On (ε)-almost paracontact metric manifolds with conformal η-Ricci solitons

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Abstract. This paper is to study (ε)-almost paracontact metric manifolds bearing conformal η-Ricci solitons with the characteristic vector field ξ. Moreover, we consider these manifolds whose the potential vector field is torse-forming and deduced some results related to the parallelity on (ε)-para Sasakian manifolds. Finally, the existence of expanding and shrinking conformal η-Ricci solitons in such manifolds are ensured by an illustrative example.

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1 Introduction

The notion of Ricci soliton which is a natural generalization of an Einstein metric (the Ricci tensor S is a constant multiple of g) was introduced by Hamiltom [15] in 1982. A pseudo Riemannian manifold (M, g) is called a Ricci soliton if it admits a smooth vector field V (potential vector field) on (M, g) such that

$$\frac{1}{2}(\ell_V g) + S(X,Y) + \lambda\, g(X,Y) = 0, \tag{1.1}$$

where ℓ_V denotes the Lie-derivative in the direction V. λ is a constant and X, Y are arbitrary vector fields on M. A Ricci soliton is said to be shrinking, steady or expanding according to λ being negative ,zero or positive respectively. It is obvious that a trivial Ricci soliton is an Einstein manifold with V zero or Killing vector field. Since Ricci solitons are the fixed points of the Ricci flow, they are important in understanding Hamilton's Ricci flow [14]. When V is the gradient of a potential function $\varphi \in C^\infty(M)$, the soliton (g, V, λ) is called a gradient Ricci soliton [21] and the equation (1) takes the form

$$\nabla\nabla\varphi = S + \lambda g, \tag{1.2}$$

where ∇ represents the Levi-Civita connection of the metric g on M. Both equations (1.1) and (1.2) can be considered as perturbations of the Einstein equation $S = \lambda g$

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and reduce to this latter in case V or $\nabla\varphi$ are Killing vector fields. When $V = 0$ or φ is constant we call the underlying Einstein manifold a trivial Ricci soliton. It is well known that, if the potential vector filed φ is zero or Killing then the Ricci soliton is an Einstein real hypersurfaces on non fat complex-space-forms.
In 2009, J. T. Cho and M. Kimura [10] introduced the notion of η-Ricci soliton and gave a classification of real hypersurfaces in non at complex-space-forms admitting η-Ricci soliton $\frac{\partial}{\partial t}g_{ij} = -2S_{ij}$, viewed as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scaling. In differential geometry, the Ricci flow is an intrinsic geometric flow. It can be viewed as a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat, smoothing out the irregularities in the metric. Geometric flows, especially Ricci flows, have become important tools in theoretical physics. Ricci soliton is known as quasi Einstein metric in physics literature [13] and the solutions of the Einstein field equations correspond to Ricci soliton [1]. Relation with the string theory and the fact that (1.1) is a particular case of Einstein field equation makes the equation of Ricci soliton interesting in theoretical physics. In spite of introducing and studying firstly in Riemannian geometry, the Ricci soliton equation has recently been investigated in pseudo-Riemannian context, especially in Lorentzian case ([3], [7], [9], [18]).
The concept of η-Ricci soliton was initiated by Cho and Kimura [10]. For a given 1-form η, an η-Ricci soliton is a data (g, V, λ, μ) on a pseudo-Riemannian manifold (M, g) satisfying

$$\frac{1}{2}(\ell_V g) + S(X, Y) + \lambda\, g(X, Y) + \mu\eta \otimes \eta(X, Y) = 0, \tag{1.3}$$

where V is a vector field, ℓ_V denotes the Lie-derivative in the direction V, S stands for the Ricci tensor field, λ and μ are constants and X, Y are arbitrary vector fields on M. In [8] the authors studied $\eta-$Ricci soliton on Hopf hypersurfaces in complex space forms. In the context of paracontact geometry $\eta-$Ricci soliton were investigated in ([3], [4], [5]).

In [12], A. E. Fischer introduced a new concept called conformal Ricci flow, which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. Since the conformal geometry plays an important role to constrain the scalar curvature and the equations is the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equations are named as the conformal Ricci flow equations. These new equations are given by

$$\frac{\partial g}{\partial t} = -2S - \left(p + \frac{2}{n}\right) g, \tag{1.4}$$

where $R(g) = -1$ and p is a non-dynamical scalar field (time dependent scalar field), $R(g)$ is the scalar curvature of the manifold and n is the dimension of the manifold M. The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics and because of this analogy the time dependent scalar field p is called a conformal pressure and, as for the real physical pressure in fluid mechanics that serves to maintain the incompressibility of the fluid, the conformal pressure serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint. The equilibrium points of the conformal Ricci flow equations are Einstein metrics with Einstein constant $\frac{-1}{n}$. Thus the conformal pressure p is zero at an equilibrium point otherwise it will be positive.

In 2015, N. Basu and A. Bhattacharyya [2] introduced the notion of conformal Ricci soliton and the equation is follows as

$$\ell_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g = 0. \tag{1.5}$$

It is an interesting and natural to see the condition in case of conformal η-Ricci soliton. Dutta et al. [11] also studied the properties of conformal Ricci soliton in Lorentzian α-Sasakian manifolds. From equations (1.3) and (1.5) we are introducing the notion of conformal η-Ricci soliton by the following equation

$$\ell_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0. \tag{1.6}$$

In particular, if $\mu = 0$ then the data (g, V, λ) is a conformal Ricci soliton [2]. Thus we can say that the conformal η-Ricci soliton is a generalization of conformal Ricci soliton.

The concept of almost Ricci soliton was first introduced by S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti in 2010 [25]. R. Sharma has also done excellent work in almost Ricci soliton [24]. A Riemannian manifold (M, g) is an almost Ricci soliton [24] if there exists a complete vector field X and a smooth soliton function λ such that $\lambda : M \to \Re$ satisfying

$$R_{ij} = \frac{1}{2}(X_{ij} + X_{ji}) = \lambda g_{ij}, \tag{1.7}$$

where R_{ij} and $X_{ij} + X_{ji}$ stands for the Ricci tensor and the Lie derivative $\ell_X g$ in local coordinates respectively. It will called shrinking, steady or expanding according as $\lambda < 0$; $\lambda = 0$ or $\lambda > 0$, respectively. The notion of η-Ricci soliton has been studied by A. M. Blaga ([3], [4]) and many others.

Therefore, motivated by these studies in the present paper we are going to study the notion of conformal η-Ricci soliton on (ε)-almost paracontact metric manifold and deduced some its geometrical results.

A data (g, V, λ, μ)on (M, g) is said to be almost conformal η-Ricci soliton if it satisfies equation (1.6), where $\lambda : M \to \Re$ is a smooth function.

In 1976, Sato [20] introduced the almost paracontact structure as a triple (ϕ, ξ, η) of a $(1, 1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying $\phi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$. The structure is an analogue of the almost contact structure [19] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well.

In 1989, Matsumoto [18] replaced the structure vector field ξ by $-\xi$ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it a Lorentzian almost paracontact manifold.

An (ε)-Sasakian manifold is always odd-dimensional. On the other hand, in a Lorentzian almost paracontact manifold given by Matsumoto, the pseudo-Riemannian metric has only index 1 and the structure vector field ξ is always timelike. These circumstances motivated the authors of [26] to associate a pseudo-Riemannian metric, not necessarily Lorentzian, with an almost para-contact structure, and this indefinite almost paracontact $\varepsilon = 1$ metric structure was called an (ε)-almost paracontact structure, where the structure vector field ξ is spacelike or timelike according as $\varepsilon = 1$ or $\varepsilon = -1$ [27].

2 Preliminaries

Let M be an n-dimensional manifold equipped with an almost paracontact structure (ϕ, ξ, η) [20] consisting of a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying

$$\phi^2 = I - \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\phi\xi = 0, \tag{2.3}$$

$$\eta \circ \phi = 0. \tag{2.4}$$

It is easy to verify that (2.1) and one of (2.2), (2.3) and (2.4) imply the other two equations. If g is a pseudo-Riemannian metric such that

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), X, Y \in (\Gamma M). \tag{2.5}$$

where $\varepsilon = \pm 1$, then M is called (ε)-almost paracontact metric manifold equipped with an (ε)-almost paracontact metric structure $(\phi, \xi, \eta, g, \varepsilon)$ [26]. In particular, if $\operatorname{index}(g) = 1$, that is when g is a Lorentzian metric, then the (ε)-almost paracontact metric manifold is called Lorentzian almost paracontact manifold. From (2.5) we have

$$g(X, \xi) = \varepsilon\, \eta(X), \tag{2.6}$$

$$g(X, \phi Y) = g(\phi X, Y), \tag{2.7}$$

for all $X, Y \in (\Gamma M)$.From (2.6) it follows that

$$g(\xi, \xi) = \varepsilon\,, \tag{2.8}$$

that is, the structure vector field ξ is never lightlike.
Let $(\phi, \xi, \eta, g, \varepsilon)$ be an (ε)-almost paracontact metric manifold (resp.a Lorentzian almost paracontact manifold). If $\varepsilon = 1$, then M is said to be a spacelike (ε)-almost paracontact metric manifold (resp. a spacelike Lorentzian almost paracontact manifold). Similarly, if $\varepsilon = -1$, then M is said to be a timelike (ε)-almost paracontact metric manifold (resp. a timelike Lorentzian almost paracontact manifold) [26].

An (ε)-almost paracontact metric structure $(\phi, \xi, \eta, g, \varepsilon)$ is called (ε)-para Sasakian structure if

$$(\nabla_X \phi) = -g(\phi X, \phi Y)\xi - \varepsilon\, \eta(Y)\phi^2 X, \tag{2.9}$$

where ∇ is the Levi-Civita connection with respect to g. A manifold endowed with an (ε)-para Sasakian structure is called (ε)-para Sasakian manifold [26].
In an (ε)-para Sasakian manifold, we have

$$\nabla\xi = \varepsilon\, \phi, \tag{2.10}$$

and the Riemann curvature tensor R and the Ricci tensor S satisfy the following equations [26]:

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.11}$$

$$R(\xi, X)Y = -\varepsilon\, g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$\eta(R(X, Y)Z) = -\varepsilon\eta(X)g(Y, Z) + \varepsilon\eta(Y)g(X, Z), \tag{2.13}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.14}$$

for all $X, Y, Z \in (\Gamma M)$.

3 Conformal η-Ricci solitons on Einstein-like (ε)-almost paracontact metric manifolds

Analogous to Einstein-like para Sasakian manifolds [21], we introduce the following definition.
Definition 3.1 *An (ε)-almost paracontact metric manifold $(\phi, \xi, \eta, g, \varepsilon)$ is said to be Einstein-like if its Ricci tensor S satisfies*

$$S(X, Y) = \alpha g(X, Y) + \beta g(\phi X, Y) + \gamma\eta(X)\eta(Y), X, Y \in (\Gamma M). \tag{3.1}$$

for some real constants α, β and γ.

Proposition 3.1 *An (ε)-para Sasakian manifold $(M, \phi, \xi, \eta, g, \varepsilon)$ bearing conformal η- Ricci soliton (g, ξ, λ, μ) is an Einstein-like manifold.*
Proof. From the definition of Lie derivative and using (2.6), (2.8) and (2.10), we have

$$(\ell_\xi g) = \{g(\nabla_X \xi) + g(\nabla_Y \xi, X)\}$$

$$= 2\varepsilon g(\phi X, Y), \forall\, X, Y \in (\Gamma M). \tag{3.2}$$

Putting the above value in (1.6) we get

$$S(X, Y) = \alpha g(X, Y) + \beta g(\phi X, Y) + \gamma\eta(X)\eta(Y), \tag{3.3}$$

where $\alpha = -\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right]$, $\beta = -2\varepsilon$, $\gamma = -2\mu$.
Thus this completes the proof.
Proposition 3.2 *In an Einstein-like (ε)-para Sasakian manifold $(M, \phi, \xi, \eta, g, \varepsilon, \alpha, \beta, \lambda)$ the following relations are held*
(i) $S(\phi X, Y) \neq S(X, \phi Y)$,
(ii) $S(\phi X, \phi Y) = S(X, Y) - (\alpha\varepsilon + \gamma)\eta(X)\eta(Y)$,
(iii) $S(X, \xi) = (\alpha\varepsilon + \gamma)\eta(X)$,
(iv) $S(\xi, \xi) = (\alpha\varepsilon + \gamma)$,
(v) $(\nabla_X S)(Y, Z) = \beta g((\nabla_X \phi)Y, Z) + \gamma\varepsilon\{\eta(Z)g(\nabla_X \xi, Y) + \eta(Y)g(\nabla_X \xi, Z)\}$,
(vi) $(\nabla_X Q)Y = \beta(\nabla_X \phi)Y + \gamma\varepsilon\{\eta(Y)\nabla_X \xi + \varepsilon\, g(\nabla_X \xi, Y)\xi\}$,
(vii) $\alpha\varepsilon + \gamma = 1 - n$,
(viii) $r = \alpha n + \beta\, trace\ (\phi) + \varepsilon\,\gamma$,
where r is the scalar curvature and S is the Ricci operator defined by $g(QX, Y) = S(X, Y)$,$X, Y \in (\Gamma M)$.

We suppose that $(M, \phi, \xi, \eta, g, \varepsilon, \alpha, \beta, \lambda)$ be an Einstein-like (ε)-para contact metric manifold bearing conformal η- Ricci soliton(g, ξ, λ, μ). Then from (1.6), we have

$$(\ell_V g) + 2S + \left\{2\lambda - \left(p + \frac{2}{n}\right)\right\} g + 2\mu\eta \otimes \eta = 0, \tag{3.4}$$

with λ and μ real constant. In view of (3.1) and (3.4), we get

$$\begin{aligned} &g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) + 2\beta g(\phi X, Y) \\ &\qquad + (2\gamma + 2\mu)\eta(X)\eta(Y) + \left\{2\lambda + 2\alpha - \left(p + \tfrac{2}{n}\right)\right\} g(X, Y) = 0, \end{aligned} \tag{3.5}$$

for all $X, Y \in (\Gamma M)$. If we replace $X = Y = \xi$ in (3.5), we obtain

$$(\gamma + \mu) + \varepsilon\left\{(\lambda + \alpha) - \frac{1}{2}\left(p + \frac{2}{n}\right)\right\} = 0. \tag{3.6}$$

Again restricting $Y = \xi$ in (3.5) and using (3.6), we get $\nabla_\xi \xi = 0$. Thus we easily prove that

$$g(\nabla_\xi \phi)\xi = 0 \text{ and } \nabla_\xi \eta = 0.$$

The above relations take into account with the case (v) and (vi) in Proposition3.2, we get the desired result.

$$(\nabla_\xi S)(Y, Z) = \beta g((\nabla_\xi \phi)Y, Z) \text{ and } \nabla_\xi Q = \beta(\nabla_\xi \phi).$$

So we give the following result.

Proposition 3.3 *Let $(M, \phi, \xi, \eta, g, \varepsilon, \alpha, \beta, \lambda)$ be an Einstein-like (ε)-almost paracontact metric manifold admitting conformal η-Ricci soliton (g, ξ, λ, μ) then*
(i) $(\gamma + \mu) + \varepsilon\left\{(\lambda + \alpha) - \frac{1}{2}\left(p + \frac{2}{n}\right)\right\} = 0$,
(ii) ξ is a geodesic vector field,
(iii) $g(\nabla_\xi \phi)\xi = 0$ and $\nabla_\xi \eta = 0$,
(iv) $(\nabla_\xi S)(Y, Z) = \beta g((\nabla_\xi \phi)Y, Z)$ and $\nabla_\xi Q = \beta(\nabla_\xi \phi)$,

(v) $\nabla_\xi S = 0$, and $\nabla_\xi Q = 0$, if the manifold is (ε)-para Sasakian.

In particular, if a vector field ξ is called torse-forming if

$$\nabla_X \xi = \psi X + \varpi(X)\xi, \tag{3.7}$$

is satisfied for some smooth function ψ and a 1-form ϖ. Taking inner product with ξ, we get

$$0 = g(\nabla_X \xi, \xi) = \varepsilon\{\psi\eta(X) + \varpi(X)\},$$

for all $X, Y \in (\Gamma M)$.
This implies

$$\varpi = -\psi\eta, \tag{3.8}$$

From (3.6) and (3.8) it follows that

$$\nabla_X \xi = \psi\phi^2 X, \tag{3.9}$$

Let $(M, \phi, \xi, \eta, g, \varepsilon, \alpha, \beta, \lambda)$ be an Einstein-like (ε)-paracontact metric manifold bearing conformal η-Ricci soliton (g, ξ, λ, μ) with the potential vector field ξ is torse-forming. Then substituting (3.9) in (3.5), it yields.

$$\begin{aligned}\psi\{g(\phi^2X, Y) + g(\phi^2Y, X)\} + 2\beta g(\phi X, Y)&\\ +(2\gamma + 2\mu)\eta(X)\eta(Y) + \left\{2\lambda + 2\alpha - \left(p + \tfrac{2}{n}\right)\right\} g(X, Y) &= 0,\end{aligned} \tag{3.10}$$

which implies

$$\beta\phi X = -\left\{\psi + \lambda + \alpha - \frac{1}{2}\left(p + \frac{2}{n}\right)\right\} X - (\gamma + \mu - \varepsilon\psi)\eta(X)\xi. \tag{3.11}$$

Thus we state the following result.

Theorem 3.1 *Let $(M, \phi, \xi, \eta, g, \varepsilon, \alpha, \beta, \lambda)$ be an Einstein-like (ε)-paracontact metric manifold bearing conformal η-Ricci soliton (g, ξ, λ, μ) with potential vector field ξ is torse vector field then M is an η-Einstein manifold.*

Besides, let M be an η-Einstein manifold (that is an Einstein-like (ε)-paracontact metric manifold with $\beta = 0$) bearing conformal η-Ricci soliton (g, ξ, λ, μ) with torse-forming potential vector field ξ. Then from (3.11) we get

$$\psi = \frac{1}{2}\left(p + \frac{2}{n}\right) - (\lambda + \alpha).$$

From (3.9), we can write

$$\nabla_X \xi = \left\{\frac{1}{2}\left(p + \frac{2}{n}\right) - (\lambda + \alpha)\right\}\phi^2 X, \tag{3.12}$$

Therefore, we have

$$R(X, Y)\xi = \left\{\frac{1}{2}\left(p + \frac{2}{n}\right) - (\lambda + \alpha)\right\}^2 \{\eta(X)Y - \eta(Y)X\}, \tag{3.13}$$

and

$$S(X,\xi)=\left\{\frac{1}{2}\left(p+\frac{2}{n}\right)-(\lambda+\alpha)\right\}^2(n-1)\eta(X), \tag{3.14}$$

In view of the case (iii) in Proposition 3.2 and (3.14), it yields

$$\gamma=\left\{\frac{1}{2}\left(p+\frac{2}{n}\right)-(\lambda+\alpha)\right\}^2(n-1)-\alpha\varepsilon, \tag{3.15}$$

Also from (3.5) and (3.15), we find

$$\mu=\left\{\frac{1}{2}\left(p+\frac{2}{n}\right)-(\lambda+\alpha)\right\}\left[\varepsilon-\left\{\frac{1}{2}\left(p+\frac{2}{n}\right)-(\lambda+\alpha)\right\}(n-1)\right]+\alpha\varepsilon. \tag{3.16}$$

Thus we have the following result.

Theorem 3.2 *Let $(M,\phi,\xi,\eta,g,\varepsilon,\alpha,0,\gamma)$ be an η-Einstein-like (ε)-paracontact metric manifold with conformal η- Ricci soliton (g,ξ,λ,μ) bearing torse-formaing potential vector field. Then ψ is constant function and*
(i) $\gamma=\left\{\frac{1}{2}\left(p+\frac{2}{n}\right)-(\lambda+\alpha)\right\}^2(n-1)-\alpha\varepsilon$,
(ii) $\mu=\left\{\frac{1}{2}\left(p+\frac{2}{n}\right)-(\lambda+\alpha)\right\}\left[\varepsilon-\left\{\frac{1}{2}\left(p+\frac{2}{n}\right)-(\lambda+\alpha)\right\}(n-1)\right]+\alpha\varepsilon$.

In particular if $\alpha=0$. Then we have the following result.

Theorem 3.3 *If $(M,\phi,\xi,\eta,g,\varepsilon,0,0,\gamma)$ be an n-dimensional $(n>1)$ non-Ricci flat η-Einstein (ε)-almost paracontact metric manifold bearing torse-formaing conformal Ricci soliton (g,ξ,λ) then the following relations hold*
(i) $\lambda=\frac{\varepsilon}{n-1}-\frac{1}{n}$ *(ii)* $\gamma=\left[\frac{2(n-1)-n\varepsilon}{n(n-1)}\right]^2(n-1)$.
Furthermore, the soliton is expanding (resp.shrinking) if the manifold is spacelike (resp. timelike).

Proposition 3.4 *In an η-Einstein-like (ε)-almost paracontact manifold admitting conformal η-Ricci soliton (g,ξ,λ,μ) with torse-formaning potential vector field, we have*
(i) $(\nabla_X S)(Y,Z)=\gamma\varepsilon\left\{\frac{1}{2}\left(p+\frac{2}{n}\right)-(\lambda+\alpha)\right\}$
$\{g(X,Y)\eta(Z)+g(X,Z)\eta(Y)-2\varepsilon\eta(X)\eta(Y)\eta(Z)\}$,
(ii) $(\nabla_X Q)Y=\gamma\left\{\frac{1}{2}\left(p+\frac{2}{n}\right)-(\lambda+\alpha)\right\}\{\varepsilon g(X,Y)\xi+\eta(Y)X-2\eta(X)\eta(Y)\xi\}$.

If the Ricci operator Q is Codazzi, then

$$(\nabla_X Q)Y=(\nabla_Y Q)X, \text{ for all } X,Y\in(\Gamma M).$$

In view of Proposition 3.4 and from above equation, we have

$$\gamma\left\{\tfrac{1}{2}\left(p+\tfrac{2}{n}\right)-(\lambda+\alpha)\right\}\{\eta(Y)X-\eta(X)Y\}=0, \text{ for all } X,Y\in(\Gamma M).$$

It is clear that $\psi=\frac{1}{2}\left(p+\frac{2}{n}\right)-(\lambda+\alpha)\neq 0$, then $\gamma=0$. Therefore M is an Einstein manifold. Thus from the above consequence we state the result as follows.

Theorem 3.4 *Let $(M,\phi,\xi,\eta,g,\varepsilon,\alpha,0,\gamma)$ be an η-Einstein (ε)-almost paracontact metric manifold admitting conformal η- Ricci soliton (g,ξ,λ,μ) with torse-formaing potential vector field. If $\psi\neq 0$ and the Ricci operator Q is Codazzi, then M is an Einstein manifold.*

4 Conformal η-Ricci soliton on (ε)-para Sasakian manifolds

Let $(M,\phi,\xi,\eta,g,\varepsilon)$ be an (ε)-para Sasakian manifold with conformal η- Ricci soliton (g,V,λ,μ) assuming that the potential vector field V is pointwise collinear with the structure field ξ, that is, $V=\kappa\xi$, for κ a smooth function on M.Therefore from (3.4), we get

$$(\ell_{\kappa\xi}g)+2S+\left\{2\lambda-\left(p+\frac{2}{n}\right)\right\}g+2\mu\eta\otimes\eta=0.$$

Applying the property of Lie derivative and Levi-Civita connection, we have

$$\begin{aligned}2\kappa g(\phi X,Y)&+\varepsilon(X\kappa)\eta(Y)+\varepsilon(Y\kappa)\eta(X)\\&+2S(X,Y)+\left\{2\lambda-\left(p+\tfrac{2}{n}\right)\right\}g(X,Y)+2\mu\eta(X)\eta(Y)=0,\end{aligned}\tag{4.1}$$

for all $X,Y\in(\Gamma M)$.
Substituting $Y=\xi$, in (4.1), using (2.14), we get

$$\begin{aligned}\varepsilon(X\kappa)&+\varepsilon(\xi\kappa)\eta(X)\\&-2(n-1)\eta(X)+\varepsilon\left\{2\lambda-\left(p+\tfrac{2}{n}\right)\right\}\eta(X)+2\mu\eta(X)=0,\end{aligned}\tag{4.2}$$

Again we replace $X=\xi$, in (4.2), we have

$$(\xi\kappa)=\varepsilon(n-1)-\left\{\lambda-\frac{1}{2}\left(p+\frac{2}{n}\right)\right\}-\varepsilon\mu,\tag{4.3}$$

Taking into account of (4.2) and (4.3), we obtain

$$(X\kappa)=\left[\varepsilon(n-1)-\left\{\lambda-\frac{1}{2}\left(p+\frac{2}{n}\right)\right\}-\varepsilon\mu\right]\eta(X),\tag{4.4}$$

It is clear that κ is constant if $\varepsilon(n-1)=\left\{\lambda-\frac{1}{2}\left(p+\frac{2}{n}\right)\right\}+\varepsilon\mu$. So from (4.1) we get

$$S(X,Y)=-\left\{\lambda-\frac{1}{2}\left(p+\frac{2}{n}\right)\right\}g(X,Y)-\kappa g(\phi X,Y)-\mu\eta(X)\eta(Y).$$

So we give the following result.

Theorem 4.1 *Let $(M,\phi,\xi,\eta,g,\varepsilon)$ be an (ε)-para Sasakian manifold with conformal η-Ricci soliton (g,V,λ,μ) and the potential vector field V is pointwise collinear with the structure field ξ, then V is constant multiple of ξ provided $\varepsilon(n-1)=\left\{\lambda-\frac{1}{2}\left(p+\frac{2}{n}\right)\right\}+\varepsilon\mu$ and M is an Einstein-like manifold.*

According to the hypothesis of Theorem 4.1, if $R(\xi,\cdot)\cdot S=0$, then V is constant multiple of ξ. Then we have

$$S(R(\xi,X),Z)+S(Y,R(\xi,X)Z=0,\text{ for all}X,Y,Z\in(\Gamma M).$$

In view of (2.10) and (2.12), the above equation reduces to

$$\begin{aligned}-\varepsilon g(X,Y)&\{\alpha\varepsilon\eta(Z)+\gamma\eta(Z)\}+\eta(Y)\{\alpha g(X,Z)\\&+\beta(g(\phi X,Z)+\gamma\eta(X)\eta(Z)\}-\varepsilon g(X,Z)\{\alpha\varepsilon\eta(Y)+\gamma\eta(Y)\}\\&+\eta(Z)\{\alpha g(X,Y)+\beta(g(\phi X,Y)+\gamma\eta(X)\eta(Y)\}=0,\end{aligned}\tag{4.5}$$

Taking $X = Y = Z = \xi$, in (4.5), for $V = \xi$, we get

$$\left\{2\lambda - \left(p + \frac{2}{n}\right)\right\} \varepsilon = 4\mu.$$

Thus we have the following result.

Theorem 4.2 *If $(M, \phi, \xi, \eta, g, \varepsilon)$ is an n-dimensional (ε)- para Sasakian manifold admitting conformal Ricci soliton (g, ξ, λ) then* we have

$$\lambda = \frac{\varepsilon}{2}\left(p + \frac{2}{n}\right).$$

Also, the soliton is expanding (resp.shrinking) if the manifold is spacelike (resp. timelike).

Remark 4.1 *If M is an Einstein-like (ε)-para Sasakian manifold and $V = \xi$, then the structure $\left(g, \xi, \frac{1}{2}\left(p + \frac{2}{n}\right) - \alpha, -\gamma\right)$ is the conformal η-Ricci soliton on M.*

Finally, we assume that Ω be a $(0, 2)$-tensor field is to be parallel with respect to Levi-Civita connection ∇, it means $\nabla\Omega = 0$, using the Ricci identity

$$\nabla^2\Omega(X, Y; Z, W) - \nabla^2\Omega(X, Y; W, Z) = 0,$$

In [23], we have

$$\Omega(R(X, Y)Z, W) + \Omega(R(X, Y)W, Z) = 0, \tag{4.6}$$

Using the symmetric property of Ω and taking $X = Y = Z = \xi$ in (4.6), we get

$$\Omega(R(X, Y)\xi, \xi) = 0. \tag{4.7}$$

With this reference we suppose that $(M, \phi, \xi, \eta, g, \varepsilon)$ Einstein-like (ε)-paracontact metric manifold with $\beta = 0)$ bearing conformal η-Ricci soliton (g, ξ, λ, μ) with torse-forming potential vector field ξ. Then from (3.13) and (4.7), we have

$$\left\{\frac{1}{2}\left(p + \frac{2}{n}\right) - (\lambda + \alpha)\right\}^2 \{\eta(X)\Omega(Y, \xi) - \eta(Y)\Omega(\xi, X)\} = 0. \tag{4.8}$$

Taking $X = \xi$ in (4.8), we yield

$$\left\{\frac{1}{2}\left(p + \frac{2}{n}\right) - (\lambda + \alpha)\right\}^2 \{\Omega(Y, \xi) - \eta(Y)\Omega(\xi, \xi)\} = 0.$$

From the above equation it is clear that if $\frac{1}{2}\left(p + \frac{2}{n}\right) \neq (\lambda + \alpha)$ then we have

$$\Omega(Y, \xi) = \eta(Y)\Omega(\xi, \xi) \tag{4.9}$$

So we can give the following result.

Theorem 4.3 *An Einstein-like (ε)-paracontact metric manifold $(M, \phi, \xi, \eta, g, \varepsilon)$ bearing conformal η-Ricci soliton (g, ξ, λ, μ) with torse-forming characteristic vector field is regular if $\frac{1}{2}\left(p + \frac{2}{n}\right) \neq (\lambda + \alpha)$.*

Corollary 4.3 *A symmetric parallel second order covariant tensor in a regular Einstein-like (ε)-paracontact metric manifold $(M,\phi,\xi,\eta,g,\varepsilon)$ with torse-forming characteristic vector field is a constant multiple of the metric tensor.*

Let $(M,\phi,\xi,\eta,g,\varepsilon)$ be a regular Einstein-like (ε)-paracontact metric manifold with torse-forming characteristic vector field. Since $2\lambda-\left(p+\frac{2}{n}\right)$ is constant in conformal η-Ricci soliton (g,ξ,λ,μ). So $\nabla\left(2\lambda-\left(p+\frac{2}{n}\right)\right)=0$, it means $\ell_\xi g+2S$ is parallel. Thus from the Corollary 4.3, we give the result related to the Ricci soliton.

Theorem 4.4 *Let $(M,\phi,\xi,\eta,g,\varepsilon)$ be a regular Einstein-like (ε)-paracontact metric manifold with torse-forming characteristic vector field. Then $\Omega=\frac{1}{2}(\ell_\xi)+S+\mu\eta\otimes\eta$ is parallel if and only if $\left(g,\xi,\lambda=-\varepsilon\,\Omega\,(\xi,\xi)+\frac{1}{2}\left(p+\frac{2}{3}\right),\mu\right)$ is a conformal η- Ricci soliton on M.*

In [6], the authors prove that on a (ε)-para Sasakian manifold any parallel symmetric $(0,2)$-tensor field is a constant multiple of the metric. Ensuring these results we can give the following results in the consequence of conformal η-Ricci soliton.

Theorem 4.5 *Let $(M,\phi,\xi,\eta,g,\varepsilon)$ be an (ε)-para Sasakian manifold. Then $\Omega=\frac{1}{2}(\ell_\xi)+S+\mu\eta\otimes\eta$ is parallel if and only if $\left(g,\xi,\lambda=-\varepsilon\,\Omega\,(\xi,\xi)+\frac{1}{2}\left(p+\frac{2}{3}\right),\mu\right)$ is a conformal η-Ricci soliton on M.*

Let $(M,\phi,\xi,\eta,g,\varepsilon,\alpha,\beta,\gamma)$ is an Einstein -like (ε)-para Sasakian manifold. Then we have

$$\begin{aligned}\frac{1}{2}(\ell_{\kappa\xi}g)(X,Y)+S(X,Y)+\mu\eta(X)\eta(Y)&\\ =\alpha g(X,Y)+(\varepsilon+\beta)g(\phi X,Y)+(\gamma+\mu)\eta(X)\eta(Y).&\end{aligned}\tag{4.10}$$

Thus we give the following results.

Corollary 4.5 *If $(M,\phi,\xi,\eta,g,\varepsilon,\alpha,\beta,\gamma)$ is an Einstein-like (ε)-para Sasakian manifold, then $\Omega=\frac{1}{2}(\ell_\xi)+S+\mu\eta\otimes\eta$ is parallel if and only if*

$$\left(g,\xi,\lambda=\frac{1}{2}\left(p+\frac{2}{n}\right)-\varepsilon\left(\varepsilon(\alpha+\varepsilon\gamma)+\mu\right)\mu\right)$$

is conformal η-Ricci soliton on M.

5 An example

Example 5.1 Let us consider a 4-dimensional manifold $M=\{(x,y,z,u)\in\Re^4:(x,y,z,u)\neq 0\}$, where (x,y,z,u) being standard coordinate in $\Re^4$. Let (e_1,e_3,e_3,e_4) be the orthogonal system of vector fields at each point of M, defined as

$$e_1=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right),e_2=\frac{\partial}{\partial y},e_3=\left(\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right),e_4=\frac{\partial}{\partial u}.$$

and

$$[e_1,e_2]=-e_2,[e_1,e_4]=-e_1,[e_2,e_4]=-e_2,[e_3,e_4]=-e_3,[e_1,e_3]=-e_3.$$

Let g be a pseudo-Riemannian metric defined as follows

$$g_{ij} = \left\{ \begin{array}{ll} 0, & i \neq j = 1,2,3,4 \\ 1, & i = j \\ \varepsilon, & i = j = 4 \end{array} \right\}$$

and given by

$$g = \frac{1}{x^2}\left[(1-y^2)dx \otimes dx + x^2 dy \otimes dy + \varepsilon\, x^2 du \otimes du\right].$$

Let be η the 1-form have the significance

$$\eta(X) = g(X, e_4)$$

for any $X \in \Gamma(TM)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi\, e_1 = e_1, \qquad \phi\, e_2 = e_2, \qquad \phi\, e_3 = e_3, \qquad \phi\, e_4 = 0.$$

Making use of the linearity of ϕ and g we have

$$\eta(e_4) = 1,$$

$$\phi^2(X) = X + \eta(X)e_4,$$
$$g(\phi X, \phi Y) = g(X,Y) - \varepsilon\, \eta(X)\eta(Y),$$

for any $X, Y \in \Gamma(TM)$.Thus for $e_4 = \xi$ the structure (ϕ, η, ξ, g) leads to timelike Lorentzian almost paraconatct structure in $\Re^4$.
The Riemannian connection ∇ of metric tensor g is given by the beauty of Koszul's formula

$$\begin{aligned} 2g(\nabla_U V, W) &= U(g(V,W)) + V(g(W,X)) - W(g(U,V)) \\ &\quad -g(U,[V,W]) - g(V,[U,W]) + g(W,[U,V]). \end{aligned}$$

Making use Koszul's formula we follows:

$$\left\{ \begin{array}{llll} \nabla_{e_1} e_4 = -e_1, & \nabla_{e_2} e_4 = -e_2, & \nabla_{e_3} e_4 = -e_3, & \nabla_{e_4} e_4 = 0, \\ \nabla_{e_3} e_1 = \frac{1}{\varepsilon} e_4, & \nabla_{e_2} e_1 = e_2, & \nabla_{e_3} e_3 = -e_1 + \frac{1}{\varepsilon} e_4, & \nabla_{e_2} e_2 = -e_1 + \frac{1}{\varepsilon} e_4, \\ \nabla_{e_4} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_2 = 0, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0. & \end{array} \right.$$

Consequently, (ϕ, η, ξ, g) timelike (ε)-almost paracontact metric manifold that satisfy,

$$(\nabla_X \phi)Y = -(g(\phi X, \phi Y)\xi - \varepsilon\eta(Y)\phi^2 X, \ \ \nabla_X \xi = \alpha\phi X.$$

where $\alpha = -1$. Hence the structure (ϕ, η, ξ, g) endowed with an (ε)-para Sasakian structure .
Using the above relations, we can easily calculate the non-vanishing component of curvature tensor as follows:

$$\left\{ \begin{array}{lll} R(e_2,e_3)e_2 = \frac{1}{\varepsilon} e_3, & R(e_2,e_3)e_3 = -\frac{1}{\varepsilon} e_2, & R(e_1,e_4)e_1 = e_2, \\ R(e_1,e_3)e_1 = \frac{1}{\varepsilon} e_3, & R(e_1,e_4)e_4 = -e_1, & R(e_2,e_4)e_2 = \frac{1}{\varepsilon} e_4 - e_1, \\ R(e_1,e_3)e_3 = -\left(e_1 + \frac{1}{\varepsilon} e_2\right), & R(e_3,e_4)e_4 = -e_3, & R(e_3,e_4)e_3 = \frac{1}{\varepsilon} e_3, \\ R(e_1,e_2)e_2 = -\left(\frac{1}{\varepsilon} e_1 + \frac{1}{\varepsilon} e_4\right), & R(e_2,e_4)e_4 = -e_2, & R(e_1,e_2)e_1 = \frac{1}{\varepsilon} e_2 + e_2, \\ R(e_2,e_1)e_1 = -\frac{1}{\varepsilon} e_3 - e_2, & R(e_3,e_1)e_1 = -\frac{1}{\varepsilon} e_2 - e_3, & R(e_4,e_3)e_3 = \frac{1}{\varepsilon} e_4 - e_1. \end{array} \right.$$

From the above expressions of the curvature tensor, we evaluate the value of the Ricci tensor as follows:

$$S(e_1, e_1) = -\left(\frac{1}{\varepsilon} + 2\right), \quad S(e_2, e_2) = -\left(\frac{1}{\varepsilon} + 2\right),$$

$$S(e_3, e_3) = -\left(\frac{1}{\varepsilon} + 2\right), \qquad S(e_4, e_4) = -2.$$

Also from (3.1), we get

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = \alpha + \beta, \quad S(e_4, e_4) = \varepsilon\, \alpha + \gamma.$$

In this case, for $\alpha = -\frac{1}{\varepsilon}, \beta = -2$ and $\gamma = -1$, the data (g, ξ, λ, μ) is a conformal η-Ricci soliton on (ε)-para Sasakian manifold. So Proposition 3.1 holds.
From this consequence, we can easily write

$$\lambda = \frac{1}{2\varepsilon}\left[2 + \varepsilon\left(p + \frac{2}{n}\right)\right], \qquad \mu = \frac{1}{2}.$$

Thus it follows that the soliton is expanding (resp.shrinking) if the manifold is space-like (resp.timelike). Thus Theorem 4.2 holds.

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