On 3-dimensional Lorentzian para-Kenmotsu manifolds

Vinod Chandra and Shankar Lal

Abstract. The object of the present paper is to study the special results on 3-dimensional Lorentzian para-Kenmotsu manifolds. In Section 1, we introduce the historical background of Kenmotsu manifolds. Next in Section 2, some rudimentary facts and related properties of Lorentzian para-Kenmotsu manifolds are discussed. In Section 3 we study of Ricci tensor of Lorentzian para-Kenmotsu manifolds. Further, in Section 4, it is shown that a 3-dimensional Lorentzian para-Kenmotsu manifold satisfying the condition $R(X, Y).S = 0$ is a special manifold. Sections 5 and 6 deal with $\varphi$-symmetry, Ricci symmetry, and in Section 7, it is shown that a 3-dimensional Lorentzian para-Kenmotsu manifold with $\eta$-parallel Ricci tensor is of the positive constant scalar curvature.

Key words: Kenmotsu manifold; Lorentz Metric; Riemannian manifold; Ricci tensor; Einstein manifold.

1 Introduction

K. Kenmotsu [8] studied a contact Riemannian manifold which satisfies a special type of condition, characterized different geometric properties of the manifolds of class (3), the obtained structure was called a Kenmotsu structure. In 1972, K. Kenmotsu gave the notion of Kenmotsu manifolds [8]. Further, G. Pitis [9], De U.C. and Pathak [2], Jun, De U.C., and G. Pathak [7] and many other authors [10] provided results on Kenmotsu manifolds. Recently Dileo G. [5] and A. M. Pastore [4] studied almost Kenmotsu manifolds satisfying $n$-parallelism and locally symmetry, respectively. In [6] was produced a complete classification of 3-dimensional almost Kenmotsu manifolds, assuming that $\xi$ belongs to the $(K, \mu)$-nullity distribution. In 1977 Takahashi [11] introduced the local $\varphi$-symmetry of manifolds [1]. A Sasakian manifold is locally $\varphi$-symmetric if

\begin{equation}
\varphi^2((\nabla_W R)(X, Y)) = 0,
\end{equation}

for each horizontal vector fields $X, Y, Z$ and $W$ on $M$. 

U.C. De and Sarkar [3] called a Sasakian manifold as being $\varphi$-Ricci symmetric if
\begin{equation}
\varphi^2(\nabla_X Q)(\mathcal{Y}) = 0 \quad \text{and} \quad S(X, \mathcal{Y}) = g(QX, \mathcal{Y}),
\end{equation}
for each $X, \mathcal{Y}$ on $M$.

The present paper studies 3-dimensional Lorentzian para-Kenmotsu manifolds.

\section{Preliminaries}

Let $M^n$ be Lorentzian metric manifold, with an (1,1) tensor field $\varphi$. We consider a vector field $\xi$, a Lorentzian metric $g$ and a 1 form $\eta$ on $M$ and assume that the structure given tensor $(\varphi, \xi, \eta, g)$ satisfies [7]:
\begin{align}
\varphi^2(X) &= X + \eta(X)\xi, \\
g(\varphi X, \varphi\mathcal{Y}) &= g(X, \mathcal{Y}) + \eta(X)\eta(\mathcal{Y}), \\
\eta(\xi) &= -1, \quad \eta(\varphi X) = 0.
\end{align}

This provides a Lorentzian almost para-contact manifold for all $X, \mathcal{Y}$ on $M$. In Lorentzian almost para-contact 3-dimensional manifolds, we have
\begin{align}
\varphi \xi &= 0, \quad \eta(\varphi X) = 0, \\
\varphi(X, \mathcal{Y}) &= \varphi(\mathcal{Y}, X), \quad \text{where } \varphi(X, \mathcal{Y}) = g(X, \varphi\mathcal{Y}).
\end{align}

The para-contact structure is called K-para-contact if $\xi$ is a Killing vector field. In such case, we have
\begin{equation}
\nabla_X \xi = 0.
\end{equation}

\textbf{Definition 2.1.} An almost Lorentzian para-contact manifold $M$ is called Lorentzian para-Sasakian 3-dimensional manifold if
\begin{equation}
(\nabla_X \varphi)\mathcal{Y} = g(X, \mathcal{Y})\xi + \eta(\mathcal{Y})X + 2\eta(X)\eta(\mathcal{Y}).
\end{equation}

\textbf{Definition 2.2.} A Lorentzian almost para-contact 3-dimensional manifold $M$ is a Lorentzian para-Kenmotsu manifold if for any vector fields $X, \mathcal{Y}$ on $M$, we have
\begin{align}
(\nabla_X \varphi)\mathcal{Y} &= -g(X, \mathcal{Y}) - \eta(\mathcal{Y})\varphi X, \\
\nabla_X \xi &= -X - \eta(X)\xi, \\
(\nabla_X \eta)\mathcal{Y} &= -g(X, \mathcal{Y}) - \eta(X)\eta(\mathcal{Y}),
\end{align}
for all vector fields $X, \mathcal{Y}$ on $M$, where $\nabla$ denotes covariant differentiation.
Remark. In any Lorentzian para-Kenmotsu 3-dimensional manifold $M$, the following relations hold:

\begin{align}
(2.11) \quad g(R(X, Y)Z, \xi) &= \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \\
(2.12) \quad R(\xi, X)Y &= g(X, Y)\xi - \eta(Y)X \\
(2.13) \quad R(X, Y)\xi &= \eta(Y)X - \eta(X)Y \\
(2.14) \quad R(\xi, X)\xi &= X + \eta(X)\xi \\
(2.15) \quad S(X, \xi) &= (n - 1)\eta(X) \\
(2.16) \quad Q\xi &= (n - 1)\xi \\
(2.17) \quad S(\varphi X, \varphi Y) &= S(X, Y) + (n - 1)\eta(X)\eta(Y).
\end{align}

We denote by $R$ and $S$ the Riemannian curvature tensor and the Ricci tensor, respectively. In an $M^3$ Riemannian manifold, we have

\begin{equation}
(2.18) \quad R(X, Y)Z = g(Y, Z)QX - g(X, Y)QY + S(Y, Z)Y - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y],
\end{equation}

where $Q(g(QX, Y) = S(X, Y))$ and $r$ are the Ricci Operator and the scalar curvature, respectively.

**Definition 2.3.** A Lorentzian para-Kenmotsu manifold is an $\eta$-Einstein manifold if its Ricci tensor $S$ takes the form

\begin{equation}
(2.19) \quad S(X, Y) = a g(X, Y) + b\eta(X)\eta(Y),
\end{equation}

where $a$ and $b$ are scalar functions on $M$.

### 3 The Ricci tensor on a 3-dimensional Lorentzian para-Kenmotsu manifold

**Theorem 3.1.** Any 3-dimensional Lorentzian para-Kenmotsu manifold $M$ which is an $\eta$-Einstein manifold, satisfies $a - b = (n - 1)$.

**Proof.** By replacing $Z = \xi$ in (2.19), we get

\begin{equation}
(3.1) \quad R(X, Y)Z = g(\xi, Z)QX - g(X, Z)Q\xi - g(X, Z)Q\xi + S(\xi, Z)X \\
- S(X, Z)\xi - \frac{r}{2}[g(\xi, Z)X - g(X, Z)\xi].
\end{equation}
Now using (2.13) and (2.15) we get

\[(3.2) \quad \eta(Y)QX - \eta(X)QY = \left(\frac{r}{2} - 1\right) [\eta(Y)X - \eta(X)Y]\]

Further replacement in (2.19), by means of (2.13) and (2.15) infer

\[(3.3) \quad QX = \frac{1}{2} [(r - 2)X + (r - 6)\eta(X)\eta(Y)], \]

(3.4) \[S(X, Y) = \frac{1}{2} [(r - 2)g(X, Y) + (r - 6)\eta(X)\eta(Y)], \]

then (3.4) concludes the proof. \(\square\)

**Lemma 3.1.** If the scalar curvature is constant \(r = 6\), then the Riemannian manifold \(M^3\) is of constant positive curvature. **Proof.** Using (3.4) in (2.19) we get

\[(3.5) \quad R(X, Y)Z = \left(\frac{r - 4}{2}\right) [g(Y, Z)X - g(X, Z)Y] + \left(\frac{r - 6}{2}\right) [g(Y, Z)\eta(X)\xi, -g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X]. \]

If \(r = 6\), we get

\[(3.6) \quad R(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \]

and Lemma 3.1 follows. \(\square\)

### 4 Special 3-dimensional Lorentzian para-Kenmotsu manifolds

We consider a 3-dimensional Riemannian manifold which satisfies the condition

\[(4.1) \quad R(X, Y).S = 0. \]

From (4.1), we obtain

\[(4.2) \quad S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0. \]

Putting \(X = \xi\) and using (3.1), we get

\[(4.3) \quad S(V, \xi)g(Y, U) - S(V, Y)\eta(U) - S(U, Y)\eta(V) + S(Y, \xi)g(U, V) = 0. \]

Given (2.15) and (4.3), we get

\[(4.4) \quad 2g(Y, U)\eta(V) - S(Y, V)\eta(U) + 2g(Y, V)\eta(U) - S(U, Y)\eta(V) = 0. \]

Let \(\{e_1, e_2, e_3\}\) be an orthonormal basis; then putting \(Y = e_i\) in the above equation and taking the sum for \(1 \leq i \leq 3\), then we get

\[(4.5) \quad S(V, \xi) - 8\eta(V) + r\eta(V) = 0. \]
Using (2.15), we have

\[(r - 6)\eta(V) = 0.\]

Since we have \(\eta(V) \neq 0\), it follows \((r - 6) = 0\), which gives \(r = 6\), which states by Lemma 3.1 that the manifold is of constant positive curvature.

Then we can state the following result:

**Theorem 4.1.** The Riemannian manifold \((M^3)\) satisfying the condition \(R(X, Y)S = 0\) is a 3-dimensional manifold of constant positive curvature 1.

which infers

**Lemma 4.1.** The manifold \((M^3, \varphi, \xi, \eta, g)\) is a Lorentzian para-Kenmotsu 3-dimensional manifold of constant curvature 1.

5 Locally \(\varphi\)-symmetric Lorentzian para-Kenmotsu 3-dimensional manifolds

**Definition 5.1.** A Lorentzian para-Kenmotsu 3-dimensional manifold is locally \(\varphi\)-symmetric if

\[\varphi^2(\nabla_W R)(X, Y)Z = 0,\]

for all vector fields \(W, X, Y,\) and \(Z\) orthogonal to \(\xi\).

Takahashi introduced the notion of \(\varphi\)-symmetry on Sasakian manifold.

By covariant differentiation concerning \(W\) of (3.5), we get

\[
(\nabla_W R)(X, Y)Z = \frac{dr(W)}{2} [g(Y, Z)X - g(X, Z)Y] + \frac{dr(W)}{2} [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi]
\]

\[
- g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + \left(\frac{r - 6}{2}\right) [g(Y, Z)(\nabla_W \eta)(X)\xi - g(X, Z)(\nabla_W \eta)(Y)\xi + (\nabla_W \eta)(Y)\eta(Z)X - (\nabla_W \eta)(X)\eta(Y)\xi]
\]

For \(X, Y, Z\) and \(W\) orthogonal to \(\xi\), from (2.9) and (2.10), we get from equation (5.2)

\[
(\nabla_W R)(X, Y)Z = \frac{dr(W)}{2} [g(Y, Z)X - g(X, Z)Y]
\]

\[
+ \left(\frac{r - 6}{2}\right) [g(X, Z)g(W, Y)\xi - g(Y, Z)g(W, X)\xi]
\]

Then it follows that

\[
\varphi^2(\nabla_W R)(X, Y)Z = \frac{dr(W)}{2} [g(Y, Z)\varphi^2 X - g(X, Z)\varphi^2 Y]
\]

Now taking \(X, Y, Z\) and \(W\) orthogonal to \(\xi\), from equation (2.1) we have

\[
\varphi^2(\nabla_W R)(X, Y)Z = \frac{1}{2} dr(W) [g(Y, Z)X - g(X, Z)Y].
\]

These lead to the following
Theorem 5.1. A 3-dimensional Lorentzian para-Kenmotsu manifold is locally $\varphi$-symmetric iff its scalar curvature is constant.

From comparing Section 3 and Section 4, we can also state the following

Theorem 5.2. If a 3-dimensional Lorentzian para-Kenmotsu manifold satisfies the condition $R(X, Y).S = 0$, then the manifold is locally $\varphi$-symmetric.

6 $\varphi$-symmetric Lorentzian para-Kenmotsu 3-dimensional manifolds

Definition 6.1. A Lorentzian para-Kenmotsu 3-dimensional manifold $M$ is said to be $\varphi$-Ricci symmetric if the Ricci operators satisfy the condition

$$\varphi^2(\nabla_X Q)(T) = 0,$$

for all $X, T$ on $M$, let $S(X, T) = g(QX, T)$.

If the manifold is $\varphi$-Ricci symmetric, then from (6.1) and (2.1),

$$(\nabla_X Q)(T) + \eta(\nabla_X Q)(T)\xi = 0.$$ (6.2)

It follows that

$$g((\nabla_X Q)(T), Z) + \eta(\nabla_X Q)(T)\eta(\xi) = 0.$$ (6.3)

Solving (6.3), we get

$$g((\nabla_X Q)(T), Z) + S(\nabla_X T, Z) + \eta(\nabla_X Q)(T)\eta(Z) = 0.$$ (6.4)

Replacing $T = \xi$ in (6.4), we get

$$g((\nabla_X Q)(\xi), Z) + S(\nabla_X \xi, Z) + \eta(\nabla_X Q)(\xi)\eta(Z) = 0.$$ (6.5)

From (2.9) and (2.13), we obtain

$$(n - 1)[g(X, Z) + \eta(X)\eta(Z)] - S(X, Z) - S(\xi, Z)\eta(X) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0.$$ (6.6)

Replacing $X$ by $\varphi X$ and $Z$ by $\varphi Z$, we get

$$S(X, Z) = (n - 1)g(X, Z),$$ (6.7)

which proves that we have an Einstein 3-dimensional manifold. We know that a symmetric Riemannian manifold is $\varphi$-Ricci symmetric.

Lemma 6.1. Any $\varphi$-symmetric Lorentzian para-Kenmotsu manifold is an Einstein manifold.

From result (6.7), we infer the following

Theorem 6.1. If a 3-dimensional Lorentzian para-Kenmotsu manifold is an Einstein manifold, then it is $\varphi$-Ricci symmetric.
Proof. From (6.2), for $n = 3$, we get
\begin{equation}
S(X, Z) = 2g(X, Z). \tag{6.8}
\end{equation}
If $S(X, Z) = g(QX, Z)$, then $QX = 2X$ and we have
\begin{equation}
\varphi^2(\nabla_{\varphi}Q)(X) = 0, \tag{6.9}
\end{equation}
which is the condition of $\varphi$-Ricci symmetry. From Theorems 4.1 and 5.1, we derive the following result.

**Theorem 6.2.** A 3-dimensional Lorentzian para-Kenmotsu manifold is $\varphi$-Ricci symmetric iff it is an Einstein Manifold.

**Corollary 6.1.** A 3-dimensional Lorentzian para-Kenmotsu manifold is $\varphi$-Ricci symmetric if its scalar curvature $r$ is constant.

7 Lorentzian para-Kenmotsu 3-dimensional manifold with $\eta$-parallel Ricci tensor

**Definition 7.1.** In a Lorentzian para-Kenmotsu 3-dimensional manifold $M$, the Ricci tensor $S$ is called $\eta$-parallel if it is satisfies
\begin{equation}
(\nabla_X S)(\varphi X, \varphi Y) = 0, \tag{7.1}
\end{equation}
for all vector fields $X$, $Y$, and $Z$.

Let us consider a 3-dimensional Lorentzian para-Kenmotsu manifold with $\eta$-parallel Ricci tensor. Then from (2.3) and using (1.1) and (1.2) we get
\begin{equation}
S(\varphi X, \varphi Y) = \left(\frac{r-2}{2}\right)[g(X, Y) - \eta(X)\eta(Y)]. \tag{7.2}
\end{equation}
By covariantly differentiating (7.2) $Z$, we yield
\begin{equation}
(\nabla_Z S)(\varphi X, \varphi Y) = \frac{dr(Z)}{2}[g(X, Y) - \eta(X)\eta(Y)]
- \left(\frac{r-2}{2}\right)[\eta(Y)(\nabla_Z \eta)(X) + \eta(X)(\nabla_Z \eta)(Y)] = 0 \tag{7.3}
\end{equation}
By using (7.1) and (7.3) we get
\begin{equation}
dr(Z)[g(X, Y) - \eta(X)\eta(Y)] - (r-2)[\eta(Y)(\nabla_Z \eta)(X) - \eta(X)(\nabla_Z \eta)(Y)] = 0. \tag{7.4}
\end{equation}
Putting $X = Y = e_i$ in (7.4) and taking summation over $1 \leq i \leq 3$, we get $dr(Z) = 0$ for all $Z$.

**Lemma 7.1.** If a 3-dimensional Lorentzian para-Kenmotsu manifold is $\eta$-parallel Ricci tensor, then the scalar curvature is constant positive.

**Theorem 7.1.** A 3-dimensional Lorentzian para-Kenmotsu Manifold with $\eta$-parallel Ricci tensor is locally $\varphi$-symmetric.
Acknowledgment. The authors are grateful to the referee for valuable improvements to the present work.

References


Authors’ address:

Vinod Chandra and Shankar Lal
Department of Mathematics, H.N.B. Garhwal University,
S. R. T. Campus, Badshahithaul, Tehri Garhwal, Uttarakhand, India.
E-mail: chandravinod8126@gmail.com, shankar_alm@yahoo.com