On a type of quarter-symmetric non-metric $\xi$-connection on 3-dimensional quasi-Sasakian manifolds

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Abstract. The object of the present paper is to study a type of quarter-symmetric non-metric $\xi$-connection on a 3-dimensional non-cosymplectic quasi-Sasakian manifold. We investigate the curvature tensor and the Ricci tensor of a 3-dimensional quasi-Sasakian manifold with respect to the quarter-symmetric non-metric $\xi$-connection. We characterize $\xi$-projectively flat and $\phi$-projectively flat 3-dimensional quasi-Sasakian manifolds with constant structure function admitting the quarter-symmetric non-metric $\xi$-connection. Also we study second order parallel tensor and Ricci semi-symmetric 3-dimensional quasi-Sasakian manifolds with constant structure function with respect to the quarter-symmetric non-metric $\xi$-connection. Finally, we give an example to verify our result.


Key words: 3-dimensional quasi-Sasakian manifold; quarter-symmetric non-metric $\xi$-connection; projective curvature tensor; $\xi$-projectively flat; $\phi$-projectively flat; Ricci semi-symmetric; $\eta$-Einstein manifold.

1 Introduction

On a 3-dimensional quasi-Sasakian manifold, the structure function $\beta$ was defined by Olszak [21] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat [22]. Next he has proved that if the manifold is additionally conformally flat with $\beta = \text{constant}$, then (a) the manifold is locally a product of $\mathbb{R}$ and a 2-dimensional Kählerian space of constant Gauss curvature (the cosymplectic case), or, (b) the manifold is of constant positive curvature (the non-cosymplectic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure). An example of a 3-dimensional quasi-Sasakian structure being conformally flat with non-constant structure function is also described in [22].

In 1924, Friedmann and Schouten [13] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\nabla$ on a differentiable
manifold \( M \) is said to be a semi-symmetric connection if the torsion tensor \( T \) of the connection \( \nabla \) satisfies

\[
T(X, Y) = u(Y)X - u(X)Y,
\]

where \( u \) is a 1-form, \( \chi(M) \) is the set of all differentiable vector fields on \( M \), for all vector fields \( X, Y \in \chi(M) \).

In 1932, Hayden [15] introduced the idea of semi-symmetric metric connection on a Riemannian manifold \((M, g)\). A semi-symmetric connection \( \nabla \) is said to be a semi-symmetric metric connection if

\[
(1.2) \quad \nabla g = 0.
\]

A relation between a semi-symmetric metric connection \( \nabla \) and the Levi-Civita connection \( \nabla \) of \((M, g)\) is given by Yano [26]:

\[
\nabla_X Y = \nabla_X Y + u(Y)X - g(X, Y)\rho_1,
\]

where \( u(X) = g(X, \rho_1) \).

The study of semi-symmetric metric connections were further developed by Amur and Pujar [1], Binh [9], De [10], Singh et al. [24], Ozgur et al. [18, 19], Ozen, Uysal Demirbag [20], Zhao ([27, 28]) and many others. After a long gap the study of a semi-symmetric connection \( \nabla \) satisfying

\[
(1.3) \quad \nabla g \neq 0.
\]

was initiated by Prvanović [23] with the name of pseudo-metric semi-symmetric connection and was just followed by Andonie [2].

A semi-symmetric connection \( \nabla \) is said to be a semi-symmetric non-metric connection if it satisfies the condition \((1.3)\).

In 1975, Golab [14] defined and studied quarter-symmetric connections in differentiable manifolds with affine connections. A linear connection \( \overline{\nabla} \) on a Riemannian manifold \( M \) is called a quarter-symmetric connection [14] if its torsion tensor \( T \) satisfies

\[
T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y,
\]

where \( \eta \) is a 1-form and \( \phi \) is a \((1,1)\)-tensor field. In particular, if \( \phi X = X \), then the quarter-symmetric connection reduces to the semi-symmetric connection [13]. Thus the notion of quarter-symmetric connection generalizes the notion of semi-symmetric connection.

A quarter-symmetric connection \( \overline{\nabla} \) is said to be a quarter-symmetric metric connection if \( \overline{\nabla} g = 0 \). If moreover, a quarter-symmetric connection \( \overline{\nabla} \) satisfies the condition \((\overline{\nabla}_X g)(Y, Z) \neq 0;\) for all \( X, Y, Z \in \chi(M) \), then \( \overline{\nabla} \) is said to be a quarter-symmetric non-metric connection.

The quarter-symmetric non-metric connections have been studied by Mishra and Pandey [16], Singh and Pandey [25], De and Mondal [12], Barman [4], [3], [5], Mondal and De [17] and many others.

In this paper we study 3-dimensional quasi-Sasakian manifolds with respect to a type of quarter-symmetric non-metric \( \xi \)-connection.

The paper is organized as follows:

After introduction, in section 2, we give a brief account of the 3-dimensional quasi-Sasakian manifolds. In section 3, we define a type of quarter-symmetric non-metric \( \xi \)-connection on 3-dimensional quasi-Sasakian manifolds. Section 4 is devoted to establish the relation between the curvature tensors with respect to a type of quarter-symmetric non-metric \( \xi \)-connection and the Levi-Civita connection. Next we study
the projective curvature tensor with respect to the type of quarter-symmetric non-metric $\xi$-connection. Among others we characterize $\phi$-projectively flat 3-dimensional quasi-Sasakian manifolds with constant structure function admitting a type of quarter-symmetric non-metric $\xi$-connection. Also we study a second order parallel tensor on a Ricci semi-symmetric 3-dimensional quasi-Sasakian manifolds with constant structure function with respect to a type of quarter-symmetric non-metric $\xi$-connection. Finally, we give an example to verify our result.

2 Preliminaries

Let $M$ be a $(2n + 1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, g)$, where $\phi$ is a tensor field of type $(1, 1)$, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is the Riemannian metric on $M$ such that ([6], [7])

\begin{align}
\phi^2 X &= -X + \eta(X)\xi, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\
\phi \xi &= 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1.
\end{align}

Let $\Phi$ be the fundamental 2-form defined by

\begin{equation}
\Phi(X, Y) = g(X, \phi Y) = -g(\phi X, Y).
\end{equation}

$M$ is said to be quasi-Sasakian if the almost contact structure $(\phi, \eta, \xi, g)$ is normal and the fundamental 2-form $\Phi$ is closed ($d\Phi = 0$), which was first introduced by Blair [8]. The normality condition gives that the induced almost contact structure of $M \otimes \mathbb{R}$ is integrable or equivalently, the torsion tensor field $N = [\phi, \phi] + 2 \xi \otimes d\eta$ vanishes identically on $M$. The rank of the quasi-Sasakian structure is always odd [8], it is equal to 1 if the structure is cosymplectic and it is equal to $2n + 1$ if the structure is Sasakian.

An almost contact metric manifold $M$ is a 3-dimensional quasi-Sasakian manifold if and only if [21]

\begin{equation}
\nabla_X \xi = -\beta \phi X, \quad X \in \chi(M),
\end{equation}

for a certain function $\beta$ on $M$, such that $\xi \beta = 0$, $\nabla$ being the operator of covariant differentiation with respect to the Levi-Civita connection of $M$. Clearly such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. From the equation (2.5) we obtain [21]

\begin{equation}
(\nabla_X \phi)(Y) = \beta(g(X, Y)\xi - \eta(Y)X),
\end{equation}
\begin{equation}
(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) = -\beta g(\phi X, Y) = \beta g(X, \phi Y).
\end{equation}

Let $M$ be a 3-dimensional quasi-Sasakian manifold. The Ricci tensor $S$ of $M$ is given by [22]
On a type of quarter-symmetric non-metric $\xi$-connection

$$S(Y, Z) = \left(\frac{r}{2} - \beta^2\right)g(Y, Z) + (3\beta^2 - \frac{r}{2})\eta(Y)\eta(Z) - \eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y),$$

(2.8)

where $r$ is the scalar curvature of $M$.

In a 3-dimensional Riemannian manifold we always have

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y - \frac{r}{2}(g(Y, Z)X - g(X, Z)Y),$$

where $Q$ is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$. Now as a consequence of (2.8), we get for the Ricci operator $Q$

$$QY = \left(\frac{r}{2} - \beta^2\right)Y + (3\beta^2 - \frac{r}{2})\eta(Y)\xi + \eta(Y)(\phi grad\beta) - d\beta(\phi Y)\xi,$$

where the gradient of a function $f$ is related to the exterior derivative $df$ by the formula $df(X) = g(\text{grad} f, X)$.

Also from (2.8) it follows that

$$S(\phi Y, \phi Z) = S(Y, Z) - 2\beta^2 \eta(Y)\eta(Z).$$

(2.9)

3 Quarter-symmetric non-metric $\xi$-connection on 3-dimensional quasi-Sasakian manifolds

This section deals with the quarter-symmetric non-metric $\xi$-connection on 3-dimensional non-cosymplectic quasi-Sasakian manifold. Let $(M, g)$ be a quasi-Sasakian manifold with the Levi-Civita connection $\nabla$ and we define a linear connection $\bar{\nabla}$ on $M$ by

$$\bar{\nabla} X Y = \nabla X Y + (\beta - 1)\eta(X)\phi Y + \beta\eta(Y)\phi X,$$

(3.1)

where $\beta$ is a certain function on $M$.

Using (3.1), the torsion tensor $T$ of the connection $\bar{\nabla}$ is given by

$$T(X, Y) = \bar{\nabla} X Y - \bar{\nabla} Y X - [X, Y] = \eta(Y)\phi X - \eta(X)\phi Y,$$

(3.2)

Thus the linear connection $\bar{\nabla}$ is a quarter-symmetric connection. So the equation (3.1) with the help of (2.4) turns into

$$(\bar{\nabla} X g)(Y, Z) = \bar{\nabla} X g(Y, Z) - g(\bar{\nabla} X Y, Z) - g(Y, \bar{\nabla} X Z) = -\beta \eta(Y)g(\phi X, Z) - \beta \eta(Z)g(Y, \phi X) \neq 0.$$

(3.3)

The linear connection $\bar{\nabla}$ satisfying (3.2) and (3.3) is called a quarter-symmetric non-metric connection on 3-dimensional non-cosymplectic quasi-Sasakian manifold.
By making use of (2.3), (2.5) and (3.1), it is obvious that

\[(3.4) \quad \nabla_X \xi = \nabla_X \xi + (\beta - 1)\eta(X)\phi \xi + \beta \eta(\xi)\phi X = 0.\]

The linear connection \(\nabla\) defined by (3.1) satisfying (3.2), (3.3) and (3.4) is the quarter-symmetric non-metric \(\xi\)-connection on 3-dimensional quasi-Sasakian manifold.

Conversely, we show that a linear connection \(\nabla\) defined on \(M\) satisfying (3.2), (3.3) and (3.4) is given by (3.1). Let \(H\) be a tensor field of type \((1,2)\) and

\[(3.5) \quad \nabla_X Y = \nabla_X Y + H(X,Y).\]

Then we conclude that

\[(3.6) \quad T(X,Y) = H(X,Y) - H(Y,X).\]

Further using (3.5), it follows that

\[(3.7) \quad (\nabla_X g)(Y,Z) = \nabla_X g(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = -g(H(X,Y), Z) - g(Y, H(X,Z)).\]

In view of (3.3) and (3.7) yields

\[(3.8) \quad g(H(X,Y), Z) + g(Y, H(X,Z)) = \beta \eta(Y)g(\phi X, Z) + \beta \eta(Z)g(Y, \phi X).\]

Also using (3.8) and (3.6), we derive that

\[g(T(X,Y), Z) + g(T(Z,X), Y) + g(T(Z,Y), X) = 2g(H(X,Y), Z) + 2\beta \eta(X)g(Y, \phi Z) + 2\beta \eta(Y)g(X, \phi Z).\]

The above equation yields

\[(3.9) \quad g(H(X,Y), Z) = \frac{1}{2}[g(T(X,Y), Z) + g(T(Z,X), Y) + g(T(Z,Y), X)] - 2\beta \eta(X)g(Y, \phi Z) - 2\beta \eta(Y)g(X, \phi Z).\]

Let \(T'\) be a tensor field of type \((1,2)\) given by

\[(3.10) \quad g(T'(X,Y), Z) = g(T(Z,X), Y).\]

Adding (2.4), (3.2) and (3.10), we obtain

\[(3.11) \quad T'(X,Y) = -\eta(X)\phi Y - g(\phi X, Y')\xi.\]
From (3.9) we have, by using (3.10) and (3.11):

\[ g(H(X, Y), Z) = \frac{1}{2} [g(T(X, Y), Z) + g(T'(X, Y), Z) + g(T'(Y, X), Z)] \]

\[ -2\beta \eta(X)g(Y, \phi Z) - 2\beta \eta(Y)g(X, \phi Z) = (\beta - 1)\eta(X)g(\phi Y, Z) \]

(3.12) \[ + \beta \eta(Y)g(\phi X, Z). \]

Now contracting \( Z \) in (3.12) and using (2.4), implies that

(3.13) \[ H(X, Y) = (\beta - 1)\eta(X)\phi Y + \beta \eta(Y)\phi X. \]

Combining (3.5) and (3.13), it follows that

\[ \bar{\nabla}_X Y = \nabla_X Y + (\beta - 1)\eta(X)\phi Y + \beta \eta(Y)\phi X. \]

Now, we are in a position to state the following theorem:

**Theorem 3.1.** On a 3-dimensional non-cosymplectic quasi-Sasakian manifold with structure function \( \beta \) there exists a unique linear connection \( \bar{\nabla} \), satisfies (3.2), (3.3) and (3.4).

If \( \beta = 0 \) in the equation (3.1), then we get

\[ \bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \]

Hence, \( \bar{\nabla} \) defines a quarter-symmetric metric connection. Such a case has been studied by De and Mandal [11].

4 Curvature tensor of 3-dimensional quasi-Sasakian manifolds admitting quarter-symmetric non-metric \( \xi \)-connection

In this section, we obtain the expressions of the curvature tensor and Ricci tensor of \( M \) with respect to the quarter-symmetric non-metric \( \xi \)-connection on a 3-dimensional non-cosymplectic quasi-Sasakian manifold defined by (3.1).

Analogous to the definition of the curvature tensor of \( M \) with respect to the Levi-Civita connection \( \nabla \), we define the curvature tensor \( \bar{R} \) of \( M \) with respect to the quarter-symmetric non-metric \( \xi \)-connection \( \bar{\nabla} \) by

\[ \bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z, \]

(4.1) where \( X, Y, Z \in \chi(M) \).
Using (2.3) and (3.1) in (4.1), we obtain

\[
\bar{R}(X,Y)Z = R(X,Y)Z - (\beta - 1)\eta(X)(\nabla_Y \phi)(Z) + (\beta - 1)\eta(Y)(\nabla_X \phi)(Z) \\
- \beta(\nabla_Y \eta)(Z)\phi X + \beta(\nabla_X \eta)(Z)\phi Y - (\beta - 1)(\nabla_Y \eta)(X)\phi Z \\
+ (\beta - 1)(\nabla_X \eta)(Y)\phi Z - \beta\eta(Z)(\nabla_Y \phi)(X) + \beta\eta(Z)(\nabla_X \phi)(Y) \\
- \beta(\beta - 1)\eta(X)\eta(Z)Y + \beta(\beta - 1)\eta(Y)\eta(Z)X + (X(\beta - 1))\eta(Y)\phi Z \\
+ (X\beta)\eta(Z)\phi Y - (Y(\beta - 1))\eta(X)\phi Z - (Y\beta)\eta(Z)\phi X.
\]

(4.2)

By making use of (2.3), (2.6) and (2.7) in (4.2), we have

\[
\bar{R}(X,Y)Z = R(X,Y)Z + \beta^2 g(\phi Y, Z)\phi X - \beta^2 g(\phi X, Z)\phi Y \\
+ 2\beta(\beta - 1)g(\phi Y, X)\phi Z + \beta(\beta - 1)\eta(Y)g(X, Z)\xi \\
- \beta(\beta - 1)\eta(X)g(Y, Z)\xi - \beta^2\eta(Y)\eta(Z)X \\
+ \beta^2\eta(X)\eta(Z)Y + (X(\beta - 1))\eta(Y)\phi Z \\
+ (X\beta)\eta(Z)\phi Y - (Y(\beta - 1))\eta(X)\phi Z - (Y\beta)\eta(Z)\phi X.
\]

(4.3)

So the equation (4.3) turns into

\[
\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z.
\]

Moreover in view of (4.3) it follows that

\[
R(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0.
\]

Let \( \{e_1, e_2, e_3\} \) be a local orthonormal basis of the tangent space at a point of the manifold \( M \). Then by putting \( X = U = e_i \) in (4.3) and taking summation over \( i \), \( 1 \leq i \leq 3 \) and also using (2.3), we get

\[
\bar{S}(Y,Z) = S(Y,Z) + \beta(2\beta - 1)g(Y, Z) - \beta(4\beta - 1)\eta(Y)\eta(Z) \\
+ (\phi Z)(\beta)\eta(Y) + (\phi Y)(\beta)\eta(Z),
\]

(4.4)

where \( \bar{S} \) and \( S \) denote the Ricci tensor of \( M \) with respect to \( \bar{\nabla} \) and \( \nabla \) respectively.

Again contracting \( Y \) and \( Z \) in the above equation (4.4) we have

\[
\bar{r} = r - 2\beta + 2\beta^2.
\]

(4.5)

Summing up all of the above equations we can state the following proposition:

**Proposition 4.1.** In a 3-dimensional quasi-Sasakian manifold \( M \) admitting the quarter-symmetric non-metric \( \xi \)-connection \( \bar{\nabla} \):

(i) The curvature tensor \( \bar{\bar{R}} \) is given by

\[
\bar{\bar{R}}(X,Y)Z = R(X,Y)Z + \beta^2 g(\phi Y, Z)\phi X - \beta^2 g(\phi X, Z)\phi Y + 2\beta(\beta - 1)g(\phi Y, X)\phi Z \\
+ \beta(\beta - 1)\eta(Y)g(Y, Z)\xi - \beta(\beta - 1)\eta(X)g(Y, Z)\xi - \beta^2\eta(Y)\eta(Z)X + \beta^2\eta(X)\eta(Z)Y \\
+ (X(\beta - 1))\eta(Y)\phi Z + (X\beta)\eta(Z)\phi Y - (Y(\beta - 1))\eta(X)\phi Z - (Y\beta)\eta(Z)\phi X,
\]
(ii) The Ricci tensor $\bar{S}$ is given by
$$\bar{S}(Y, Z) = S(Y, Z) + \beta(2\beta - 1)g(Y, Z) - \beta(4\beta - 1)\eta(Y)\eta(Z) + (\phi Z)(\beta)\eta(Y) + (\phi Y)(\beta)\eta(Z),$$

(iii) $\bar{R}(X, Y)Z = -\bar{R}(Y, X)Z$,

(iv) $\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0$,

(v) $\bar{r} = r - 2\beta + 2\beta^2$.

Moreover, if the structure function $\beta$ is constant, then we conclude the following:

**Corollary 4.2.** In a 3-dimensional non-cosymplectic quasi-Sasakian manifold $M$ with constant structure function admitting the quarter-symmetric non-metric $\xi$-connection $\bar{\nabla}$:

(i) The curvature tensor $\bar{R}$ is given by
$$\bar{R}(X, Y)Z = R(X, Y)Z + \beta^2g(\phi Y, Z)\phi X - \beta^2g(\phi X, Z)\phi Y + 2\beta(\beta - 1)g(\phi Y, X)\phi Z + \beta(\beta - 1)\eta(Y)g(X, Z)\xi - \beta(\beta - 1)\eta(X)g(Y, Z)\xi - \beta^2\eta(Y)\eta(Z)X + \beta^2\eta(X)\eta(Z)Y,$$

(ii) The Ricci tensor $\bar{S}$ is given by
$$\bar{S}(Y, Z) = S(Y, Z) + \beta(2\beta - 1)g(Y, Z) - \beta(4\beta - 1)\eta(Y)\eta(Z),$$

(iii) $\bar{R}(X, Y)Z = -\bar{R}(Y, X)Z$,

(iv) $\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0$,

(v) $g(\bar{R}(X, Y)Z, U) = -g(\bar{R}(X, Y)U, Z)$,

(vi) $\bar{r} = r - 2\beta + 2\beta^2$.

5 Projective curvature tensor on 3-dimensional non-cosymplectic quasi-Sasakian manifolds with respect to the quarter-symmetric non-metric $\xi$-connection $\nabla$

In this section we characterize $\xi$-projectively flat and $\phi$-projectively flat 3-dimensional non-cosymplectic quasi-Sasakian manifold with respect to the quarter-symmetric non-metric $\xi$-connection $\nabla$.

After the conformal curvature tensor, the projective curvature tensor is an important tensor from the differential geometric point of view. Let $M$ be a $(2n + 1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of $M$ and a domain in the Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then $M$ is said to be locally projectively flat. For $n \geq 1$, $M$ is locally projectively flat if and only if the well-known projective curvature tensor $P$
vanishes. The projective curvature tensor is defined by \[ \text{(5.1)} \]
\[ P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y], \]
where \( S \) is the Ricci tensor of \( M \).

Let \( M \) be an almost contact metric manifold equipped with an almost contact metric structure \((\phi, \xi, \eta, g)\). At each point \( p \in M \), we decompose the tangent space \( T_pM \) into direct sum \( T_pM = \phi(T_pM) \oplus \{\xi_p\} \), where \( \{\xi_p\} \) is the 1-dimensional linear subspace of \( T_pM \) generated by \( \{\xi_p\} \). Thus the conformal curvature tensor \( C \) is a map
\[ C : T_pM \times T_pM \times T_pM \longrightarrow \phi(T_pM) \oplus \{\xi_p\}, \quad p \in M. \]

It may be natural to consider the following particular cases:

1. \( (1) \quad C : T_pM \times T_pM \times T_pM \longrightarrow \{\xi_p\} \), i.e., the projection of the image of \( C \) in \( \phi(T_pM) \) is zero.

2. \( (2) \quad C : T_pM \times T_pM \times T_pM \longrightarrow \phi(T_pM) \), i.e., the projection of the image of \( C \) in \( \{\xi_p\} \) is zero. This condition is equivalent to
\[ \text{(5.2)} \quad C(X, Y)\xi = 0. \]

3. \( (3) \quad C : \phi(T_pM) \times \phi(T_pM) \times \phi(T_pM) \longrightarrow \{\xi_p\} \), i.e., when \( C \) is restricted to \( \phi(T_pM) \times \phi(T_pM) \times \phi(T_pM) \), the projection of the image of \( C \) in \( \phi(T_pM) \) is zero. This condition is equivalent to
\[ \text{(5.3)} \quad \phi^2C(\phi X, \phi Y)\phi Z = 0. \]

A \( K \)-contact manifold satisfying \( (5.2) \) and \( (5.3) \) are called \( \xi \)-conformally flat and \( \phi \)-conformally flat respectively. A \( K \)-contact manifold satisfying the cases \( (1), (2) \) and \( (3) \) are considered in [29], [30] and [31] respectively.

In an analogous way we define projective curvature tensor \( \tilde{P} \) on 3-dimension non-cosymplectic quasi-Sasakian manifold with respect to a special type of quarter-symmetric non-metric \( \xi \)-connection \( \tilde{\nabla} \), by
\[ \text{(5.4)} \quad \tilde{P}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{2}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y]. \]

**Definition 5.1.** A 3-dimension quasi-Sasakian manifold \( M \) with respect to the quarter-symmetric non-metric \( \xi \)-connection \( \tilde{\nabla} \) is said to be \( \xi \)-projectively flat if the condition \( \tilde{P}(X, Y)\xi = 0 \) holds.

So using \( (4.3), (4.4) \) and \( (2.3) \) the equation \( (5.4) \) becomes
\[ \text{(5.5)} \quad \tilde{P}(X, Y)Z = P(X, Y)Z - \beta(\beta - 1)\eta(Y)(Z)g(Y, Z)X + \beta(\beta - 1)\eta(Y)g(X, Z)\xi + \beta^2\eta(Y)g(X, Z)\phi X - \beta^2\eta(Y)\phi X - 2\beta(\beta - 1)g(\phi X, Y)\phi Z + \beta^2\eta(Y)\eta(Z)X - \beta(2\beta - 1)g(Y, Z)X - \beta(2\beta - 1)g(X, Z)Y + \beta(4\beta - 1)\eta(X)\eta(Z)Y - \beta(4\beta - 1)\eta(Y)\eta(Z)X, \]
where \( P(X,Y)Z = R(X,Y)Z - \frac{1}{2}[S(Y,Z)X - S(X,Z)Y] \) is the projective curvature tensor with respect to the Levi-Civita connection on a 3–dimensional quasi-Sasakian manifold.

Putting \( Z = \xi \) in equation (5.5) and using (2.3), we conclude that

\[
P(X,Y)\xi = \bar{P}(X,Y)\xi.
\]

Thus, we can state the following theorem:

**Theorem 5.1.** A 3–dimensional non-cosymplectic quasi-Sasakian manifold \( M \) with constant structure function admitting the quarter-symmetric non-metric \( \xi \)-connection \( \bar{\nabla} \) is \( \xi \)-projectively flat if and only if the Levi-Civita connection \( \nabla \) is so.

**Definition 5.2.** A 3–dimensional quasi-Sasakian manifold \( M \) with respect to the quarter-symmetric non-metric \( \xi \)-connection \( \bar{\nabla} \) is said to be \( \phi \)-projectively flat if it satisfies the condition

\[
g(\bar{P}(\phi X, \phi Y)\phi Z, \phi U) = 0.
\]

From (5.4), it follows that

\[
\bar{P}(X,Y,Z,U) = \bar{R}(X,Y,Z,U) - \frac{1}{2}[\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U)],
\]

where \( \bar{P}(X,Y,Z,U) = g(\bar{P}(X,Y)Z,U), \) for \( X, Y, Z, U \in \chi(M). \)

Putting \( X = \phi X, Y = \phi Y, Z = \phi Z \) and \( U = \phi U \) in (5.7), it is obvious that

\[
\bar{P}(\phi X, \phi Y, \phi Z, \phi U) = \bar{R}(\phi X, \phi Y, \phi Z, \phi U) - \frac{1}{2}[\bar{S}(\phi Y, \phi Z)g(\phi X, \phi U) - \bar{S}(\phi X, \phi Z)g(\phi Y, \phi U)].
\]

Let \( \{e_1, e_2, \xi\} \) be a local orthonormal basis of vector fields in \( M \). Then \( \{\phi e_1, \phi e_2, \xi\} \) is also a local orthonormal basis. Putting \( X = \phi = e_i \) in (5.8), taking summation over \( i, \ 1 \leq i \leq 2 \) and also using (2.3), we get

\[
\bar{P}(\phi e_i, \phi Y, \phi Z, \phi e_i) = \bar{R}(\phi e_i, \phi Y, \phi Z, \phi e_i) - \frac{1}{2}[\bar{S}(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - \bar{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i)].
\]

In view of (5.6) and (5.9) and using (2.3), we obtain

\[
\bar{S}(\phi Y, \phi Z) = 0.
\]

Combining (2.9), (4.4) and (5.10) and using (2.3), we have

\[
S(Y, Z) - 2\beta^2 \eta(Y) \eta(Z) + \beta(2\beta - 1)g(\phi Y, \phi Z) = 0.
\]
By making use of (2.1), (2.2) and (2.3) in (5.11), we can write

\[(5.12) \quad S(Y, Z) - \beta(4\beta - 1)\eta(Y)\eta(Z) + \beta(2\beta - 1)g(Y, Z) = 0.\]

Interchanging \(Y\) and \(Z\) in (5.12) and using (2.7), it follows that

\[(5.13) \quad S(Y, Z) - \beta(4\beta - 1)\eta(Y)\eta(Z) + \beta(2\beta - 1)g(Y, Z) = 0.\]

Adding the equations (5.12) and (5.13), we conclude that

\[S(Y, Z) = \beta(4\beta - 1)\eta(Y)\eta(Z) - \beta(2\beta - 1)g(Y, Z),\]

which means that a \(\phi\)-projectively flat 3-dimensional quasi-Sasakian manifold with constant structure function admitting the quarter-symmetric non-metric \(\xi\)-connection \(\bar{\nabla}\) is an \(\eta\)-Einstein manifold with respect to the Levi-Civita connection.

In view of the above discussions we state the following theorem:

**Theorem 5.2.** A \(\phi\)-projectively flat 3-dimensional non-cosymplectic quasi-Sasakian manifold \(M\) with constant structure function admitting the quarter-symmetric non-metric \(\xi\)-connection \(\bar{\nabla}\) is an \(\eta\)-Einstein manifold with respect to the Levi-Civita connection \(\nabla\).

6 The second order symmetric parallel tensor

**Definition 6.1.** A tensor \(\alpha\) of second order is said to be a parallel tensor if \(\nabla\alpha = 0\), where \(\nabla\) denotes the operator of covariant differentiation with respect to the metric tensor \(g\).

Let \(\bar{\alpha}\) be a \((0,2)\)-symmetric tensor field on a 3-dimensional non-cosymplectic quasi-Sasakian manifold with respect to the quarter-symmetric non-metric \(\xi\)-connection \(\bar{\nabla}\) such that \(\bar{\nabla}\alpha = 0\). Then it follows that

\[(6.1) \quad \bar{\alpha}(\bar{R}(W, X)Y, Z) + \bar{\alpha}(Y, \bar{R}(W, X)Z) = 0,\]

for all vector fields \(W, X, Y, Z\).

Substitution of \(W = Y = Z = \xi\) in (6.1) yields

\[(6.2) \quad \bar{\alpha}(R(\xi, X)\xi, \xi) = 0,\]

since \(\bar{\alpha}\) is symmetric. Using (4.3) in (6.2) we have

\[(6.3) \quad \bar{\alpha}(\xi, \xi) = 0.\]

Differentiating (6.3) covariantly along \(X\), we get

\[(6.4) \quad \bar{\alpha}(\phi X, \xi) = 0.\]
Substituting \( X \) by \( \phi X \) in (6.4) yields
\[
\bar{\alpha}(X, \xi) = 0.
\]
Again differentiating (6.4) covariantly along \( Y \) and making use of (6.3) and (6.4), we get
\[
\bar{\alpha}(\phi X, \phi Y) = 0.
\]
Substitution of \( X = \phi X \) and \( Y = \phi Y \) in (6.6) yields
\[
\bar{\alpha}(X, Y) = 0.
\]
In view of (6.7) we can state the following:

**Theorem 6.1.** On a 3-dimensional non-cosymplectic quasi-Sasakian manifold there does not exist a nonzero symmetric parallel tensor of second order with respect to a quarter-symmetric non-metric \( \xi \)-connection.

### 7 Example

In this section, we give an example of a 3-dimensional quasi-Sasakian manifold \( M \) with constant structure function admitting the quarter-symmetric non-metric \( \xi \)-connection \( \bar{\nabla} \).

We consider the 3-dimensional manifold \( \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq 0\} \), where \((x, y, z)\) are the standard coordinates in \( \mathbb{R}^3 \).

We choose the vector fields
\[ e_1 = \frac{\partial}{\partial z} - y \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = 2 \frac{\partial}{\partial x}, \]
which are linearly independent at each point of \( M \).

Let \( g \) be the Riemannian metric defined by
\[
g(e_i, e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j; \ i, j = 1, 2, 3. \end{cases}
\]

Let \( \eta \) be the 1-form defined by
\[
\eta(Z) = g(Z, e_3),
\]
for any \( Z \in \chi(M) \).

Let \( \phi \) be the \((1,1)\)-tensor field defined by
\[
\phi(e_1) = -e_2, \ \phi(e_2) = e_1, \ \phi(e_3) = 0.
\]

Using the linearity of \( \phi \) and \( g \), we have
\[
\eta(e_3) = 1, \ \phi^2 Z = -Z + \eta(Z)e_3
\]
and
\[
g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),
\]
for any vector fields $Z, U \in \chi(M)$. Thus for $e_3 = \xi$, the structure $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$.

Then we have

$$[e_1, e_2] = \frac{1}{2} e_3, [e_1, e_3] = 0, [e_2, e_3] = 0. \quad (7.1)$$

The Levi-Civita connection $\nabla$ of the metric tensor $g$ is given by Koszul’s formula

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using Koszul’s formula, we get the following:

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = \frac{1}{4} e_3, \quad \nabla_{e_1} e_3 = \frac{1}{4} e_2,$$

$$\nabla_{e_2} e_1 = \frac{1}{4} e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = -\frac{1}{4} e_1,$$

$$\nabla_{e_3} e_1 = \frac{1}{4} e_2, \quad \nabla_{e_3} e_2 = -\frac{1}{4} e_1, \quad \nabla_{e_3} e_3 = 0.$$

In view of the above relations, we see that

$$\nabla_X \xi = -\beta \phi X, \quad (\nabla_X \phi) Y = \beta g(X, Y) \xi - \eta(Y) X.$$

Therefore, the manifold is a 3-dimensional quasi-Sasakian manifold $M$ with constant structure function $\beta = \frac{1}{4}$.

Using (3.1) in above equations, we obtain

$$\bar{\nabla}_{e_1} e_1 = 0, \quad \bar{\nabla}_{e_1} e_2 = -\frac{1}{4} e_3, \quad \bar{\nabla}_{e_1} e_3 = 0,$$

$$\bar{\nabla}_{e_2} e_1 = \frac{1}{4} e_3, \quad \bar{\nabla}_{e_2} e_2 = 0, \quad \bar{\nabla}_{e_2} e_3 = 0,$$

$$\bar{\nabla}_{e_3} e_1 = e_2, \quad \bar{\nabla}_{e_3} e_2 = -e_1, \quad \bar{\nabla}_{e_3} e_3 = 0.$$

The above arguments tell us that $M$ is a 3–dimensional quasi-Sasakian manifold with constant structure function admitting a type of quarter-symmetric non-metric $\xi$-connection $\bar{\nabla}$.

The expressions of the curvature tensor with respect to $\nabla$ are:

$$R(e_1, e_2) e_3 = 0, \quad R(e_2, e_3) e_3 = \frac{1}{16} e_2, \quad R(e_1, e_3) e_3 = \frac{1}{16} e_1,$$

$$R(e_1, e_2) e_2 = -\frac{3}{16} e_1, \quad R(e_2, e_3) e_2 = -\frac{1}{16} e_3, \quad R(e_1, e_3) e_2 = 0,$$

$$R(e_1, e_2) e_1 = \frac{3}{16} e_2, \quad R(e_2, e_3) e_1 = 0, \quad R(e_1, e_3) e_1 = -\frac{1}{16} e_3.$$

From the above expressions the non-zero components of the Ricci tensor with respect to $\nabla$ are given by

$$S(e_1, e_1) = -\frac{1}{8}, \quad S(e_2, e_2) = -\frac{1}{8}, \quad S(e_3, e_3) = \frac{1}{8}.$$
Similarly, the expressions of the curvature tensor with respect to $\nabla$ are:

$$
\bar{R}(e_1, e_2)e_3 = 0, \quad \bar{R}(e_1, e_3)e_3 = 0, \\
\bar{R}(e_1, e_2)e_1 = -\frac{1}{2} e_2, \quad \bar{R}(e_1, e_2)e_2 = 0, \quad \bar{R}(e_2, e_3)e_3 = 0, \\
R(e_2, e_1)e_1 = 0.
$$

From the above expressions the components of the Ricci tensor with respect to $\nabla$ are given by

$$
\bar{S}(e_1, e_1) = 0, \quad \bar{S}(e_2, e_2) = 0, \quad \bar{S}(e_3, e_3) = 0.
$$

In view of the above equations we can easily obtain

$$
P(e_i, e_j)e_3 = 0 = \bar{P}(e_i, e_j)e_3,
$$

for all $1 \leq i, j \leq 3$. Therefore theorem 5.1 is verified.

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**References**


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