Ricci curvature of a homogeneous Finsler space with exponential metric

Kirandeep Kaur and Gauree Shanker

Abstract. In the current paper, first we discuss the concept of Ricci curvature in Finsler geometry. Next, we derive the formula for Ricci curvature of a homogeneous Finsler space with exponential metric. Based on this formula, we give a necessary condition for compact homogeneous space with the afore said metric to be Einstein metric and with vanishing S-curvature.

M.S.C. 2010: 22E60, 53C30, 53C60.
Key words: Homogeneous Finsler space; exponential metric; Ricci curvature; Einstein metric.

1 Introduction

According to S. S. Chern ([4]), Finsler geometry is just the Riemannian geometry without the quadratic restriction. The notion of (α, β)-metric in Finsler geometry was introduced by M. Matsumoto in 1972 ([9]). An (α, β)-metric is a Finsler metric of the form $F = \alpha\phi(s),\ s = \dfrac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric on a connected smooth n-manifold M and $\beta = b_i(x)y^i$ is a 1-form on M. It is well known fact that (α, β)-metrics are the generalizations of the Randers metric introduced by G. Randers in ([11]). (α, β)-metrics have various applications in physics and biology ([1]). Some important class of (α, β)-metrics are Randers metric, Kropina metric, Matsumoto metric, exponential metric and infinite series metric etc. Many authors ([10], [12], [13], [14], [15] etc.) have studied various properties of (α, β)-metrics. The study of various types of curvatures of Finsler spaces such as S-curvature, mean Berwald curvature, flag curvature and Ricci curvature always remain the central idea in the Finsler geometry.

In Finsler geometry, one of the interesting open problems is the problem "Does every smooth manifold admit an Einstein metric?" posed by S. S. Chern. If the Ricci curvature of a Finsler space (M, F) can be written as $Ric(x, y) = \lambda(x)F^2(x, y)$, where λ is a smooth function on M, then F is called an Einstein metric [2]. Zhou [18] has given the formulae of Ricci curvature and Riemannian curvature for (α, β)-metrics. Cheng et al. [3] have proved that the formulae given in [18] are incorrect and they have

Differential Geometry - Dynamical Systems, Vol.22, 2020, pp. 130-140.

given the corrected formulae of Ricci curvature and Riemannian curvature for (α, β)-metrics. Homgeneous Einstein-Randers metrics have been studied in ([5], [6], [16]). In 2016, Yan and Deng [17] have discussed homogeneous Einstein (α, β)-metrics. They have derived a formula for Ricci curvature of a homogeneous (α, β)-metric. Also, they have deduced a necessary and sufficient condition for a compact homogeneous (α, β)-metric with zero S-curvature to be Einstein metric. Further, they have shown that any homogeneous Ricci flat (α, β)-metric with zero S-curvature must be a Minkowski metric. Recently, Deng and Liu [8] have studied homogeneous square Einstein metrics.

2 Preliminaries

Now, we discuss an important geometrical entity, known as Ricci curvature. For a non-zero vector $y \in T_xM$, the Riemannian curvature [3] is a linear map $R_y\colon T_xM \to T_xM$ defined as follows:

$$R_y(u) = R^i_k(y)u^k \frac{\partial}{\partial x^i}, \quad u = u^i \frac{\partial}{\partial x^i},$$

where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

Definition 2.1. [3] Let (M, F) be a Finsler space and $tr(R_y)$ be the trace of its Riemannian curvature, where $y \in TM$. A scalar function $Ric\colon TM \longrightarrow \mathbb{R}$ such that $Ric(y) = tr(R_y)$ is called Ricci curvature of (M, F).

Now, we give some notations going to be used in the next theorem.

$$r_{ij} = \frac{b_{i;j} + b_{j;i}}{2}, \quad r^i_j = a^{ik} r_{kj}, \quad r_i = b^k r_{ki} = b_j r^j_i, \quad r = r_{ij} b^i b^j = b^i r_i,$$
$$s_{ij} = \frac{b_{i;j} - b_{j;i}}{2}, \quad s^i_j = a^{ik} s_{kj}, \quad s_i = b^k s_{ki} = b_j s^j_i,$$
$$r_{i0} = r_{ij} y^j, \quad s_{i0} = s_{ij} y^j, \quad r_{00} = r_{ij} y^i y^j, \quad r_0 = r_i y^i, \quad s_0 = s_i y^i.$$

Theorem 2.1. *[3] Let F be an (α, β)-metric on a Finsler space M. Then its Ricci curvature is given by $Ric(Y) = Ric^\alpha(Y) + RT^k_k$, where*

$$\begin{aligned} RT^k_k =& \frac{1}{\alpha^2}\Big\{ (n-1)\, A_1 + A_2 \Big\} r^2_{00} + \frac{1}{\alpha}\Big[\{(n-1)\, A_3 + A_4\}\, r_{00} s_0 \\ &+ \{(n-1)\, A_5 + A_6\}\, r_{00} r_0 + \{(n-1)\, A_7 + A_8\}\, r_{00;0} \Big] \\ &+ \{(n-1)\, A_9 + A_{10}\}\, s^2_0 + A_{11}\left(r r_{00} - r^2_0\right) + \{(n-1)\, A_{12} + A_{13}\}\, r_0 s_0 \\ &+ A_{14}\Big\{ r_{00} r^k_k - r_{0k} r^k_0 + r_{00;k} b^k - r_{0k;0} b^k \Big\} + \{(n-1)\, A_{15} + A_{16}\}\, r_{0k} s^k_0 \\ &+ \{(n-1)\, A_{17} + A_{18}\}\, s_{0;0} + A_{19} s_{0k} s^k_0 + \alpha\big[A_{20} r s_0 + \{(n-1)\, A_{21} + A_{22}\} \big] \\ &+ \alpha\big\{ A_{23}\left(3 s_k r^k_0 - 2 s_0 r^k_k + 2 r_k s^k_0 - 2 s_{0;k} b^k + s_{k;0} b^k\right) + A_{24} s^k_{0;k} \big\} \\ &+ \alpha^2\left(A_{25} s_k s^k + A_{26} s^i_k s^k_i\right), \end{aligned}$$

where

$$
\begin{aligned}
A_1 &= 2\left(B-s^2\right)\Psi\Theta_s - 2s\Psi\Theta - \Theta_s + \Theta^2,\\
A_2 &= \left(B-s^2\right)^2\left(2\Psi\Psi_{ss}-\Psi_s^2\right) - \left(B-s^2\right)\left(6s\Psi\Psi_s+\Psi_{ss}\right) + 2s\Psi_s,\\
A_3 &= -4\left(B-s^2\right)\left(2Q\Psi\Theta_s + Q_s\Psi\Theta\right) + 2\left(Q_s\Theta + 2Q\Theta_s\right) + 4Q\Theta\left(s\Psi-\Theta\right) - 2\Theta_B,\\
A_4 &= \left(B-s^2\right)^2\Big\{-4\Psi\left(2Q\Psi_{ss}+Q_s\Psi_s+Q_{ss}\Psi\right) + 4Q\Psi_s^2\Big\} + \left(B-s^2\right)\\
&\quad \Big\{-4\Psi^2\left(Q-sQ_s\right) + 2\left(2Q_{ss}\Psi + Q_s\Psi_s + 2Q\Psi_{ss}\right) + 20sQ\Psi\Psi_s - 2\Psi_{sB}\Big\}\\
&\quad + 2\Psi\left(Q-sQ_s\right) - 10sQ\Psi_s - Q_{ss} - 4\Psi_s,\\
A_5 &= 4\Psi\Theta - 2\Theta_B,\\
A_6 &= 2\left(B-s^2\right)\left(2\Psi\Psi_s - \Psi_{sB}\right) - 2\Psi_s,\\
A_7 &= -\Theta,\\
A_8 &= -\left(B-s^2\right)\Psi_s,\\
A_9 &= 8Q\Psi\left(B-s^2\right)\left(Q_s\Theta + Q\Theta_s\right) + 4Q\left(\Theta_B - Q_s\right) + 4Q^2\left(\Theta^2 - \Theta_s\right),\\
A_{10} &= 4sQ\Psi_B - Q_s^2 + 2QQ_{ss} - 8Q^2\Psi - 4s^2Q^2\Psi^2 + 4\left(2+3sQ\right)\left(Q\Psi_s + Q_s\Psi\right)\\
&\quad + 4\left(B-s^2\right)\Big\{Q\Psi_{sB} + Q_s\Psi_B - Q\left(Q\Psi_{ss} + Q_s\Psi_s\right) - \Psi\left(2QQ_{ss} - Q_s^2\right)\\
&\quad - 4sQ\Psi\left(Q\Psi_s + Q_s\Psi\right)\Big\} + 4\left(B-s^2\right)^2\Big\{2Q\Psi\left(Q\Psi_{ss} + Q_s\Psi_s\right)\\
&\quad + \Psi^2\left(2QQ_{ss} - Q_s^2\right) - Q^2\Psi_s^2\Big\},\\
A_{11} &= 4\Psi_B + 4\Psi^2,\\
A_{12} &= 4Q\left(\Theta_B - 2\Psi\Theta\right),\\
A_{13} &= 4Q\Psi_s + 8sQ\Psi^2 - 4\left(1-sQ\right)\Psi_B\\
&\quad + \left(B-s^2\right)\Big\{4\left(Q_s\Psi_B + Q\Psi_{sB}\right) + 8\Psi\left(Q_s\Psi - Q\Psi_s\right)\Big\},\\
A_{14} &= 2\Psi,\\
A_{15} &= 4\Theta Q,\\
A_{16} &= 2Q_s - 2\Psi\left(1+2sQ\right) - 4\left(B-s^2\right)\left(Q_s\Psi - Q\Psi_s\right),\\
A_{17} &= 2\Theta Q,\\
A_{18} &= 2\left(B-s^2\right)\left(Q\Psi_s + \Psi Q_s\right) + 2s\Psi Q - Q_s,\\
A_{19} &= 2\left(1+sQ\right)Q_s - 2Q^2,\\
A_{20} &= -8Q\left(\Psi_B + \Psi^2\right),\\
A_{21} &= -4\Theta Q^2,\\
A_{22} &= 2\Psi Q - 4\left(B-s^2\right)Q^2\Psi_s,\\
A_{23} &= 2\Psi Q,\\
A_{24} &= 2Q,\\
A_{25} &= -4\Psi Q^2,\\
A_{26} &= -Q^2,\\
\Psi &= \frac{\phi''}{2\big\{\phi - s\phi' + (B-s^2)\,\phi''\big\}},\\
\Theta &= \frac{\phi\phi' - s\left(\phi\phi'' + \phi'\phi'\right)}{2\phi\big\{\phi - s\phi' + (B-s^2)\,\phi''\big\}},\\
Q &= \frac{\phi'}{\phi - s\phi'},\\
B &= b^2.
\end{aligned}
$$

3 Ricci curvature of a homogeneous Finsler space with exponential metric

Now, we calculate Ricci curvature of a homogeneous Finsler space. Consider a G-invariant vector field v in $\mathfrak{m}$ corresponding to the 1-form β with length $c = |v|$ and an orthonormal basis $\left\{v_1, v_2, ..., v_n = \frac{v}{c}\right\}$ of $\mathfrak{m}$ with respect to the restriction $\langle\ ,\ \rangle$ of Riemannian metric α. The Christoffel symbols γ_{ij}^k are given by $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}$. Let $C_{ij}^k = \left\langle [v_i, v_j]_{\mathfrak{m}}, v_k \right\rangle$ be the structure constants of $\mathfrak{g}$. We denote $\sum_{a=1}^n C_{ij}^a C_{kl}^a$ by $C_{ij}^a C_{kl}^a$ and $\left\langle [v_i, v_j]_{\mathfrak{m}}, y \right\rangle$ by C_{ij}^0.

Further, we define

$$f(j,k) = \begin{cases} 1, & \text{if } \ j < k, \\ 0, & \text{if } \ j \geq k. \end{cases}$$

We require the following lemma for further computation.

Lemma 3.1. *[7] For $v_i, v_j, v_k, v_l \in \mathfrak{m}$, we have the following values at the origin eH:*

$$\begin{aligned}
b_j &= c\delta_{nj}, \\
s_{ij} &= \frac{c}{2} C_{ij}^n, \\
s_i &= \frac{c^2}{2} C_{ni}^n, \\
r_{ij} &= -\frac{c}{2}\left(C_{nj}^i + C_{ni}^j\right), \\
s_{ij;l} &= \frac{c}{2}\left\{C_{ji}^a C_{la}^n + \frac{1}{2} C_{ia}^n \left(C_{lj}^a + C_{ja}^l + C_{la}^j\right) + \frac{1}{2} C_{aj}^n \left(C_{li}^a + C_{ia}^l + C_{la}^i\right)\right\}, \\
s_{i;j} &= c\left(s_{ni;j} + \frac{c}{2} C_{ai}^n \Gamma_{nj}^a\right), \\
b_{i;j;l} &= c\Big(\hat{v}_l \left\langle \nabla_{\hat{v}_n} \hat{v}_j, \hat{v}_i \right\rangle + C_{ln}^a \left\langle \nabla_{\hat{v}_a} \hat{v}_j, \hat{v}_i \right\rangle - \Gamma_{na}^i \left\langle \nabla_{\hat{v}_l} \hat{v}_j, \hat{v}_a \right\rangle - \Gamma_{nj}^a \left\langle \nabla_{\hat{v}_l} \hat{v}_i, \hat{v}_a \right\rangle\Big), \\
r_{ij;l} &= s_{ij;l} + b_{j;i;l}, \\
\left\langle \nabla_{\hat{v}_i} \hat{v}_j, \hat{v}_k \right\rangle &= -\frac{1}{2}\left(C_{jk}^i + C_{ik}^j + C_{ij}^k\right), \\
\left\langle \nabla_{\hat{v}_i} \hat{v}_j, \hat{v}_k \right\rangle \hat{v}_l &= \frac{1}{2}\left(C_{la}^i C_{jk}^a + C_{la}^j C_{ik}^a + C_{la}^k C_{ij}^a + C_{jk}^a C_{li}^a + C_{ik}^a C_{lj}^a + C_{ij}^a C_{lk}^a\right), \\
\Gamma_{ij}^k &= f(i,j) C_{ij}^k + \left\langle \nabla_{\hat{v}_i} \hat{v}_j, \hat{v}_k \right\rangle, \\
\frac{\partial \Gamma_{ij}^k}{\partial x^l} &= \left\langle \nabla_{\hat{v}_i} \hat{v}_j, \hat{v}_k \right\rangle \hat{v}_l + f(l,i) \left\langle \nabla_{\hat{v}_a} \hat{v}_j, \hat{v}_k \right\rangle C_{li}^a + f(l,j) \left\langle \nabla_{\hat{v}_i} \hat{v}_a, \hat{v}_k \right\rangle C_{lj}^a \\
&\quad - \left(\Gamma_{la}^k + \left\langle \nabla_{\hat{v}_l} \hat{v}_k, \hat{v}_a \right\rangle\right) \Gamma_{ij}^a.
\end{aligned}$$

Using Lemma 3.1, we calculate the quantities used in the statement of Theorem 2.1, at the origin for homogeneous Finsler spaces.

$$
\begin{aligned}
r_{00} &= -\frac{c}{2}\left(C^0_{n0} + C^0_{n0}\right) = -cC^0_{n0} = cC^0_{0n},\\
s_i &= b^k s_{ki} = a^{kj} b_j s_{ki} = cs_{ni},\\
s_0 &= cs_{n0} = c\frac{c}{2}C^n_{n0} = \frac{c^2}{2}C^n_{n0},\\
r_i &= b^k r_{ki} = a^{kj} b_j r_{ki} = cr_{ni},\\
r_0 &= cr_{n0} = -\frac{c^2}{2}\left(C^n_{n0} + C^0_{nn}\right) = -\frac{c^2}{2}\left(C^n_{n0} + 0\right) = -\frac{c^2}{2}C^n_{n0} = \frac{c^2}{2}C^n_{0n},\\
r &= r_{ij} b^i b^j = b^i r_i = cr_n = ccr_{nn} = -c^2\frac{c}{2}\left(C^n_{nn} + C^n_{nn}\right) = 0,\\
r_{00;0} &= cC^0_{i0}\left(C^0_{ni} + C^i_{n0}\right),\\
r^k_k &= a^{kl} r_{lk} = -\frac{c}{2}a^{kl}\left(C^l_{nk} + C^k_{nl}\right) = -\frac{c}{2}\delta^{kl}\left(C^l_{nk} + C^k_{nl}\right)\\
&= -\frac{c}{2}\left(C^k_{nk} + C^k_{nk}\right) = -cC^k_{nk} = cC^k_{kn},\\
r_{0k} r^k_0 &= r_{0k} r_{k0} = \left\{-\frac{c}{2}\left(C^0_{nk} + C^k_{n0}\right)\right\}\left\{-\frac{c}{2}\left(C^k_{n0} + C^0_{nk}\right)\right\} = \frac{c^2}{4}\left(C^0_{nk} + C^k_{n0}\right)^2,\\
r_{00;k} b^k &= r_{00;n} b^n = cr_{00;n} = \frac{c^2}{2}\left(C^a_{0n} + C^0_{na} + C^n_{0a}\right)\left(C^a_{0n} + C^0_{an}\right),\\
r_{0k;0} b^k &= r_{0n;0} b^n = cr_{0n;0} = \frac{c^2}{2}\left\{C^n_{na}C^0_{a0} + \frac{1}{2}\left(C^a_{0n} + C^n_{a0} + C^0_{an}\right)\left(C^0_{na} + C^a_{n0}\right)\right\},\\
r_{0k} s^k_0 &= r_{0k} s_{k0} = -\frac{c}{2}\left(C^0_{nk} + C^k_{n0}\right)\frac{c}{2}C^n_{k0} = \frac{c^2}{4}C^n_{0k}\left(C^0_{nk} + C^k_{n0}\right),\\
s_{0;0} &= c\left(s_{n0;0} + \frac{c}{2}C^n_{k0}\Gamma^k_{n0}\right) = \frac{c^2}{2}C^n_{na}C^0_{0a},\\
s_{0k} s^k_0 &= s_{0k} s_{k0} = \frac{c}{2}C^n_{0k}\frac{c}{2}C^n_{k0} = -\frac{c^2}{4}\left(C^n_{k0}\right)^2,\\
s_k s^k_0 &= s_k s_{k0} = \frac{c^2}{2}C^n_{nk}\frac{c}{2}C^n_{k0} = \frac{c^3}{4}C^n_{nk}C^n_{k0},\\
s_k r^k_0 &= \frac{c^2}{2}C^n_{nk}\left\{-\frac{c}{2}\left(C^k_{n0} + C^0_{nk}\right)\right\} = \frac{c^3}{4}C^n_{kn}\left(C^k_{n0} + C^0_{nk}\right),\\
r_k s^k_0 &= cr_{nk} s_{k0} = c\left\{-\frac{c}{2}\left(C^k_{n0} + C^0_{nk}\right)\right\}\frac{c}{2}C^n_{k0} = \frac{c^3}{4}C^n_{0k}\left(C^n_{nk} + C^k_{nn}\right) = \frac{c^3}{4}C^n_{0k}C^n_{nk},\\
s_k s^k &= \frac{c^2}{2}C^n_{nk}\frac{c^2}{2}C^n_{nk} = \frac{c^4}{4}\left(C^n_{nk}\right)^2,\\
s^k_{0;k} &= s_{k0;k} = \frac{c}{2}\left\{C^a_{0k}C^n_{ka} + \frac{1}{2}C^n_{ka}\left(C^a_{k0} + C^k_{0a} + C^0_{ka}\right) + \frac{1}{2}C^n_{a0}\left(C^a_{kk} + C^k_{ka} + C^k_{ka}\right)\right\}\\
&= \frac{c}{2}\left\{C^a_{0k}C^n_{ka} - \frac{1}{2}C^n_{ka}C^a_{0k} + \frac{1}{2}C^n_{ka}\left(C^k_{0a} + C^0_{ka}\right) + C^n_{a0}C^k_{ka}\right\}\\
&= \frac{c}{2}\left\{C^n_{a0}C^k_{ka} + \frac{1}{2}C^n_{ka}\left(C^a_{0k} + C^k_{0a} + C^0_{ka}\right)\right\},\\
s_{0;k} b^k &= s_{0;n} b^n = cs_{0;n} = c^2\left(s_{n0;n} + \frac{c}{2}C^n_{a0}\Gamma^a_{nn}\right) = \frac{c^3}{4}C^n_{na}\left(C^a_{0n} + C^n_{0a} + C^0_{na}\right),\\
s_{k;0} b^k &= s_{n;0} b^n = cs_{n;0} = \frac{c^3}{4}C^n_{na}\left(C^a_{n0} + C^n_{0a} + C^0_{na}\right),\\
s^i_k s^k_i &= s_{ik} s_{ki} = \frac{c}{2}C^n_{ik}\frac{c}{2}C^n_{ki} = -\frac{c^2}{4}\left(C^n_{ik}\right)^2.
\end{aligned}
$$

Next, we calculate A_1 to A_{26} for exponential metric as follows:

$$
\begin{aligned}
F &= \alpha e^{\beta/\alpha} = \alpha e^s, \\
Q &= \frac{e^s}{e^s - s e^s} = \frac{1}{1-s}, \\
Q_s &= \frac{d}{ds}\left(\frac{1}{1-s}\right) = \frac{1}{(1-s)^2}, \\
Q_{ss} &= \frac{d}{ds}\left[\frac{1}{(1-s)^2}\right] = \frac{2}{(1-s)^3}, \\
\Theta &= \frac{e^s e^s - s\left(e^s e^s + e^s e^s\right)}{2e^s\left\{e^s - s e^s + (B - s^2)\, e^s\right\}} = \frac{(1-2s)}{2\left\{(1-s) + (B-s^2)\right\}}, \\
\Psi &= \frac{e^s}{2\left\{e^s - s e^s + (B - s^2)\, e^s\right\}} = \frac{1}{2\left\{(1-s) + (B-s^2)\right\}}, \\
\Psi_s &= \frac{1}{2}\frac{\partial}{\partial s}\left[\frac{1}{\left\{(1-s) + (B-s^2)\right\}}\right] = \frac{(1+2s)}{2\left\{(1-s) + (B-s^2)\right\}^2}, \\
\Psi_B &= \frac{1}{2}\frac{\partial}{\partial B}\left[\frac{1}{\left\{(1-s) + (B-s^2)\right\}}\right] = \frac{-1}{2\left\{(1-s) + (B-s^2)\right\}^2}, \\
\Psi_{sB} &= \frac{1}{2}\frac{\partial}{\partial s}\left[\frac{-1}{\left\{(1-s) + (B-s^2)\right\}^2}\right] = \frac{-(1+2s)}{\left\{(1-s) + (B-s^2)\right\}^3}, \\
\Psi_{ss} &= \frac{1}{2}\frac{\partial}{\partial s}\left[\frac{(1+2s)}{\left\{(1-s) + (B-s^2)\right\}^2}\right] = \frac{\left(3s^2 + 3s + 2 + B\right)}{\left\{(1-s) + (B-s^2)\right\}^3}, \\
\Theta_s &= \frac{\partial}{\partial s}\left[\frac{(1-2s)}{2\left\{(1-s) + (B-s^2)\right\}}\right] = \frac{\left(-2s^2 + 2s - 2B - 1\right)}{2\left\{(1-s) + (B-s^2)\right\}^2}, \\
\Theta_B &= \frac{\partial}{\partial B}\left[\frac{(1-2s)}{2\left\{(1-s) + (B-s^2)\right\}}\right] = \frac{(2s-1)}{2\left\{(1-s) + (B-s^2)\right\}^2}, \\
A_1 &= \frac{\left(B - s^2\right)\left(-2s^2 + 2s - 2B - 1\right)}{2\left\{(1-s) + (B-s^2)\right\}^3} + \frac{\left(12s^2 - 10s + 4B + 3\right)}{4\left\{(1-s) + (B-s^2)\right\}^2}, \\
A_2 &= \frac{\left(8s^2 + 8s + 7 + 4B\right)\left(B - s^2\right)^2}{4\left\{(1-s) + (B-s^2)\right\}^4} - \frac{\left(12s^2 + 9s + 4 + 2B\right)\left(B - s^2\right)}{2\left\{(1-s) + (B-s^2)\right\}^3} \\
&\quad + \frac{s\,(1+2s)}{\left\{(1-s) + (B-s^2)\right\}^2}, \\
A_3 &= \frac{2s\,(1-2s)}{(1-s)\left\{(1-s) + (B-s^2)\right\}^2} + \frac{\left(4s^3 - 3s^2 - B\right)}{(1-s)^2\left\{(1-s) + (B-s^2)\right\}^2} \\
&\quad + \frac{\left(-6s^3 + 7s^2 - 3s - 2sB + 1 + 4B\right)\left(B - s^2\right)}{(1-s)^2\left\{(1-s) + (B-s^2)\right\}^3},
\end{aligned}
$$

$$A_4 = \left(B-s^2\right)^2 \left[\frac{\left(-16s^4+11s^3+10s^2+2Bs^2+8s+11sB-11-5B-2B^2\right)}{(1-s)^3 \left\{ (1-s)+(B-s^2) \right\}^4} - \frac{2(1+2s)^2}{(1-s)\left\{ (1-s)+(B-s^2) \right\}^5} \right] + \left[\frac{\left(18s^2+19s+10+4B\right)}{(1-s)\left\{ (1-s)+(B-s^2) \right\}^3} + \frac{4\left(-2s^2+B+1\right)}{\left\{ (1-s)+(B-s^2) \right\}^2 (1-s)^3} \right] \left(B-s^2\right) + \frac{(1-2s)}{(1-s)^2 \left\{ (1-s)+(B-s^2) \right\}} - \frac{5s(1+2s)}{\left\{ (1-s)+(B-s^2) \right\}^2 (1-s)} - \frac{4(1/2+s)}{\left\{ (1-s)+(B-s^2) \right\}^2} - \frac{2}{(1-s)^3},$$

$$A_5 = \frac{2(1-2s)}{\left\{ (1-s)+(B-s^2) \right\}^2},$$

$$A_6 = \frac{3(1+2s)\left(B-s^2\right)}{\left\{ (1-s)+(B-s^2) \right\}^3} - \frac{(1+2s)}{\left\{ (1-s)+(B-s^2) \right\}^2},$$

$$A_7 = \frac{(2s-1)}{2\left\{ (1-s)+(B-s^2) \right\}},$$

$$A_8 = \frac{-(1+2s)\left(B-s^2\right)}{2\left\{ (1-s)+(B-s^2) \right\}^2},$$

$$A_9 = \frac{2\left(B-s^2\right)\left(4s^3-3s^2-B\right)}{(1-s)^3 \left\{ (1-s)+(B-s^2) \right\}^3} + \frac{\left(8s^2-8s+4B+3\right)}{(1-s)^2 \left\{ (1-s)+(B-s^2) \right\}^2} + \frac{2(2s-1)}{(1-s)\left\{ (1-s)+(B-s^2) \right\}^2} - \frac{4}{(1-s)^3},$$

$$A_{10} = \frac{3}{(1-s)^4} + 3\left(B-s^2\right)^2 \left[\frac{1}{(1-s)^4 \left\{ (1-s)+(B-s^2) \right\}^2} + \frac{\left(-4s^3-2s^2+s+3+2B\right)}{(1-s)^3 \left\{ (1-s)+(B-s^2) \right\}^4} \right] + \left(B-s^2\right) \left[\frac{2\left(14s^3+3s^2-7s-2Bs-3B-5\right)}{(1-s)^3 \left\{ (1-s)+(B-s^2) \right\}^3} - \frac{6}{(1-s)^4 \left\{ (1-s)+(B-s^2) \right\}} + \frac{2\left(5s^2-s-3-B\right)}{\left\{ (1-s)+(B-s^2) \right\}^3 (1-s)^2} \right] + \frac{\left(-11s^3-9s^2+10s+6sB+4\right)}{(1-s)^3 \left\{ (1-s)+(B-s^2) \right\}^2},$$

$$A_{11} = \frac{-1}{\left\{ (1-s)+(B-s^2) \right\}^2},$$

$$A_{12} = \frac{4(2s-1)}{(1-s)\left\{ (1-s)+(B-s^2) \right\}^2},$$

$$A_{13} = \frac{6(1+2s)(s-1)\left(B-s^2\right)}{(1-s)^2 \left\{ (1-s)+(B-s^2) \right\}^3},$$

$$A_{14} = \frac{1}{\left\{ (1-s)+(B-s^2) \right\}},$$

$$
\begin{aligned}
A_{15} &= \frac{2\,(1-2s)}{\big\{\,(1-s)+(B-s^2)\,\big\}\,(1-s)},\\
A_{16} &= \frac{3\,(1+s)\left(B-s^2\right)+\left(1-2s+s^2\right)}{\big\{\,(1-s)+(B-s^2)\,\big\}^2\,(1-s)},\\
A_{17} &= \frac{(1-2s)}{(1-s)\,\big\{\,(1-s)+(B-s^2)\,\big\}},\\
A_{18} &= \frac{3s\left(B-s^2\right)-(1-s)^2}{(1-s)\,\big\{\,(1-s)+(B-s^2)\,\big\}^2},\, A_{19} = \frac{2s}{(1-s)^3},\\
A_{20} &= \frac{2}{(1-s)\,\big\{\,(1-s)+(B-s^2)\,\big\}^2},\\
A_{21} &= \frac{2\,(2s-1)}{\big\{\,(1-s)+(B-s^2)\,\big\}\,(1-s)^2},\\
A_{22} &= \frac{(1-s)^2-(1+5s)\left(B-s^2\right)}{\big\{\,(1-s)+(B-s^2)\,\big\}^2\,(1-s)^2},\\
A_{23} &= \frac{1}{(1-s)\,\big\{\,(1-s)+(B-s^2)\,\big\}},\, A_{24} = \frac{2}{(1-s)},\\
A_{25} &= \frac{-2}{(1-s)\,\big\{\,(1-s)+(B-s^2)\,\big\}},\quad A_{26} = \frac{-1}{(1-s)^2}.
\end{aligned}
\tag{3.1}
$$

Summarizing the above calculations, we get

Theorem 3.2. *The Ricci curvature of the homogeneous Finsler space $(G/H, F)$ with exponential metric $F = \alpha e^{\beta/\alpha}$ is given by*

$$
\begin{aligned}
Ric(Y) = {} & Ric^{\alpha}(Y) + \frac{c^2\left(C^0_{0n}\right)^2}{\alpha^2(Y)}\Big\{\,(n-1)\,A_1 + A_2\Big\} + \frac{c^3 C^0_{0n} C^n_{n0}}{2\alpha(Y)}\Big\{\,(n-1)\,(A_3 - A_5)\\
& + A_4 - A_6\Big\} + \frac{c C^0_{k0}\left(C^0_{nk} + C^k_{n0}\right)}{\alpha(Y)}\Big\{\,(n-1)\,A_7 + A_8\Big\}\\
& + \frac{c^4\left(C^n_{n0}\right)^2}{4}\Big\{\,(n-1)\,(A_9 - A_{12}) + A_{10} - A_{11} - A_{13}\Big\}\\
& + \frac{c^2}{4}\Big\{4C^0_{0n}C^k_{kn} + \left(C^k_{0n} + C^0_{kn}\right)\left(2C^0_{nk} + C^n_{0k} + 2C^k_{0n}\right) + 2C^n_{nk}C^0_{0k}\Big\}A_{14}\\
& + \frac{c^2}{4}C^n_{0k}\left(C^k_{n0} + C^0_{nk}\right)\Big\{\,(n-1)\,A_{15} + A_{16}\Big\}\\
& + \frac{c^2}{2}C^n_{nk}C^0_{0k}\Big\{\,(n-1)\,A_{17} + A_{18}\Big\}\\
& - \frac{c^2}{4}\left(C^n_{k0}\right)^2 A_{19} + \frac{c^3}{4}\alpha(Y) C^n_{k0} C^n_{nk}\Big\{\,(n-1)\,A_{21} + A_{22}\Big\}\\
& + \frac{c^3}{4}\alpha(Y)\Big\{C^n_{kn}\left(4C^0_{nk} + C^n_{k0}\right) + 4C^n_{n0}C^k_{nk}\Big\}A_{23}\\
& + \frac{c}{4}\alpha(Y)\Big\{2C^n_{a0}C^k_{ka} + C^n_{ka}C^0_{ka}\Big\}A_{24} + \frac{c^2}{4}\alpha^2(Y)\Big\{c^2\left(C^n_{nk}\right)^2 A_{25} - \left(C^n_{ik}\right)^2 A_{26}\Big\},
\end{aligned}
$$

where $Y\,(\neq 0) \in \mathfrak{m}$, A_1 to A_{26} for exponential metric are given by (3.1).

Definition 3.1. [17] Let $F = \alpha\phi(s)$ be an (α, β)-metric on a homogeneous Finsler space. The smooth function $\phi(s)$ is said to be normal if it satisfies the following condition:

$$\sum_{i=1}^{7} r_i \phi_i(s) = constant \implies r_i = 0 \; \forall \; i = 1, 2, ..., 7,$$

where

$$\begin{aligned}
\phi_1(s) &= \Big\{ (n-1)\, A_1 + A_2 \Big\} \left(c^2 - s^2\right), \\
\phi_2(s) &= \Big\{ (n-1)\, A_1 + A_2 \Big\} s^2 - \frac{c^2}{2} \Big\{ (n-1)\,(A_3 - A_5) + A_4 - A_6 \Big\} s \\
&\quad + \frac{c^4}{4} \Big\{ (n-1)\,(A_9 - A_{12}) + A_{10} - A_{11} - A_{13} \Big\}, \\
\phi_3(s) &= -\Big\{ (n-1)\, A_7 + A_8 \Big\} s + \frac{c^2}{2} \Big\{ A_{14} + (n-1)\, A_{17} + A_{18} \Big\}, \\
\phi_4(s) &= -\Big\{ (n-1)\, A_7 + A_8 \Big\} s + \frac{c^2}{4} \Big\{ -A_{14} + (n-1)\, A_{15} + A_{16} \Big\}, \\
\phi_5(s) &= -\Big\{ (n-1)\, A_7 + A_8 \Big\} s - \frac{c^2}{2} A_{14}, \\
\phi_6(s) &= -\frac{1}{2} \Big\{ (n-1)\, A_7 + A_8 \Big\} s, \\
\phi_7(s) &= -\frac{c^2}{2} A_{19}.
\end{aligned}$$

Using Theorem 3.2 and Lemma 4.1 of [17], we obtain the following result.

Theorem 3.3. *Let $(G/H, F)$ be a compact connected homogeneous Finsler space with G-invariant exponential metric $F = \alpha e^{\beta/\alpha}$. Then S-curvature of $(G/H, F)$ vanishes if and only if $\langle X, [v, X]_{\mathfrak{m}} \rangle = 0 \; \forall \; X \in \mathfrak{m}$. Further, if F is Einstein metric, then either S-curvature of $(G/H, F)$ vanishes or ϕ is not normal.*

Corollary 3.4. *Let $(G/H, F)$ be a compact connected homogeneous Finsler space with G-invariant exponential metric $F = \alpha e^{\beta/\alpha}$. Further, suppose that S-curvature of $(G/H, F)$ vanishes. Then Ricci scalar is given by*

$$\begin{aligned}
Ric(Y) &= Ric^{\alpha}(Y) - \frac{c^2}{4} \left(C^n_{k0}\right)^2 A_{19} + \alpha(Y) \left(\frac{c}{2} C^n_{a0} C^k_{ka} + \frac{c}{4} C^n_{ka} C^0_{ka} \right) A_{24} \\
&\quad - \frac{c^2}{4} \alpha^2(Y) \left(C^n_{ik}\right)^2 A_{26}.
\end{aligned}$$

Proof. Suppose S-curvature of $(G/H, F)$ vanishes, then by Theorem 3.3,

$$\langle Y, [v, Y]_{\mathfrak{m}} \rangle = 0 \; \forall \; Y \in \mathfrak{m}.$$

Therefore, for non-zero $Z \in \mathfrak{m}$, we have

$$\begin{aligned}
C^0_{0n} &= \langle Z, [v_n, Z]_{\mathfrak{m}} \rangle \\
&= \left\langle Z, \left[\frac{v}{c}, Z\right]_{\mathfrak{m}} \right\rangle = 0, \\
C^i_{nj} + C^j_{ni} &= \left\langle v_i, [v_n, v_j]_{\mathfrak{m}} \right\rangle + \langle v_j, [v_n, v_i]_{\mathfrak{m}} \rangle \\
&= \left\langle v_i, \left[\frac{v}{c}, v_j\right]_{\mathfrak{m}} \right\rangle + \left\langle v_j, \left[\frac{v}{c}, v_i\right]_{\mathfrak{m}} \right\rangle \\
&= \frac{1}{c}\Big\{ \left\langle v_i, [v, v_j]_{\mathfrak{m}} \right\rangle + \left\langle v_j, [v, v_j]_{\mathfrak{m}} \right\rangle + \langle v_j, [v, v_i]_{\mathfrak{m}} \rangle + \langle v_i, [v, v_i]_{\mathfrak{m}} \rangle \Big\} \\
&= \frac{1}{c}\Big\{ \left\langle v_i + v_j, [v, v_j]_{\mathfrak{m}} \right\rangle + \langle v_i + v_j, [v, v_i]_{\mathfrak{m}} \rangle \Big\} \\
&= \frac{1}{c}\Big\{ \left\langle v_i + v_j, [v, v_j]_{\mathfrak{m}} + [v, v_i]_{\mathfrak{m}} \right\rangle \Big\} \\
&= \frac{1}{c} \left\langle v_i + v_j, [v, v_i + v_j]_{\mathfrak{m}} \right\rangle \\
&= 0 \ \forall\ 1 \leq i, j \leq n.
\end{aligned}$$

Next,

$$\begin{aligned}
C^n_{ni} &= \langle [v_n, v_i]_{\mathfrak{m}}, v_n \rangle \\
&= \left\langle \left[\frac{v}{c}, v_i\right]_{\mathfrak{m}}, \frac{v}{c} \right\rangle \\
&= \frac{1}{c^2}\big\{ \langle [v, v_i]_{\mathfrak{m}}, v \rangle + \langle [v, v_i]_{\mathfrak{m}}, v_i \rangle \big\} \\
&= \frac{1}{c^2} \langle [v, v_i]_{\mathfrak{m}}, v + v_i \rangle \\
&= \frac{1}{c^2} \langle [v, v]_{\mathfrak{m}} + [v, v_i]_{\mathfrak{m}}, v + v_i \rangle \\
&= \frac{1}{c^2} \langle [v, v + v_i]_{\mathfrak{m}}, v + v_i \rangle \\
&= 0.
\end{aligned}$$

Finally, using Theorem 3.2, we get the required result. □

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Authors' addresses:

Kirandeep Kaur
Department of Mathematics,
Punjabi University College Ghudda, Bathinda, Punjab, India.
Email: kirandeepiitd@gmail.com

Gauree Shanker (corresponding author)
Department of Mathematics and Statistics,
School of Basic and Applied Sciences,
Central University of Punjab,
Bathinda, Punjab-151001, India.
Email: gshankar@cup.ac.in