.1.** The full curvature associated with the Chern connection $\nabla$ on the vector bundle $\pi^*TM$ over the manifold $TM$ is the application

\[ \phi : \chi(\tilde{T}M) \times \chi(\tilde{T}M) \times \Gamma(\pi^*TM) \rightarrow \Gamma(\pi^*TM) \]

\[ (X, Y, \xi) \mapsto \phi(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi. \]

By the relation (2.2), we have

\begin{equation}
\nabla_X = \nabla_{\hat{X}} + \nabla_X,
\end{equation}

where $X = \hat{X} + X$ with $\hat{X} \in \Gamma(\mathcal{H})$ and $X \in \Gamma(\mathcal{V})$.

Using the metric $F$, one can define the full curvature of $\nabla$ as:

\begin{equation}
\Phi(\xi, \eta, X, Y) = g(\phi(X, Y)\xi, \eta) = g(\phi(\hat{X}, \hat{Y})\xi + \phi(\hat{X}, \hat{Y})\xi + \phi(\hat{X}, \hat{Y})\xi, \eta) = R(\xi, \eta, X, Y) + P(\xi, \eta, X, Y) + Q(\xi, \eta, X, Y),
\end{equation}

where $R(\xi, \eta, X, Y) = g(\phi(\hat{X}, \hat{Y})\xi, \eta)$, $P(\xi, \eta, X, Y) = g(\phi(\hat{X}, \hat{Y})\xi, \eta) + g(\phi(\hat{X}, \hat{Y})\xi, \eta)$ and $Q(\xi, \eta, X, Y) = g(\phi(\hat{X}, \hat{Y})\xi, \eta)$ are respectively the first (horizontal) curvature, mixed curvature and vertical curvature.

In particular, if $\nabla$ is the Chern connection, the $Q$-curvature vanishes.

In a local coordinate system, the components of the Chern curvature are:

\begin{equation}
\Phi(\partial_i, \partial_j, \partial_k + \partial_l, \partial_l + \partial_l) = R(\partial_i, \partial_j, \partial_k + \partial_l, \partial_l + \partial_l) + P(\partial_i, \partial_j, \partial_k + \partial_l, \partial_l + \partial_l) + Q(\partial_i, \partial_j, \partial_k + \partial_l, \partial_l + \partial_l) = \left( \frac{\delta \Gamma_{ik}}{\delta x^j} - \frac{\delta \Gamma_{ik}}{\delta x^j} \right) g_{js} + \left( \Gamma_{ik} \Gamma_{ls} - \Gamma_{il} \Gamma_{ks} \right) g_{jr} - F \frac{\partial \Gamma_{ik}}{\partial y^j} g_{js}
\end{equation}

where $\partial_i := \frac{\partial}{\partial \xi^i} \in \pi^*TM$, $\partial_k := \frac{\partial}{\partial \xi^k} \in \mathcal{H}$ and $\partial_k := \frac{\partial}{\partial \eta^k} \in \mathcal{V}$.

**Remark 3.2.** In natural coordinates, the curvatures $R$ and $P$ can also be found in [2].
For the Berwald metric \( F(x, y) = \sqrt{(y^1)^2 + f(x)F^2(x, y)} \), by the Definition 2.6, the Christoffel symbols are

\[
\begin{align*}
\Gamma^i_{ij} &= 0 \text{ for } i, j \in [1, n], \\
\Gamma^k_{11} &= 0 \text{ for } k \in [1, n], \\
\Gamma^a_{ib} &= \frac{1}{2f} \frac{\partial f}{\partial x^1} \delta^a_b \text{ for } a, b \in [2, n], \\
\Gamma^a_{bc} &= \frac{1}{2} g^{ad} \left( \frac{\partial g_{bd}}{\partial x^c} + \frac{\partial g_{cd}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^d} \right) \text{ for } a, b, c, d \in [2, n] \\
&= \frac{1}{2} f^{-1} g^{ad} \left[ \frac{\partial (f g_{bd})}{\partial x^c} + \frac{\partial (f g_{cd})}{\partial x^b} - \frac{\partial (f g_{bc})}{\partial x^d} \right] \\
&= \Gamma^a_{bc} + \frac{1}{2} f^{-1} \left( \frac{\partial f}{\partial x^c} \delta^a_b + \frac{\partial f}{\partial x^b} \delta^a_c - \frac{\partial f}{\partial x^d} g^{ad} \delta^a_{bc} \right).
\end{align*}
\]

If \( F \) is a Berwald metric then the relation (3.6) becomes

\[
\Phi_{ijkl} = \left( \frac{\partial \Gamma^a_{il}}{\partial x^j} - \frac{\partial \Gamma^a_{ij}}{\partial x^l} \right) g_{ka} + \left( \Gamma^a_{ik} \Gamma^i_{la} - \Gamma^a_{il} \Gamma^i_{ka} \right) g_{jr}
\]

\[
= \left( \frac{\partial \Gamma^a_{il}}{\partial x^j} - \frac{\partial \Gamma^a_{ij}}{\partial x^l} \right) g_{ka} + \left( \Gamma^a_{ik} \Gamma^i_{la} - \Gamma^a_{il} \Gamma^i_{ka} \right) g_{jr}
\]

where \( \Phi_{ijkl} = \Phi(\partial_i, \partial_j, \partial_k, \partial_l) \). In particular, by the relations (3.7)-(3.10)

\[
\begin{align*}
\Phi_{i1k1} &= 0, \\
\Phi_{i2k1} &= 0, \\
\Phi_{11k1} &= 0, \\
\Phi_{12k1} &= 0, \\
\Phi_{abc} &= \left( \frac{\partial \Gamma^a_{il}}{\partial x^j} - \frac{\partial \Gamma^a_{ij}}{\partial x^l} \right) g_{ka} + \left( \Gamma^a_{ik} \Gamma^i_{la} - \Gamma^a_{il} \Gamma^i_{ka} \right) g_{jr}
\end{align*}
\]
It follows that

\[
\Psi_{abcd} = \sqrt{g_{abcd}} + \frac{1}{4f} \left( \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^b} g_{bd} + \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^c} g_{ab} - \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^d} g_{cd} 
+ \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^d} g_{bd} + \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^c} g_{ac} - \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^b} g_{cd}
\right)
- \frac{1}{2 f} \left( \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^c} y^{ac} g_{bd} - \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^d} y^{ad} g_{bc} + \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^b} y^{ab} g_{cd}
\right)
- \frac{1}{2 f} \left( \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^d} y^{ad} g_{bc} - \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^c} y^{ac} g_{bd} + \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^b} y^{ab} g_{cd}
\right)
- \frac{1}{2 f} \left( \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^c} y^{ac} g_{bd} - \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^d} y^{ad} g_{bc} + \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^b} y^{ab} g_{cd}
\right)
\]

That prove the following.

(3.16)
Proposition 3.1. Let \((I \times f \overline{M}, F)\) be an \(n\)-dimensional twisted product Berwald manifold with \(I = (0, \infty)\). Then, in natural coordinates, the full curvature coefficients of \((I \times f \overline{M}, F)\) are given by

\[
\Phi_{abcd} = \int \overline{g}_{abcd} + \frac{1}{4f} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \overline{g}^{ij} (\overline{g}_{ad} \overline{g}_{bc} - \overline{g}_{ac} \overline{g}_{bd})
\]

\[
+ \frac{1}{4f} \left( \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^b} \overline{g}_{ac} - \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^c} \overline{g}_{ad} - \frac{\partial f}{\partial x^b} \frac{\partial f}{\partial x^d} \overline{g}_{bd} - \frac{\partial f}{\partial x^c} \frac{\partial f}{\partial x^d} \overline{g}_{ac} \right)
\]

\[
+ \frac{1}{2} \left( \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^b} \overline{g}_{ad} \overline{g}_{bc} - \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^c} \overline{g}_{ac} \overline{g}_{bd} + \frac{\partial f}{\partial x^b} \frac{\partial f}{\partial x^d} \overline{g}_{bd} \overline{g}_{ac} - \frac{\partial f}{\partial x^c} \frac{\partial f}{\partial x^d} \overline{g}_{ad} \overline{g}_{bc} \right)
\]

\[
+ \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^a \partial x^b} \overline{g}_{ab} - \frac{\partial^2 f}{\partial x^a \partial x^c} \overline{g}_{ac} + \frac{\partial^2 f}{\partial x^b \partial x^d} \overline{g}_{bd} - \frac{\partial^2 f}{\partial x^c \partial x^d} \overline{g}_{cd} \right)
\]

\[
+ \frac{1}{2} \left[ \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^b} \left( \overline{g}^{ij} \overline{g}^{kl} \right) \overline{g}_{bc} + \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^c} \left( \overline{g}^{ij} \overline{g}^{kl} \right) \overline{g}_{bd} \right],
\]

\[
\Phi_{ijkl} = \frac{1}{2} \left[ \frac{\partial^2 f}{\partial (x^i)^2} - \frac{3}{2f} \left( \frac{\partial f}{\partial x^i} \right)^2 \right],
\]

\[
\Phi_{abcd} = \Phi_{ijkl} = 0 \quad \text{for} \quad a, b, c, d \in [2, n] \quad \text{and} \quad i, j, k, l \in [1, n].
\]

Theorem 3.2. Let \((I \times f \overline{M}, F)\) be an \(n\)-dimensional twisted product Berwald manifold with \(I = (0, \infty)\) and \(F\) a local Minkowskian metric on \(\overline{M}\). Then \((I \times f \overline{M}, F)\) is locally Minkowskian manifold if and only if the twisted function \(f\) satisfies the following equations:

\[
\frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^b} \overline{g}^{ij} \overline{g}^{kl} \overline{g}_{ad} \overline{g}_{bc} + \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^c} \overline{g}^{ij} \overline{g}^{kl} \overline{g}_{ac} \overline{g}_{bd} + \frac{\partial f}{\partial x^b} \frac{\partial f}{\partial x^d} \overline{g}^{ij} \overline{g}^{kl} \overline{g}_{bd} \overline{g}_{ac} + 2f \frac{\partial^2 f}{\partial x^a \partial x^b} \overline{g}_{ad} \overline{g}_{bd} + 2f \frac{\partial^2 f}{\partial x^a \partial x^c} \overline{g}_{ac} \overline{g}_{bd} + 2f \frac{\partial^2 f}{\partial x^b \partial x^d} \overline{g}_{bd} \overline{g}_{ac} + 2f \frac{\partial^2 f}{\partial x^c \partial x^d} \overline{g}_{ac} \overline{g}_{bd} + 2f \frac{\partial^2 f}{\partial x^a \partial x^c} \overline{g}_{ac} \overline{g}_{bd} + 2f \frac{\partial^2 f}{\partial x^b \partial x^d} \overline{g}_{bd} \overline{g}_{ac} + 2f \frac{\partial^2 f}{\partial x^c \partial x^d} \overline{g}_{ac} \overline{g}_{bd} + 2f \frac{\partial^2 f}{\partial x^a \partial x^d} \overline{g}_{ad} \overline{g}_{bd} + 2f \frac{\partial^2 f}{\partial x^b \partial x^c} \overline{g}_{ac} \overline{g}_{bd} + 2f \frac{\partial^2 f}{\partial x^c \partial x^a} \overline{g}_{ac} \overline{g}_{bd} + 2f \frac{\partial^2 f}{\partial x^d \partial x^a} \overline{g}_{ad} \overline{g}_{bd} + 2f \frac{\partial^2 f}{\partial x^d \partial x^b} \overline{g}_{bd} \overline{g}_{ac}
\]

\[
= \frac{2f}{\partial x^a} \frac{\partial^2 f}{\partial (x^i)^2} \left( \frac{\partial f}{\partial x^i} \right)^2.
\]

Theorem 3.3. Let \((I \times f \overline{M}, F)\) be an \(n\)-dimensional warped product Berwald manifold with \(I = (0, \infty)\) and \(F\) a Riemannian metric on \(\overline{M}\). Then \((I \times f \overline{M}, F)\) is Riemannian manifold if and only if the following function \(f\) and the full curvatures satisfy the following equations:

\[
\begin{align*}
\Phi_{abcd} &= \int \overline{g}_{abcd}, \quad \text{for} \quad a, b, c, d \in [2, n] \\
\Phi_{ijkl} &= \frac{1}{2} \left[ \frac{\partial^2 f}{\partial (x^i)^2} - \frac{3}{2f} \left( \frac{\partial f}{\partial x^i} \right)^2 \right] \overline{g}_{bc}, \quad \text{for} \quad b, c \in \{2, \ldots, n\} \\
\Phi_{ijkl} &= \Phi_{ijkl} = 0, \quad \text{for} \quad i, j, k \in [1, n].
\end{align*}
\]

With respect to the Chern connection, we have the following.
The Berwald Ricci tensor $\text{Ric}$ of $(M, F)$ is defined by

\begin{equation}
\text{Ric}(\xi, X) := \text{trace}_g \left[ \eta \mapsto R(X, h(\eta) + v(\eta)) \xi \right].
\end{equation}

Locally, we have

\begin{equation}
\text{Ric}(\partial_t, \partial_t + \partial_k) = \frac{\partial \Gamma^t_{ik}}{\partial x^k} - \frac{\partial \Gamma^t_{ik}}{\partial x^l} + \Gamma^s_{ik} \Gamma^t_{ls} - \Gamma^s_{il} \Gamma^t_{ks}.
\end{equation}

The Berwald scalar curvature $\text{Scal}$ of $(M, F)$ is defined by

\begin{equation}
\text{Scal} := \text{trace}_g \left( \text{Ric} \right), \quad g := \pi^* g.
\end{equation}

Locally, we have

\begin{equation}
\text{Scal} = \left( \frac{\partial \Gamma^t_{ik}}{\partial x^k} - \frac{\partial \Gamma^t_{ik}}{\partial x^l} + \Gamma^s_{ik} \Gamma^t_{ls} - \Gamma^s_{il} \Gamma^t_{ks} \right) g^{tk}.
\end{equation}

**Proposition 3.5.** Let $(I \times_f \overline{M}, F)$ be an $n$-dimensional twisted Finslerian manifold with $I = (0, \infty)$ and $\overline{F}$ a Finslerian metric on $\overline{M}$. Then $(I \times_f \overline{M}, F)$ is a local Berwald manifold if and only if the twisted function $f$ and the Ricci curvatures satisfy the following equations:

\begin{equation}
\text{Ric}_{ac} = \overline{\text{Ric}}_{ac} - \frac{n - 2}{4f^2} \left( \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^c} \right) + \frac{1}{2f} \left( (n - 3) \frac{\partial^2 f}{\partial x^a \partial x^c} + \frac{1}{2f} \frac{\partial f}{\partial x^a} \right) \Gamma^s_{ac} \Gamma^t_{st} + \frac{1}{2f} \left[ \frac{\partial^2 f}{\partial x^a \partial x^c} \right] \Gamma^s_{ac} \Gamma^t_{st} - \frac{1}{2f} \frac{\partial f}{\partial x^a} \frac{\partial f}{\partial x^c} \left( \overline{\text{Ric}}_{ac} \right)
\end{equation}

for $a, b \in [2, n]$ and

\begin{equation}
\text{Ric}_{11} = \frac{n - 1}{2} \left( \frac{\partial^2 f}{\partial x^1 \partial x^1} \right) - \frac{3}{2f} \left( \frac{\partial f}{\partial x^1} \right)^2.
\end{equation}

**Proposition 3.5.** Let $(I \times_f \overline{M}, F)$ be an $n$-dimensional twisted Finslerian manifold with $I = (0, \infty)$ and $\overline{F}$ a Finslerian metric on $\overline{M}$. Then $(I \times_f \overline{M}, F)$ is a local Berwald manifold if and only if the twisted function $f$ and the scalar curvatures satisfy the following equations:

\begin{equation}
\text{Scal} = \overline{\text{Scal}} - \frac{1}{2f} \left( \frac{\partial^2 f}{\partial x^1 \partial x^1} \right) - \frac{n(n - 2)}{4f^3} \frac{\partial f}{\partial x^1} \frac{\partial f}{\partial x^1} \overline{\text{Ric}}_{ac}
\end{equation}

for $a, c, r, s \in [2, n]$. 

\[ \]
Proof. By the relations (3.7)-(3.10) in (3.24).

Corollary 3.6. Let \((I \times f \mathbb{M}, F)\) be an \(n\)-dimensional twisted product Berwald manifold with \(I = (0, \infty)\). If \(f(x^1, x^2, \ldots, x^n) = f(x^1)\) then the scalar curvatures \(\text{Scal}\) of \((I \times f \mathbb{M}, F)\) and \(\text{scal}\) of \((\mathbb{M}, F)\) are related by the followin equation

\[
(3.28) \quad \text{Scal} = \frac{\text{scal}}{f} - \frac{1}{2f} \left[ \frac{\partial f}{\partial x^1} \right]^2 - \frac{\partial^2 f}{\partial (x^1)^2}.
\]

4 Example of a twisted product Berwald metric of polar type

Let \(U = \{ \pi = (x^2, x^3) \in \mathbb{R}^2 \}, \varphi = (y^2, y^3) \in T_{\pi}U\) with \(y^2, y^3 > 0\) and \(f : (0, \infty) \times U \rightarrow (0, \infty)\) a \(C^\infty\) positive function. Consider \(p, q \in \mathbb{R} \setminus \{0\}\) such that \(p + q = 1\) and \(pq < 0\). Then the application \(F : T((0, \infty) \times U) \rightarrow \mathbb{R}\) defined by

\[
(4.1) \quad F(x^1, x^2, x^3; y^1, y^2, y^3) = \sqrt{(y^1)^2 + f e^\rho(x^2, x^3)(y^2)^{2p}(y^3)^{2q}},
\]

where \(\rho\) is a \(C^\infty\) function on \(U\) is a Berwald metric on \((0, \infty) \times U\).

By direct calculations, using the relation (2.1), (3.2) and (3.3), we obtain

\[
(g_{ij}(x, y)) = \begin{pmatrix}
1 & 0 \\
0 & 2p q e^\rho(y^2)^{2p-1}(y^3)^{2q-1}
\end{pmatrix}
\]

and

\[
(g^{ij}(x, y)) = \begin{pmatrix}
1 & - \frac{2q-1}{f} e^{-\rho} (y^2)^{-2p+2}(y^3)^{-2q+1} \\
0 & \frac{2q-1}{f} e^{-\rho} (y^2)^{-2p+1}(y^3)^{-2q+1}
\end{pmatrix}
\]

Hence, since \(g_{(x,y)}(v, v) = g_{ij}(x, y)v^i v^j > 0\) for every \(v \in T_x M \setminus \{0\}\), \(F\) is a Finslerian metric on \((0, \infty) \times U\).

Further, the 27 functions \(\Gamma_{ij}^k\) are independent of \(y\) and satisfy to the relations (3.7), (3.8),(3.9) and (3.10). Hence, \(F\) defined in (4.1), is a twisted Berwald metric of polar form.

References


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