Prolongation of Jacobi 2-form on Weil bundles

A.J. Nguéngué Louvouandou, O. Mabiala Mikanou and B.G.R. Bossoto

Abstract. Let M be a smooth manifold, A a local algebra and M^A the associated Weil bundle. When (M, ω_M) is a Jacobi manifold, we construct the prolongation of the Jacobi 2-form, ω_{M^A} on M^A and we give the necessary and sufficient condition such that $(M^A, \boldsymbol{\omega}_{M^A})$ be the A-Jacobi manifold.

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1 Introduction

In what follows, all structures are assumed to be of class C^∞. We denote by M a smooth differential manifold, $C^\infty(M)$ the algebra of smooth functions on M and by $\mathfrak{X}(M)$, the $C^\infty(M)$-module of vectors field on M.

1.1 Weil algebra and Weil bundle

A Weil algebra (or a local algebra in the sense of André Weil) is a real commutative, associative, unitary algebra of finite dimension with unique maximal ideal $\mathfrak{m}$ of codimension 1. We have

$$A = \mathbb{R} \oplus \mathfrak{m}.$$

Let M be a smooth manifold of dimension n. We recall that an infinitely near point to $x \in M$ of kind A is a homomorphism of $\mathbb{R}$-algebras

$$\xi : C^\infty(M) \longrightarrow A$$

such that $\xi(f) - f(x) \in \mathfrak{m}$, for all $f \in C^\infty(M)$ [17]. We denote by M^A_x the set of all infinitely near points to $x \in M$ of kind A and

$$M^A = \bigcup_{x \in M} M^A_x.$$

The set M^A is a smooth manifold of dimension $\dim A \times \dim M$ and the triplet (M^A, π, M) is a bundle called bundle of infinitely near points or simply Weil bundle

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[4], [14].

If $f : M \longrightarrow \mathbb{R}$ is a smooth function, then the mapping

$$f^A : M^A \longrightarrow A, \xi \longmapsto \xi(f)$$

is also a smooth function. The set $C^\infty(M^A, A)$ of smooth functions on M^A with values on A, is a commutative algebra over A with unit $1_{C^\infty(M^A,A)}$ and the mapping

$$C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto f^A$$

is an injective homomorphism of algebras [1]. Then, for $f, g \in C^\infty(M)$ and for $\lambda \in \mathbb{R}$, we have:

$$(f+g)^A = f^A + g^A;\ (\lambda \cdot f)^A = \lambda \cdot f^A;\ (f \cdot g)^A = f^A \cdot g^A.$$

The mapping

$$C^\infty(M^A) \times A \longrightarrow C^\infty(M^A, A), (F, a) \longmapsto F \cdot a : \xi \longmapsto F(\xi) \cdot a$$

is bilinear and induces one and only one linear mapping

$$\sigma : C^\infty(M^A) \otimes A \longrightarrow C^\infty(M^A, A).$$

When $(a_\alpha)_{\alpha=1,2,\dots,\dim A}$ is a basis of A and when $(a^*_\alpha)_{\alpha=1,2,\dots,\dim A}$ is a dual basis of the basis $(a_\alpha)_{\alpha=1,2,\dots,\dim A}$, the mapping

$$\sigma^{-1} : C^\infty(M^A, A) \longrightarrow A \otimes C^\infty(M^A), \varphi \longmapsto \sum_{\alpha=1}^{\dim A} a_\alpha \otimes (a^*_\alpha \circ \varphi)$$

is an isomorphism of A-algebras and the mapping

$$\gamma : C^\infty(M) \longrightarrow A \otimes C^\infty(M^A), f \longmapsto \sigma^{-1}(f^A),$$

is a homomorphism of algebras [1],[12].
For any $\varphi \in C^\infty(M^A, A)$, we have

$$\varphi = \sum_{\alpha=1}^{\dim A} (a^*_\alpha \circ \varphi) \cdot a_\alpha.$$

Let M be a smooth manifold, A and B both be Weil algebras. Then the mapping

$$(M^A)^B \longrightarrow M^{A \otimes B}, \eta \longmapsto (id_A \otimes \eta) \circ \gamma$$

is an isomorphism of smooth manifolds [17].

1.2 Differential operators and Kähler differentials

Let R be a commutative algebra with unit over a commutative field K with characteristic *zero* and let E be a R-module. We denote by $\mathcal{D}^{[1]}_K(R, E)$ the R-module of differential operators of order ≤ 1 from R into E i.e. the set of K-linear maps

$$\delta : R \longrightarrow E$$

such that, for any $a,\, b \in R$,

$$\delta(ab) = \delta(a)\cdot b + a\cdot\delta(b) - ab\cdot\delta(1_R)$$

where 1_R is the unit of R.

When $E = R$, we denote by $\mathcal{D}_K^{[1]}(R)$ instead of $\mathcal{D}_K^{[1]}(R,R)$, the R-module of differential operators of order ≤ 1 from R into R. The R-module of K-derivations of R, denoted by $Der_K(R)$ is simultaneously a R-submodule and a K-Lie subalgebra of $\mathcal{D}_K^{[1]}(R)$.

Let $\Omega_{\mathbb{R}}[C^\infty(M)] = \Omega_{C^\infty(M)/\mathbb{R}}$ be the $C^\infty(M)$-module of Kälher differentials forms of $C^\infty(M)$ and

$$\delta_M : C^\infty(M) \longrightarrow \Omega_{\mathbb{R}}[C^\infty(M)], f \longmapsto \overline{f\otimes 1_{C^\infty(M)} - 1_{C^\infty(M)}\otimes f}$$

the canonical derivation which the image of δ_M generates the $C^\infty(M)$-module $\Omega_{\mathbb{R}}[C^\infty(M)]$ i.e. for $x \in \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$x = \sum_{i\in I:finite} f_i\cdot\delta_M(g_i),$$

with $f_i, g_i \in C^\infty(M)$ for any $i \in I$[3],[10], [11],[8].

The map

$$\Delta_M \ = id_{C^\infty(M)} \oplus \delta_M : C^\infty(M) \longrightarrow C^\infty(M) \oplus_{\mathbb{R}} \Omega_{\mathbb{R}}[C^\infty(M)], f \longmapsto f + \delta_M(f)$$

is a differential operator of order ≤ 1 and the image of Δ_M generates the $C^\infty(M)$-module $C^\infty(M) \oplus_{\mathbb{R}} \Omega_{\mathbb{R}}[C^\infty(M)]$.

We recall that a Jacobi structure on a smooth manifold M is due to the existence of a bracket $\{,\}_M$ on $C^\infty(M)$ such that the pair $(C^\infty(M), \{,\}_M)$ is a real Lie algebra such that, for any $f \in C^\infty(M)$ the inner derivation

$$\mathrm{ad}(f) : C^\infty(M) \longrightarrow C^\infty(M), g \longmapsto \{f, g\}_M$$

is a differential operator of order ≤ 1 i.e.

$$\mathrm{ad}(f)(gh) = g\cdot\mathrm{ad}(f)(h) + \mathrm{ad}(f)(g)\cdot h - gh\cdot\mathrm{ad}(f)(1_{C^\infty(M)})$$

i.e

$$\{f, gh\} = g\{f, h\} + \{f, g\}\cdot h - gh\{f, 1_{C^\infty(M)}\}$$

for $f, g, h \in C^\infty(M)$. In this case we say that $C^\infty(M)$ is a Jacobi algebra and M is a Jacobi manifold [6].

In [8], Mahoungou and Bossoto study the prolongation of Poisson 2-form on Weil bundles. In this paper we use same technick to study the prolongation of the Jacobi 2-form ω_M on Weil bundles.

In the following, M^A denote the Weil bundle of M of kind A, $R = C^\infty(M^A)$ the set of smooth functions on M^A with real values and $\mathbf{R} = C^\infty(M^A, A)$ the set of smooth functions on M^A with values on A.

2 Lie derivative with respect to a differential operator on M^A

2.1 Differential operator and module of Kähler differential on Weil bundle

If M^A denotes the Weil bundle of M of kind A, let $\Omega_A[\mathbf{R}] = \Omega_{\mathbf{R}/A}$ be the $\mathbf{R}$-module of Kälher differentials of $\mathbf{R}$ which are A-linear (with $\mathbf{R} = C^\infty(M^A, A)$) and

$$\delta_{M^A} : \mathbf{R} \longrightarrow \Omega_A[\mathbf{R}], \varphi \longmapsto \overline{\varphi \otimes 1_{C^\infty(M^A,A)} - 1_{C^\infty(M^A,A)} \otimes \varphi}$$

the canonical derivation which the image of δ_{M^A} generates $\Omega_A[\mathbf{R}]$. According to [8],

Proposition 2.1. *The mapping $\delta_{M^A} : \mathbf{R} \longrightarrow \Omega_A[\mathbf{R}]$ is a unique A-linear derivation such that for any $f \in C^\infty(M)$,*

$$[\delta_M(f)]^A = \delta_{M^A}(f^A).$$

Consider the $\mathbf{R}$-module, $\mathbf{R} \oplus \Omega_A[\mathbf{R}]$ definite for any φ, ψ in $\mathbf{R}$ and for any $\omega \in \Omega_A[\mathbf{R}]$ by

$$\psi(\varphi + \omega) = (\varphi + \omega)\psi = \psi\varphi + \psi\omega = \varphi\psi + \omega\psi.$$

Let E be a $\mathbf{R}$-module. We recall [2] that a differential operator on M^A with values in E is a A-lineair mapping

$$\Phi : \mathbf{R} \longrightarrow E$$

such that for any $\varphi, \psi \in R$,

$$\Phi(\varphi\psi) = \Phi(\varphi) \cdot \psi + \varphi \cdot \Phi(\psi) - \varphi \cdot \psi \cdot \Phi(1_{\mathbf{R}}).$$

Proposition 2.2. *The mapping*

$$\mathbf{\Delta}_{M^A} : \mathbf{R} \longrightarrow \mathbf{R} \oplus \Omega_A[\mathbf{R}], \varphi \longmapsto \varphi + \delta_{M^A}(\varphi)$$

is a unique differential operator on M^A with values in the $\mathbf{R}$-module $\mathbf{R} \oplus \Omega_A[\mathbf{R}]$ such that, for any $f \in C^\infty(M)$,

$$\mathbf{\Delta}_{M^A}(f^A) = [\Delta_M(f)]^A.$$

Proof. - For any φ, ψ in $\mathbf{R}$,

$$\begin{aligned} \mathbf{\Delta}_{M^A}(\varphi + \psi) &= \varphi + \psi + \delta_{M^A}(\varphi + \psi) \\ &= \varphi + \psi + \delta_{M^A}(\varphi) + \delta_{M^A}(\psi) \\ &= \mathbf{\Delta}_{M^A}(\varphi) + \mathbf{\Delta}_{M^A}(\psi). \end{aligned}$$

- For any $\varphi \in \mathbf{R}$ and for $a \in A$,

$$\begin{aligned} \mathbf{\Delta}_{M^A}(a \cdot \varphi) &= a \cdot \varphi + \delta_{M^A}(a \cdot \varphi) \\ &= a \cdot \varphi + a \cdot \delta_{M^A}(\varphi) \\ &= a \cdot (\varphi + \delta_{M^A}(\varphi)) \\ &= a \cdot \mathbf{\Delta}_{M^A}(\varphi). \end{aligned}$$

- For any φ, ψ in $\mathbf{R}$,

$$\begin{aligned}
& \boldsymbol{\Delta}_{M^A}\ (\varphi\cdot\psi) - \boldsymbol{\Delta}_{M^A}\ (\varphi)\cdot\psi - \varphi\cdot\boldsymbol{\Delta}_{M^A}\ (\psi) + \varphi\cdot\psi\cdot\boldsymbol{\Delta}_{M^A}\ (1_{\mathbf{R}}) \\
= \ & \varphi\cdot\psi + \delta_{M^A}(\varphi\cdot\psi) - [\varphi\cdot\psi + \delta_{M^A}(\varphi)\cdot\psi] - [\varphi\cdot\psi + \varphi\cdot\delta_{M^A}(\psi)] + \varphi\cdot\psi\cdot[1_{\mathbf{R}} + 0] \\
= \ & \delta_{M^A}(\varphi\cdot\psi) - \delta_{M^A}(\varphi)\cdot\psi - \varphi\cdot\delta_{M^A}(\psi) \\
= \ & 0
\end{aligned}$$

i.e.

$$\boldsymbol{\Delta}_{M^A}\ (\varphi\cdot\psi) = \boldsymbol{\Delta}_{M^A}\ (\varphi)\cdot\psi + \varphi\cdot\boldsymbol{\Delta}_{M^A}\ (\psi) - \varphi\cdot\psi\cdot\boldsymbol{\Delta}_{M^A}\ (1_{C^\infty(M^A,A)}).$$

- For any f in $C^\infty(M)$,

$$\begin{aligned}
\boldsymbol{\Delta}_{M^A}\ (f^A) &= f^A + \delta_{M^A}(f^A) \\
&= f^A + [\delta_M(f)]^A \\
&= [f + [\delta_M(f)]]^A \\
&= [\Delta_M(f)]^A.
\end{aligned}$$

Since $\mathbf{R}$ is generated by the functions f^A, we conclude that the proposition is proved. □

According to [10],[12], we have the following theorem:

Theorem 2.3. *The pair* $(\mathbf{R}\oplus\Omega_A[\mathbf{R}], \boldsymbol{\Delta}_{M^A})$ *satisfies the following universal property: for all* $\mathbf{R}$*-module* E *and for any differential operator*

$$\delta_E : \mathbf{R} \longrightarrow E,$$

with values in E*, there exists a unique* $\mathbf{R}$*-linear mapping*

$$\delta_E^* : \mathbf{R}\oplus\Omega_A[\mathbf{R}] \longrightarrow E$$

such that

$$\delta_E^* \circ \boldsymbol{\Delta}_{M^A} = \delta_E.$$

Proof. To construct the action of δ_E^*, it suffices to explain the action of δ_E^* on $f^A \in \mathbf{R} = C^\infty(M^A, A)$ and on $[d_M(f)]^A \in \Omega_{\mathbf{R}/A}$. We must necessarily have:

$$\delta_E^*(f^A) = f^A\ \delta_E(1_{M^A}).$$

On the other hand,

$$\delta_E^*([d_M(f)]^A) = \delta_E^*(\boldsymbol{\Delta}_{M^A}(f^A) - f^A) = \delta_E(f^A) - f_A\delta_E(\mathbf{1}_{M^A}).$$

The action of δ_E^* on f^A and on $[d_M(f)]^A$ for all $f \in C^\infty(M)$, is thus (only) determined by

$$\delta_E^*(f^A) = f_A\,\delta_E(\mathbf{1}_{M^A}), \quad \delta_E^*([d_M f]^A) = \delta_E(f^A) - f_A\,\delta_E(\mathbf{1}_{M^A}).$$

Since f^A for all $f \in C^\infty(M)$ generates $C^\infty(M^A, A)$ as A-module, the action thus defined is prolonged by A-linearity in that of an A-linear operator of $\mathbf{R}\oplus\Omega_{\mathbf{R}/A}$ in E such

$$\delta_E = \delta_E^* \circ \boldsymbol{\Delta}_{M^A}.$$

□

This fact implies the existence of a natural isomorphism of $\mathbf{R}$-modules

$$\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}\oplus\Omega_A[\mathbf{R}],E)\longrightarrow\mathcal{D}_A^{[1]}(\mathbf{R},E),\Theta\longmapsto\Theta\circ\Delta_{M^A}$$

where $\mathcal{D}_A^{[1]}(\mathbf{R},E)$ denotes the R-module of differential operators of order ≤ 1 from $\mathbf{R}$ into E which are A-linear.

In particular, if $E=\mathbf{R}$, we have

$$\begin{aligned}[\mathbf{R}\oplus\Omega_A[\mathbf{R}]]^* &\simeq \mathcal{D}_A^{[1]}(\mathbf{R},\mathbf{R})=\mathcal{D}_A^{[1]}(C^\infty(M^A,A),C^\infty(M^A,A))\\ &= \mathcal{D}_A^{[1]}(M^A).\end{aligned}\tag{2.1}$$

2.2 Lie derivative with respect to a differential operator on M^A

We introduce in this section the exterior algebra

$$\bigwedge\nolimits_A[\mathbf{R}\oplus\Omega_{\mathbf{R}/A}].$$

We will also be led to consider algebra

$$\bigwedge_{\mathbb{R}}[R\oplus\Omega_{R/\mathbb{R}}],$$

where R denotes this time the vector $\mathbb{R}$-space $C^\infty(M)$ attached to the smooth manifold M above which, M^A is presented as a real bundle of rank $n\times(\dim(A)-1)$, but is also equipped with an A-manifold structure[14],[1][15].

The canonical dérivation $\boldsymbol{\delta}_{M^A}:\mathbf{R}\to\Omega_{\mathbf{R}/A}$ induices a differential complex

$$\mathbf{R}\overset{\boldsymbol{\delta}^0_{M^A}=\boldsymbol{\delta}_{M^A}}{\longrightarrow}\Omega_{\mathbf{R}/A}\overset{\boldsymbol{\delta}^1_{M^A}}{\longrightarrow}\bigwedge^2\Omega_{\mathbf{R}/A}\longrightarrow\cdots\overset{\boldsymbol{\delta}^p_{M^A}}{\longrightarrow}\bigwedge^p\Omega_{\mathbf{R}/A}\longrightarrow\ldots$$

For any $\boldsymbol{z}^{[p]}\in\bigwedge^p(\mathbf{R}\oplus\Omega_A[\mathbf{R}])$, $\boldsymbol{z}^{[p]}$ has the form

$$\boldsymbol{z}^{[p]}=\sum_{\iota\in I,\ \#I<+\infty}\varphi_\iota\bigwedge_{j=1}^p\boldsymbol{\Delta}_{M^A}(\psi_{\iota,j}),$$

where φ_ι and $\psi_{\iota,j}$, $(j=1,\ldots,p)$, are in $\mathbf{R}$.

We define an operator

$$\boldsymbol{\Delta}^1_{M^A}:\mathbf{R}\oplus\Omega_{\mathbf{R}/A}\longrightarrow\bigwedge^2(\mathbf{R}\oplus\Omega_{\mathbf{R}/A})$$

thus: $forall\boldsymbol{z}\in\mathbf{R}\oplus\Omega_{\mathbf{R}/A}$,

$$\boldsymbol{z}=\sum_{\iota\in I,\#I<+\infty}\varphi_\iota\boldsymbol{\Delta}_{M^A}(\psi_\iota)\Longrightarrow\boldsymbol{\Delta}^1_{M^A}(\boldsymbol{z})=\sum_{\iota\in I,\#I<+\infty}\boldsymbol{\Delta}_{M^A}(\varphi_\iota)\wedge\boldsymbol{\Delta}_{M^A}(\psi_\iota)-\boldsymbol{z}\,.$$

This operator is well defined. Indeed, if

$$\sum_{\iota\in I,\,\#I<+\infty} \varphi_\iota \mathbf{\Delta}_{M^A}(\psi_\iota) = 0,$$

we observe that

$$\begin{aligned} \sum_{\iota\in I,\,\#I<+\infty} \varphi_\iota\, \psi_\iota) &= 0 \\ \sum_{\iota\in I,\,\#I<+\infty} \varphi_\iota\, \boldsymbol{\delta}_{M^A}(\psi_\iota) &= 0 \\ \sum_{\iota\in I,\,\#I<+\infty} \psi_\iota\, \boldsymbol{\delta}_{M^A}(\varphi_\iota) &= 0. \end{aligned}$$

We deduce immediately that

$$\sum_{\iota\in I,\,\#I<+\infty} \mathbf{\Delta}_{M^A}(\varphi_\iota) \wedge \mathbf{\Delta}_{M^A}(\psi_\iota) = 0.$$

This implies that $\mathbf{\Delta}^1_{M^A}$ is an A-linear operator from $\mathbf{R}\oplus\Omega_{\mathbf{R}/A}$ to $\bigwedge^2(\mathbf{R}\oplus\Omega_{\mathbf{R}/A})$ whose action is well defined.

An important observation is that

$$\begin{aligned} \mathbf{\Delta}^1_{M^A}\big(\varphi\,\mathbf{\Delta}_{M^A}(\psi)\big) &= \mathbf{\Delta}_{M^A}(\varphi)\wedge\mathbf{\Delta}_{M^A}(\psi) - \varphi\,\mathbf{\Delta}_{M^A}(\psi) \\ &= \big(\mathbf{\Delta}_{M^A}(\varphi) - \varphi\,\mathbf{\Delta}_{M^A}(\mathbf{1}_{M^A})\big)\wedge\mathbf{\Delta}_{M^A}(\psi) \end{aligned} \tag{2.2}$$

for any $\varphi,\psi\in\mathbf{R}$. In particular, we have $\mathbf{\Delta}^1_{M^A}\big(\mathbf{\Delta}_{M^A}(\varphi)\big) = 0$, which is in agreement with the realization of a complex. The condition $\mathbf{\Delta}^1_{M^A}\circ\mathbf{\Delta}_{M^A} = 0$ on $\mathbf{R}$ actually replaces here the nilpotency clause $\boldsymbol{\delta}^2_{M^A} = 0$ (this nilpotence clause is obviously false for $\mathbf{\Delta}_{M^A}$).

We can redo all this work (construction of the differential complex, realization of an $\mathbb{R}$-linear operator $\Delta^1_M\ : R\oplus\Omega_{R/\mathbb{R}}\to\bigwedge^2(R\oplus\Omega_{R/\mathbb{R}})$ in the context where this time $R = C^\infty(M)$ denotes the vector R-space of the functions of class C^∞ on the differential manifold M.

More generally, we have, when $\varphi,\psi_1,\ldots,\psi_p\in\mathbf{R}$:

$$\mathbf{\Delta}_{M^A}\big(\varphi\bigwedge_{j=1}^{p}\mathbf{\Delta}_{M^A}(\psi_j)\big) = \mathbf{\Delta}_{M^A}(\varphi)\wedge\bigwedge_{j=1}^{p}\mathbf{\Delta}_{M^A}(\psi_j). \tag{2.3}$$

We can also define the action of the operator $\mathbf{\Delta}^*_{M^A}$ by duality according to the formula

$$\langle\mathbf{\Delta}^*_{M^A}(\omega),\boldsymbol{z}\rangle = \langle\omega,\mathbf{\Delta}_{M^A}(\boldsymbol{z})\rangle \qquad \forall\,\boldsymbol{z}\in\mathbf{R}\oplus\Omega_{\mathbf{R}/A}, \forall\,\omega\in\big((\mathbf{R}\oplus\Omega_{\mathbf{R}/A})^2\big)^*.$$

So we have

$$\langle\mathbf{\Delta}^*_{M^A}(\varphi\,\omega),\boldsymbol{z}\rangle = \langle\omega,\mathbf{\Delta}_{M^A}(\varphi\boldsymbol{z})\rangle,$$

hence the formula

$$\mathbf{\Delta}^*_{M^A}(\varphi\,\omega) = \varphi\,\mathbf{\Delta}^*_{M^A}(\omega) + \mathbf{\Delta}^*_{M^A}(\varphi)\wedge\omega. \tag{2.4}$$

Because $C^\infty(M^A, A)$ is generated as A-module by the elements f^A (with $f \in C^\infty(M)$), that $\mathbf{R} \oplus \Omega_{\mathbf{R}/A}$ is also generated as A-module by $\Delta_{M^A}(f^A)$ (with $f \in C^\infty(M)$) and that the operator $\boldsymbol{\Delta}_{M^A}$ is an A-linear operator, the algebra $\bigwedge_A(\mathbf{R} \oplus \Omega_{\mathbf{R}/A})$ is generated (always like A-module) by the elements of the form

$$f_0^A, \ \bigwedge_{j=1}^{p} \boldsymbol{\Delta}_{M^A}(f_j^A), \quad p \in \mathbb{N}, \ f_0, f_1, ..., f_p \in C^\infty(M).$$

The application $(a, f) \mapsto a\, f^A$ for $A \times R$ into $\mathbf{R}$ induces besides a $\mathbb{R}$-morphism of the differential $\mathbb{R}$-complex

$$A \otimes \bigwedge_{\mathbb{R}} \Omega_{R/\mathbb{R}} \stackrel{\mathrm{id}_A \otimes d_M}{\longrightarrow} A \otimes \bigwedge_{\mathbb{R}} \Omega_{R/\mathbb{R}}$$

in the differential $\mathbb{R}$-complex

$$\bigwedge\nolimits_A \Omega_{\mathbf{R}/A} \stackrel{\boldsymbol{\delta}_{M^A}}{\longrightarrow} \bigwedge\nolimits_A [\mathbf{R} \oplus \Omega_{\mathbf{R}/A}].$$

Recall that for $\boldsymbol{\partial} \in \mathcal{D}_A^{[1]}(M^A)$, according to (2.1), there exists $\boldsymbol{\partial}^* \in \mathrm{Hom}_{\mathbf{R}}(\mathbf{R} \oplus \Omega_A[\mathbf{R}], \mathbf{R}) = (\mathbf{R} \oplus \Omega_A[\mathbf{R}])^*$ such that $\boldsymbol{\partial}^* \circ \Delta_{M^A} = \boldsymbol{\partial}$.

Proposition 2.4. *If $\boldsymbol{\partial} \in \mathcal{D}_A^{[1]}(M^A)$, the map*

$$K_{\boldsymbol{\partial}} \ : (\mathbf{R} \oplus \Omega_A[\mathbf{R}]) \times (\mathbf{R} \oplus \Omega_A[\mathbf{R}]) \longrightarrow \mathbf{R} \oplus \Omega_A[\mathbf{R}]$$

such that for any X and Y in $\mathbf{R} \oplus \Omega_A[\mathbf{R}]$,

$$K_{\boldsymbol{\partial}} \ (X, Y) = \boldsymbol{\partial}^*(X) \cdot Y - \boldsymbol{\partial}^*(Y) \cdot X$$

is skew-symmetric $\mathbf{R}$-bilinear.

Proof. The fact that $K_{\boldsymbol{\partial}}$ is $\mathbf{R}$-linear results from the fact that $\boldsymbol{\partial}^*$ is an $\mathbf{R}$-homomorphism from $\mathbf{R} \oplus \Omega_{\mathbf{R}/A}$ into $\mathbf{R}$. The fact that it is an $\mathbf{R}$-linear application which is antisymmetric is immediate by construction. □

We denote by

$$i_{\boldsymbol{\partial}} : \Lambda^2(\mathbf{R} \oplus \Omega_A[\mathbf{R}]) \longrightarrow \mathbf{R} \oplus \Omega_A[\mathbf{R}]$$

the unique $\mathbf{R}$-linear map such that

$$i_{\boldsymbol{\partial}}(X \wedge Y) = K_{\boldsymbol{\partial}} \ (X, Y).$$

For any $\varphi \in C^\infty(M^A, A)$,

$$i_{\varphi \cdot \boldsymbol{\partial}} = \varphi \cdot i_{\boldsymbol{\partial}}.$$

According to [2], if $\theta : C^\infty(M) \longrightarrow C^\infty(M)$ is a differential operator on M, then there exist an unique differential operator, $\theta^A : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$ which is A-linear, such that for any $f \in C^\infty(M)$,

$$\theta^A(f^A) = [\theta(f)]^A \, .$$

Let $\partial \in \mathcal{D}_A^{[1]}(M^A)$ be a differential operator on M^A which is A-linear. The A-linear map

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\partial}} = i_{\boldsymbol{\partial}} \circ \boldsymbol{\Delta}^1_{M^A} + \boldsymbol{\Delta}_{M^A} \circ \boldsymbol{\partial}^* : \mathbf{R} \oplus \Omega_A[\mathbf{R}] \longrightarrow \mathbf{R} \oplus \Omega_A[\mathbf{R}]$$

is the Lie derivative with respect to a differential operator $\boldsymbol{\partial}$.

Proposition 2.5. *For any φ and ψ in $C^\infty(M^A, A)$ and for any $\boldsymbol{\partial} \in \mathcal{D}_A^{[1]}(M^A)$, we have*

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\partial}}\big(\varphi\, \boldsymbol{\Delta}_{M^A}(\psi)\big) = \boldsymbol{\partial}(\varphi)\, \boldsymbol{\Delta}_{M^A}(\psi) + \varphi\, \boldsymbol{\Delta}_{M^A}\big(\boldsymbol{\partial}(\psi)\big) - \big(\varphi\, \boldsymbol{\Delta}_{M^A}(\psi)\big)\, \boldsymbol{\partial}(\mathbf{1_R}),$$

and for $\boldsymbol{z} = \sum_{\iota \in I,\, \# I < +\infty} \varphi_\iota\ \boldsymbol{\Delta}_{M^A}(\psi_\iota)$,

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\partial}}(\boldsymbol{z}) = \sum_{\iota \in I,\, \# I < +\infty} \Big(\boldsymbol{\partial}(\varphi_\iota)\ \boldsymbol{\Delta}_{M^A}(\psi_\iota) + \varphi_\iota\, \boldsymbol{\Delta}_{M^A}\big(\boldsymbol{\partial}(\psi)\big)\Big) - \boldsymbol{z}\, \boldsymbol{\partial}(\mathbf{1_R}).$$

Proof. For $\boldsymbol{z} = \varphi\, \boldsymbol{\Delta}_{M^A}(\psi)$, we have

$$\boldsymbol{\Delta}^1_{M^A}(\boldsymbol{z}) = \big(\boldsymbol{\Delta}_{M^A}(\varphi) - \varphi\, \boldsymbol{\Delta}_{M^A}(\mathbf{1_R})\big) \wedge \boldsymbol{\Delta}_{M^A}(\psi),$$

since $\boldsymbol{\partial} = \boldsymbol{\partial}^* \circ \boldsymbol{\Delta}_{M^A}$. Thus,

$$\begin{aligned} & \boldsymbol{i}^*_{\boldsymbol{\partial}}\big(\big(\boldsymbol{\Delta}_{M^A}(\varphi) - \varphi\, \boldsymbol{\Delta}_{M^A}(\psi)\big) \wedge \boldsymbol{\Delta}_{M^A}(\psi)\big) \\ = \;& \big(\boldsymbol{\partial}(\varphi) - \varphi\, \boldsymbol{\partial}(\mathbf{1}_{C^\infty(M^A,A)})\big)\, \boldsymbol{\Delta}_{M^A}(\psi) - \boldsymbol{\delta}_{M^A}(\varphi)\, \boldsymbol{\partial}(\psi) \\ = \;& \boldsymbol{\partial}(\varphi)\, \boldsymbol{\Delta}_{M^A}(\psi) - \boldsymbol{\delta}_{M^A}(\varphi)\, \boldsymbol{\partial}(\psi) - \boldsymbol{\partial}(\mathbf{1_R})\, \boldsymbol{z}. \end{aligned}$$

Afterwards,

$$\begin{aligned} \big(\boldsymbol{\Delta}_{M^A} \circ \boldsymbol{\partial}^*\big)\big(\varphi \boldsymbol{\Delta}_{M^A}(\psi)\big) &= \boldsymbol{\Delta}_{M^A}\big(\varphi\, \boldsymbol{\partial}^*(\boldsymbol{\Delta}_{M^A}(\psi))\big) = \boldsymbol{\Delta}_{M^A}\big(\varphi\, \boldsymbol{\partial}(\psi)\big) \\ &= \varphi\, \boldsymbol{\partial}(\psi) + \boldsymbol{\delta}_{M^A}\big(\varphi\, \boldsymbol{\partial}(\psi)\big) \\ &= \varphi\, \boldsymbol{\partial}(\psi) + \varphi\, \boldsymbol{\delta}_{M^A}\big(\boldsymbol{\partial}(\psi)\big) + \boldsymbol{\partial}(\psi)\, \boldsymbol{\delta}_{M^A}(\varphi). \end{aligned}$$

We obtain

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\partial}}\big(\varphi\, \boldsymbol{\Delta}_{M^A}(\psi)\big) = \boldsymbol{\partial}(\varphi)\, \boldsymbol{\Delta}_{M^A}(\psi) + \varphi\, \boldsymbol{\Delta}_{M^A}\big(\boldsymbol{\partial}(\psi)\big) - \big(\varphi\, \boldsymbol{\Delta}_{M^A}(\psi)\big)\, \boldsymbol{\partial}(\mathbf{1_R}).$$

□

Proposition 2.6. *For any $\boldsymbol{\partial} \in \mathcal{D}_A^{[1]}(M^A)$ and for any $\varphi \in \mathbf{R}$, we have*

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\partial}}\big(\boldsymbol{\Delta}_{M^A}(\varphi)\big) = \Delta_{M^A}\big(\boldsymbol{\partial}(\varphi)\big).$$

Proof. We observe that if $\boldsymbol{z} = \boldsymbol{\Delta}_{M^A}(\varphi)$, we have $\boldsymbol{\Delta}^1_{M^A}(\boldsymbol{z}) = 0$. The calculations made in the proof of the proposition 2.5 are simplified in this case and only the contribution of the second term remains in the expression of $\boldsymbol{\mathcal{L}}_{\boldsymbol{\partial}}(\boldsymbol{z})$, which immediately provides the result. □

For any φ in $\mathbf{R}$, for any $\boldsymbol{\partial} \in \mathcal{D}_A^{[1]}(M^A)$ and $\boldsymbol{z} = \sum_{\iota \in I} \varphi_\iota \, \boldsymbol{\Delta}_{M^A}(\psi_\iota) \in \mathbf{R} \oplus \Omega_A[\mathbf{R}]$, we have

$$\boldsymbol{\partial}(\varphi \varphi_\iota) = \varphi \, \boldsymbol{\partial}(\varphi_\iota) + \boldsymbol{\partial}(\varphi)\varphi_\iota - \varphi_\iota \varphi \, \boldsymbol{\partial}(\mathbf{1}_{C^\infty(M^A, A)}),$$

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\partial}}(\varphi \, \boldsymbol{z}) = \varphi \, \boldsymbol{\mathcal{L}}_{\boldsymbol{\partial}}(\boldsymbol{z}) + \left(\boldsymbol{\partial}(\varphi) - \varphi \, \boldsymbol{\partial}(\mathbf{1}_{C^\infty(M^A, A)})\right) \boldsymbol{z}, \tag{2.5}$$

$$\boldsymbol{\mathcal{L}}_{\varphi \cdot \boldsymbol{\partial}}(\boldsymbol{z}) = \varphi \, \boldsymbol{\mathcal{L}}_{\boldsymbol{\partial}}(\boldsymbol{z}) + \boldsymbol{\partial}^*(\boldsymbol{z}) \, \boldsymbol{\delta}_{M^A}(\varphi). \tag{2.6}$$

Proposition 2.7. *For any x in $C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$ and $\theta \in \mathcal{D}^{[1]}(M)$,*

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\theta}^A}(x^A) = \left[\boldsymbol{\mathcal{L}}_{\boldsymbol{\theta}}(x)\right]^A .$$

3 Jacobi structure

When M is a smooth manifold, A a Weil algebra and M^A the associated Weil bundle, the A-algebra $\mathbf{R} = C^\infty(M^A, A)$ is a Jacobi algebra over A if there exists a bracket $\{.,.\}_A$ on $\mathbf{R} = C^\infty(M^A, A)$ such that the pair $\left(\mathbf{R} = C^\infty(M^A, A), \{.,.\}_A\right)$ is a Lie algebra over A satisfying

$$\{\varphi, \psi_1 \, \psi_2\}_A = \{\varphi, \psi_1\}_A \cdot \psi_2 + \psi_1 \cdot \{\varphi, \psi_2\}_A - \psi_1 \cdot \psi_2 \, \left\{\varphi, 1_{C^\infty(M^A, A)}\right\}_A$$

for any $\varphi, \psi_1, \psi_2 \in \mathbf{R}$. When $\mathbf{R}$ is a Jacobi A-algebra, we will say that the manifold $(M^A, \{.,.\}_A)$ is a A-Jacobi manifold [2].

For any $f \in C^\infty(M)$, there exists the unique differential operator A-linear denoted by $\mathbf{ad}(f) = [\mathrm{ad}(f)]^A \in \mathcal{D}_A^{[1]}(M^A)$ such that $[\mathbf{ad}(f)]\left(g^A\right) = \{f, g\}^A$ for any $g \in C^\infty(M)$[2].

3.1 Jacobi 2-form and Jacobi 1-form on Weil Bundle

When $(M, \{,\})$ is a Jacobi manifold, the mapping

$$\mathrm{ad} : C^\infty(M) \longrightarrow \mathcal{D}^{[1]}(M), f \longmapsto \mathrm{ad}(f)$$

such that $[ad(f)]\,(g) = \{f, g\}$ for any $g \in C^\infty(M)$, is a differential operator. Thus :

Proposition 3.1. *There exists a differential operator A-linear*

$$\mathrm{ad}^A \; : \; \mathbf{R} \longrightarrow \mathcal{D}_A^{[1]}(M^A)$$

such that

$$\forall \, f \in C^\infty(M), \quad \mathrm{ad}^A(f^A) = [\mathrm{ad}(f)]^A. \tag{3.1}$$

Proof. Let

$$\begin{array}{ccc} C^\infty(M^A, A) & \xrightarrow{\widetilde{\tau}} & \mathcal{D}_A^{[1]}(M^A) \\ \uparrow \gamma_M & & \uparrow K \\ C^\infty(M) & \xrightarrow{\mathrm{ad}} & \mathcal{D}^{[1]}(M) \end{array}$$

be the commutative diagram i.e.

$$\widetilde{\tau} \circ \gamma_M = K \circ \mathrm{ad},$$

where

$$\gamma_M : C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto f^A,$$

$$\widetilde{\tau} : C^\infty(M^A, A) \longrightarrow \mathcal{D}_A^{[1]}(M^A), \varphi \longmapsto \widetilde{\tau}(\varphi)$$

such that $\widetilde{\tau}(\varphi)(\psi) = \{\varphi, \psi\}_A$ and

$$K : \mathcal{D}^{[1]}(M) \longrightarrow \mathcal{D}_A^{[1]}(M^A),\ \theta \longmapsto \theta^A.$$

We have for any $f \in C^\infty(M)$,

$$\widetilde{\tau} \circ \gamma_M(f) = \widetilde{\tau}(f^A)$$

and

$$\begin{aligned} K \circ \mathrm{ad}(f) &= K[\mathrm{ad}(f)] \\ &= [\mathrm{ad}(f)]^A. \end{aligned}$$

Thus, there exists $\mathrm{ad}^A = \widetilde{\tau}$ such that

$$\mathrm{ad}^A(f^A) = [\mathrm{ad}(f)]^A.$$

□

We will ask, given two elements $\varphi, \psi \in \mathbf{R} = C^\infty(M^A, A)$:

$$\{\varphi, \psi\}_A = \left[\mathrm{ad}^A(\varphi)\right](\psi).$$

Since

$$\mathrm{ad}^A\ :\ C^\infty(M^A, A) \longrightarrow \mathcal{D}_A^{[1]}(M^A)$$

is a A-linear differential operator, according to the universal property of the pair $(\mathbf{R} \oplus \Omega_A[\mathbf{R}], \mathbf{\Delta}_{M^A})$, there exists a unique $\mathbf{R}$-linear mapping

$$[\mathrm{ad}^A]^*\ : \mathbf{R} \oplus \Omega_A[\mathbf{R}] \longrightarrow \mathcal{D}_A^{[1]}(M^A)$$

such that

$$[\mathrm{ad}^A]^* \circ \mathbf{\Delta}_{M^A} = \mathrm{ad}^A.$$

Let

$$\boldsymbol{\sigma}^*_{M^A}\ :\ \phi^* \in [\mathbf{R} \oplus \Omega_A[\mathbf{R}]]^* \longmapsto \phi^* \circ \mathbf{\Delta}_{M^A} \in \mathcal{D}_A^{[1]}(M^A).$$

be the canonical isomorphism and let

$$(\sigma^*_{M^A})^{-1} \circ [\mathrm{ad}^A]^* : \mathbf{R} \oplus \Omega_A[\mathbf{R}] \overset{[\mathrm{ad}^A]^*}{\longrightarrow} \mathcal{D}_A^{[1]}(M^A) \overset{(\sigma^*_{M^A})^{-1}}{\longrightarrow} [\mathbf{R} \oplus \Omega_A[\mathbf{R}]]^*$$

be the mapping.

Proposition 3.2. *The mapping*

$$\boldsymbol{\omega}_{M^A} \ : (\mathbf{R} \oplus \Omega_A[\mathbf{R}]) \times (\mathbf{R} \oplus \Omega_A[\mathbf{R}]) \longrightarrow C^\infty(M^A, A)$$

such that

$$\boldsymbol{\omega}_{M^A} \ (\boldsymbol{t}, \boldsymbol{z}) = -\big[(\sigma^*_{M^A})^{-1} \circ [\mathrm{ad}^A]^*(\boldsymbol{t})\big](\boldsymbol{z})$$

is a skew-symmetric 2-form on $\mathbf{R} \oplus \Omega_A[\mathbf{R}]$. Moreover, for any $\varphi, \psi \in \mathbf{R}$, we have

$$\{\varphi, \psi\}_A = -\boldsymbol{\omega}_{M^A}\big(\boldsymbol{\Delta}_{M^A}(\varphi), \boldsymbol{\Delta}_{M^A}(\psi)\big). \tag{3.2}$$

Proof. For any φ and ψ in $\mathbf{R}$, we have:

1.
$$\begin{aligned}
\boldsymbol{\omega}_{M^A}\big(\boldsymbol{\Delta}_{M^A}(\varphi), \boldsymbol{\Delta}_{M^A}(\psi)\big) &= -\Big[(\sigma^*_{M^A})^{-1}\big[\mathrm{ad}^A]^* \circ \boldsymbol{\Delta}_{M^A}(\varphi)\big]\Big]\big(\boldsymbol{\Delta}_{M^A}(\psi)\big) \\
&= -\Big[(\sigma^*_{M^A})^{-1}\big[\mathrm{ad}^A(\varphi)\big]\Big]\big(\boldsymbol{\Delta}_{M^A}(\psi)\big) \\
&= -\Big[\mathrm{ad}^A(\varphi)\Big](\psi) \\
&= -\{\varphi, \psi\}_A.
\end{aligned}$$

2.
$$\begin{aligned}
\boldsymbol{\omega}_{M^A}\big(\boldsymbol{\Delta}_{M^A}(\varphi), \boldsymbol{\Delta}_{M^A}(\varphi)\big) &= -\{\varphi, \varphi\}_A. \\
&= 0.
\end{aligned}$$

□

Proposition 3.3. *If*

$$\omega_M \ : (C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]) \times (C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]) \longrightarrow C^\infty(M)$$

is a Jacobi 2-form of a Jacobi manifold M, then, for any $f, g \in C^\infty(M)$

$$\boldsymbol{\omega}_{M^A}(f^A, g^A) = [\omega_M(f, g)]^A. \tag{3.3}$$

and

$$\boldsymbol{\omega}_{M^A}\big(\boldsymbol{\Delta}_{M^A}(f^A), \boldsymbol{\Delta}_{M^A}(g^A)\big) = -\Big[\mathrm{ad}^A(f^A)\Big]\big(g^A\big) = -\{f, g\}^A = -\{f^A, g^A\}_A$$

In this case, we will say that $\boldsymbol{\omega}_{M^A}$ is the Jacobi 2-form of the A-Jacobi manifold M^A and we denote $(M^A, \boldsymbol{\omega}_{M^A})$ the A-Jacobi manifold of Jacobi 2-form $\boldsymbol{\omega}_{M^A}$.

Proposition 3.4. *If $(M, \{.,.\})$ is a Jacobi manifold and*

$$\boldsymbol{\omega}_{M^A} \ : (\mathbf{R} \oplus \Omega_A[\mathbf{R}]) \times (\mathbf{R} \oplus \Omega_A[\mathbf{R}]) \longmapsto \mathbf{R},$$

the Jacobi 2-form of the A-Jacobi manifold M^A, then, for any $\varphi, \psi \in C^\infty(M^A, A)$,

$$\boldsymbol{\mathcal{L}}_{\mathrm{ad}^A(\varphi)}[\boldsymbol{\Delta}_{M^A}(\psi)] = \boldsymbol{\Delta}_{M^A}\big(\{\varphi, \psi\}_A\big). \tag{3.4}$$

Proof. For any $\varphi \in \mathbf{R}$, $\mathrm{ad}^A(\varphi) \in \mathcal{D}_A^{[1]}(M^A)$. Thus,

$$\begin{aligned} \boldsymbol{\mathcal{L}}_{\mathrm{ad}^A(\varphi)}[\boldsymbol{\Delta}_{M^A}(\psi)] &= \boldsymbol{\Delta}_{M^A}\big(\left[\mathrm{ad}^A(\varphi)\right](\psi)\big) \\ &= \boldsymbol{\Delta}_{M^A}\big(\{\varphi,\psi\}_A\big). \end{aligned}$$

□

Proposition 3.5. *When $(M,\{.,.\})$ is a Jacobi manifold, then the mapping*

$$i_{1_\mathbf{R}}\boldsymbol{\omega}_{M^A} : \mathbf{R}\oplus\Omega_A[\mathbf{R}] \longmapsto C^\infty(M^A,A)$$

such that

$$i_{1_\mathbf{R}}\boldsymbol{\omega}_{M^A}(X) = \boldsymbol{\omega}_{M^A}\ (1_\mathbf{R}, X)$$

is a 1*-form on the* $\mathbf{R}$*-module* $\mathbf{R}\oplus\Omega_A[\mathbf{R}]$*. Moreover, for any $x \in \Omega_\mathbb{R}[C^\infty(M)]$, we have*

$$i_{1_\mathbf{R}}\boldsymbol{\omega}_{M^A}(x^A) = \left[i_{1_{C^\infty(M)}}\boldsymbol{\omega}(x)\right]^A .$$

Proof. - For any $X, Y \in \mathbf{R}\oplus\Omega_A[\mathbf{R}]$ and $a \in A$,

$$\begin{aligned} i_{1_\mathbf{R}}\boldsymbol{\omega}_{M^A}(X+Y) &= \boldsymbol{\omega}_{M^A}\ (1_\mathbf{R}, X+Y) \\ &= \boldsymbol{\omega}_{M^A}\ (1_\mathbf{R}, X+Y) \\ &= \boldsymbol{\omega}_{M^A}\ (1_\mathbf{R}, X) + \boldsymbol{\omega}_{M^A}\ (1_\mathbf{R}, Y) \\ &= i_{1_\mathbf{R}}\boldsymbol{\omega}_{M^A}(X) + i_{1_\mathbf{R}}\boldsymbol{\omega}_{M^A}(Y). \end{aligned}$$

$$\begin{aligned} i_{1_\mathbf{R}}\boldsymbol{\omega}_{M^A}(a\cdot X) &= \boldsymbol{\omega}_{M^A}\ (1_{C^\infty(M^A,A)}, a\cdot X) \\ &= a\cdot\boldsymbol{\omega}_{M^A}\ (1_\mathbf{R}, X) \\ &= a\cdot i_{1_\mathbf{R}}\boldsymbol{\omega}_{M^A}(X). \end{aligned}$$

- For any $x \in \Omega_\mathbb{R}[C^\infty(M)]$,

$$\begin{aligned} i_{1_\mathbf{R}}\boldsymbol{\omega}_{M^A}(x^A) &= \boldsymbol{\omega}_{M^A}\ (1_\mathbf{R}, x^A) \\ &= \left[\omega_M\ (1_{C^\infty(M)}, x)\right]^A \\ &= \left[i_{1_{C^\infty(M)}}\omega_M\boldsymbol{\omega}(x)\right]^A . \end{aligned}$$

□

The 1-form $i_{1_\mathbf{R}}\boldsymbol{\omega}_{M^A}$ is the prolongation on M^A of the Jacobi 1-form $i_{1_{C^\infty(M)}}\omega_M$ of the Jacobi algebra $(C^\infty(M),\{.,.\})$.

Theorem 3.6. *The bracket*

$$[.,.]_A\ : (\mathbf{R}\oplus\Omega_A[\mathbf{R}])\times(\mathbf{R}\oplus\Omega_A[\mathbf{R}]) \longrightarrow \mathbf{R}\oplus\Omega_A[\mathbf{R}]$$

such that

$$[\boldsymbol{t},\boldsymbol{z}]_A = \boldsymbol{\Delta}_{M^A}[\boldsymbol{\omega}_{M^A}(\boldsymbol{t},\boldsymbol{z})] - \boldsymbol{\mathcal{L}}_{[\mathrm{ad}^A]^*(\boldsymbol{t})}(\boldsymbol{z}) - \boldsymbol{\mathcal{L}}_{[\mathrm{ad}^A]^*(\boldsymbol{z})}(\boldsymbol{t})$$

defines a structure of A-Lie algebra (the bracket $[.,.]_A$ is A-linear) on $\mathbf{R} \oplus \Omega_A[\mathbf{R}]$. Moreover,

$$[\boldsymbol{t}, \varphi \boldsymbol{z}]_A = \Big([\mathrm{ad}^A]^*(\boldsymbol{t})(\varphi) - \varphi\, [\mathrm{ad}^A]^*(\boldsymbol{t})(\mathbf{1}_{C^\infty(M^A,A)})\Big)\, \boldsymbol{z} + \varphi\, [\boldsymbol{t}, \boldsymbol{z}]_A \tag{3.5}$$

for any $\boldsymbol{t}, \boldsymbol{z}$ in $\mathbf{R} \oplus \Omega_{\mathbf{R}/A}$ and for any $\varphi \in \mathbf{R}$. Finally, the maps

$$\begin{aligned} \boldsymbol{\Delta}_{M^A} &: (\mathbf{R}, \{.,.\}_A) \longrightarrow (\mathbf{R} \oplus \Omega_A[\mathbf{R}], [.,.]_A) \\ [\mathrm{ad}^A]^* &: (\mathbf{R} \oplus \Omega_A[\mathbf{R}], [.,.]_A) \longrightarrow (\mathcal{D}_A^{[1]}(M^A), [.,.]) \end{aligned}$$

are two morphisms of A-Lie algebras.

Proof. The verification that $[.,.]_A$ is a Lie bracket on $\mathbf{R} \oplus \Omega_{\mathbf{R}/A}$, that is, the A-bilinearity (immediate), of the condition $[\boldsymbol{t}, \boldsymbol{t}]_A$ (immediate as well) and Jacobi's identity are not a problem. The mapping

$$\boldsymbol{\Delta}_{M^A} : (\mathbf{R}, \{.,.\}_A) \longrightarrow (\mathbf{R} \oplus \Omega_{\mathbf{R}/A}, [.,.]_A)$$

is indeed A-linear. Moreover, we have, for all, $\varphi,\psi \in \mathbf{R}$, that:

$$\begin{aligned} [\boldsymbol{\Delta}_{M^A}(\varphi), \boldsymbol{\Delta}_{M^A}(\psi)]_A &= \boldsymbol{\Delta}_{M^A}\left[\boldsymbol{\omega}_{M^A}(\boldsymbol{\Delta}_{M^A}(\varphi), \boldsymbol{\Delta}_{M^A}(\psi))\right] \\ &\quad + \mathcal{L}_{[\mathrm{ad}^A]^*(\boldsymbol{\Delta}_{M^A}(\varphi))}(\boldsymbol{\Delta}_{M^A}(\psi)) - \mathcal{L}_{[\mathrm{ad}^A]^*(\boldsymbol{\Delta}_{M^A}(\psi))}(\boldsymbol{\Delta}_{M^A}(\varphi)) \\ &= \boldsymbol{\Delta}_{M^A}\{\varphi, \psi\}_A . \end{aligned}$$

Similarly, the mapping

$$[\mathrm{ad}^A]^* : (\mathbf{R} \oplus \Omega_{\mathbf{R}/A}, [.,.]_A) \longrightarrow (\mathcal{D}_A^{[1]}(M^A), [.,.])$$

is $\mathbf{R}$-linear. For all $\varphi,\psi \in \mathbf{R}$, if we pose $\boldsymbol{t} = \boldsymbol{\Delta}_{M^A}(\varphi)$ and $\boldsymbol{z} = \boldsymbol{\Delta}_{M^A}(\psi)$, we have

$$\begin{aligned} [\mathrm{ad}^A]^*(\boldsymbol{\Delta}_{M^A}\{\varphi, \psi\}_A) &= \left(\mathrm{ad}^A\right)\{\varphi, \psi\}_A \\ [\mathrm{ad}^A]^*[\boldsymbol{\Delta}_{M^A}(\varphi), \boldsymbol{\Delta}_{M^A}(\psi)]_A &= \left[\left(\mathrm{ad}^A\right)(\varphi), \left(\mathrm{ad}^A\right)(\psi)\right] \\ [\mathrm{ad}^A]^*[\boldsymbol{t}, \boldsymbol{z}]_A &= \left[[\mathrm{ad}^A]^*(\boldsymbol{t}), [\mathrm{ad}^A]^*(\boldsymbol{z})\right]. \end{aligned}$$

Hence the assertion. One can be satisfied here to prove the A-bilinearity, the identity of Jacobi, so that $[.,.]_A$ realizes well a Lie hook, to consider only the elements of $\mathbf{R} \oplus \Omega_{\mathbf{R}/A}$ of type $\varphi\, \boldsymbol{\Delta}_{M^A}(\psi)$.

Let's now establish the conditions (3.5) (in fact, we will prove the

Now let's establish the conditionss (3.5) (in fact, we will prove the last assertion of the theorem, which will prove it completely). Given the 2-form of Jacobi $\boldsymbol{\omega}_{M^A}$ on $((\mathbf{R} = C^\infty(M^A, A), \{.,.\}_A))$ and the operator $\mathbf{R}$-linear

$$[\mathrm{ad}^A]^* : \mathbf{R} \oplus \Omega_{\mathbf{R}/A} \longrightarrow \mathcal{D}_A^{[1]}(M^A),$$

we can also introduce operators

$$\boldsymbol{d}^{A,p}_{[\mathrm{ad}^A]^*} : \bigwedge_A^p \mathrm{Hom}_{\mathbf{R}}\left(\mathbf{R} \oplus \Omega_{\mathbf{R}/A}\right) \longrightarrow \bigwedge_A^{p+1} \mathrm{Hom}_{\mathbf{R}}\left(\mathbf{R} \oplus \Omega_{\mathbf{R}/A}\right)$$

associated with representation

$$[\mathrm{ad}^A]^* : (\mathbf{R} \oplus \Omega_{\mathbf{R}/A}, [.,.]_A) \longrightarrow (\mathcal{D}_A^{[1]}(M^A), [.,.].)$$

The action of these operators is defined as follows: for all $\boldsymbol{\eta}^* \in \bigwedge_A^p \mathrm{Hom}_{\mathbf{R}}\left(\mathbf{R} \oplus \Omega_{\mathbf{R}/A}\right)$ and for all $\boldsymbol{t}_1, ..., \boldsymbol{t}_{p+1} \in \mathbf{R} \oplus \Omega_{\mathbf{R}/A}$, we have

$$\begin{aligned} \boldsymbol{d}^A_{[\mathrm{ad}^A]^*}\boldsymbol{\eta}^*(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_{p+1}) &= \sum_{j=1}^{p+1} (-1)^{j-1} [\mathrm{ad}^A]^*(\boldsymbol{t}_j)\left[\boldsymbol{\eta}\left(\boldsymbol{t}_1, ..., \widehat{\boldsymbol{t}_j}, ..., \boldsymbol{t}_{p+1}\right)\right] \\ &\quad + \sum_{1 \le k < \ell \le p+1} (-1)^{k+\ell}\, \boldsymbol{\eta}\left([\boldsymbol{t}_k, \boldsymbol{t}_\ell]_A, \boldsymbol{t}_1, ..., \widehat{\boldsymbol{t}}_k, ..., \widehat{\boldsymbol{t}}_\ell, ..., \boldsymbol{t}_{p+1}\right) \\ &= -\sum_{j=1}^{p+1} (-1)^{j-1}\, \boldsymbol{\omega}_{M^A}\left(\boldsymbol{t}_j,\, \boldsymbol{\Delta}_{M^A}\left[\boldsymbol{\eta}\left(\boldsymbol{t}_1, ..., \widehat{\boldsymbol{t}_j}, ..., t_{p+1}\right)\right]\right) \\ &\quad + \sum_{1 \le k < \ell \le p+1} (-1)^{k+\ell}\, \boldsymbol{\eta}\left([\boldsymbol{t}_k, \boldsymbol{t}_\ell]_A, \boldsymbol{t}_1, ..., \widehat{\boldsymbol{t}}_{\boldsymbol{k}}, ..., \widehat{\boldsymbol{t}}_\ell, ..., \boldsymbol{t}_{p+1}\right) \end{aligned}$$

The form $\boldsymbol{\omega}_{M^A}$ is a closed form for this operator $\boldsymbol{d}^A_{[\mathrm{ad}^A]^*}$, which means that

$$\boldsymbol{d}^A_{[\mathrm{ad}^A]^*}\boldsymbol{\omega}_{M^A} = 0. \tag{3.6}$$

Indeed, if the trivial representation

$$\mathrm{id}^A \;: \partial \in (\mathcal{D}^{[1]}_{M^A}(M^A), [.,.]) \to \partial \in (\mathcal{D}^{[1]}_{M^A}(M^A), [.,.])$$

has the associated cohomology operator $\boldsymbol{d}^A_{[\mathrm{id}^A]^*}$ and that we note

$$\boldsymbol{\alpha} = \boldsymbol{\omega}_{M^A}(1_{M^A} + 0, \cdot)_{|\mathfrak{X}(M)} \in \Omega^1_{M^A}(M^A),$$

we have the "twisted" relationship:

$$\boldsymbol{d}^A_{[\mathrm{ad}^A]^*}\boldsymbol{\omega}_{M^A} = \boldsymbol{d}^A_{[\mathrm{id}^A]^*}\,\boldsymbol{\omega}_{M^A} + \boldsymbol{\alpha} \wedge \boldsymbol{\omega}_{M^A}.$$

As

$$\boldsymbol{d}^A_{[\mathrm{id}^A]^*}\,\boldsymbol{\omega}_{M^A} = -\boldsymbol{\alpha} \wedge \boldsymbol{\omega}_{M^A},$$

we deduce the closing clause (3.6). We are here in the framework of the notion of Lie-Rinehart-Jacobi structure introduced in [10] and [11].

The "anchor" morphism of vector $\mathbb{R}$-bundles

$$\rho = \rho_{\mathbb{R}} \oplus \rho_{\mathfrak{X}(M^A)} : \mathcal{D}^{[1]}_{\mathbb{R}}(M) \longrightarrow \mathbb{R} \oplus \mathfrak{X}(M^A)$$

(by $\mathbb{R}$, here we mean the trivial bundle above M^A) induced at the A-modules of global sections (once we move from the R-varieties framework to the A-manifolds framework) the A-linear morphism here precisely noted $[\mathrm{ad}^A]^*$ and we thus have the relations(3.5) if we observe precisely that, for all $\boldsymbol{t} \in \mathcal{D}^{[1]}_{\mathbb{R}}(M)$,

$$\left[\varphi \in \mathbf{R} \oplus \Omega_{\mathbf{R}/A} \longmapsto \left([\mathrm{ad}^A]^*(\boldsymbol{t})(\varphi) - \varphi\, [\mathrm{ad}^A]^*(\boldsymbol{t})(\mathbf{1}_{C^\infty(M^A,A)})\right)\right] = \rho_{\mathbb{R}}(\boldsymbol{t}) \in Der_A(\mathbf{R})$$

The triplet $(\mathbf{R}\oplus\Omega_{\mathbf{R}/A}, [ad^A]^*, \boldsymbol{\omega}_{M^A})$ corresponds to a structure of Lie-Rinehart-Jacobi A-algebra in the sense of [?], [?] on $\mathbf{R}\oplus\Omega_{\mathbf{R}/A}$, considered as the algebra of the global sections of an A-bundle of rank $n+1$ above the differential manifold M. This completes the proof of the proposal. □

In fact, the triplet

$$(\mathbf{R}\oplus\Omega_{\mathbf{R}/A}, [\mathrm{ad}^A]^*, \boldsymbol{\omega}_{M^A})$$

corresponds to a Lie-Rinehart-Jacobi A-algebra structure in the sense of [?],[?] on $\mathbf{R}\oplus\Omega_{\mathbf{R}/A}$, considered as the algebra of the global sections of an A- fiber of rank $n+1$ above the differential manifold M.

3.2 Characterization of Jacobi algebras $(M^A, \{.,.\}_A)$on Weil bundle

In this section, we propose to establish the following theorem:

Theorem 3.7. *Let M be a smooth manifold, A a Weil algebra, and M^A the Weil bundle of near points from M of kind A. Conditions below are equivalent:*

1. *It exists a bracket $\{.,.\}_A$ on M^A allowing to equip $\mathbf{R} = C^\infty(M^A, A)$ with a structure of A-Jacobi algebra.*
2. *There is a structure $(\mathbf{R}\oplus\Omega_{\mathbf{R}/A}, [.,.]_A)$ of Lie algebra on $\mathbf{R}\oplus\Omega_{\mathbf{R}/A}$ as well as a differential form of degree 2 on M^A (with values in A), i.e an antisymmetric $\mathbf{R}$-bilinear mapping*

$$\boldsymbol{\omega} \ : (\mathbf{R}\oplus\Omega_{\mathbf{R}/A}, [.,.]_A) \times (\mathbf{R}\oplus\Omega_{\mathbf{R}/A}, [.,.]_A) \longrightarrow \mathbf{R}$$

such that the triplet

$$(\mathbf{R}\oplus\Omega_{\mathbf{R}/A}, F_{\boldsymbol{\omega}_A}, \boldsymbol{\omega}_A)$$

generates a Lie-Rinehart-Jacobi structure when

$$F_{\boldsymbol{\omega}_A} \ : \mathbf{R}\oplus\Omega_{\mathbf{R}/A} \longrightarrow \mathcal{D}_A^{[1]}(M^A)$$

is the operator whose action is defined by

$$F_{\boldsymbol{\omega}_A}(\boldsymbol{z}) \ : \varphi\in\mathbf{R} \longmapsto \boldsymbol{\omega}_A(\boldsymbol{\Delta}_{M^A}(\varphi), \boldsymbol{z}) \qquad \forall \boldsymbol{z}\in\mathbf{R}\oplus\Omega_{\mathbf{R}/A}. \tag{3.7}$$

Proof. In the sense (1) ⇒ (2), the assertion has been the object of the theorem 3.6.. The only thing to observe is that the application $[\mathrm{ad}^A]^*$, corresponding to the "anchor" application of the Lie-Rinehart structure is precisely in this case the mapping

$$\boldsymbol{z}\in\mathbf{R}\oplus\Omega_{\mathbf{R}/A} \longrightarrow \mathcal{D}_{M^A}^{[1]}(M^A), \boldsymbol{z}\longmapsto F_{\boldsymbol{\omega}_A}(\boldsymbol{z})$$

defined by(3.7). Indeed, the Jacobi bracket $\{.,.\}_A$ on M^A is connected to the form $\boldsymbol{\omega}_{M^A}$ precisely by the relation

$$\{\varphi,\psi\}_A = \boldsymbol{\omega}(\boldsymbol{\Delta}_{M^A}(\varphi), \boldsymbol{\Delta}_{M^A}(\psi)) \qquad \forall\, \varphi,\psi\in C^\infty(M^A, A). \tag{3.8}$$

It should be noted the very important role played here by the application "anchor":

$$[\mathrm{ad}^A]^* \ : \mathbf{R} \oplus \Omega_{\mathbf{R}/A} \longrightarrow \mathcal{D}^{[1]}_{M^A}(M^A).$$

This application should be seen as a morphism of A-bundles, once we consider M^A as equipped with its A-manifold structure (before considering the action induced on the A-modules of sections fibred above the open of M^A).

.

In the sens (2) $\Rightarrow$ (1), we observe that if the 2-form $\boldsymbol{\omega}$ is closed relative to $\boldsymbol{d}^{A,2}_{[\mathrm{ad}^A]^*}$, the bracket defined by (3.8) is a Jacobi bracket on the A-algebra $C^\infty(M^A, A)$, which makes it possible to equip this A-algebra of an A-structure of Jacobi. This completes the proof of the theorem. □

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Authors' addresses:

Apépé Jugeandène Nguéngué Louvouandou,
Department of Mathematics, Marien Ngouabi University,
and
Institut de Recherche en Sciences Exactes et Naurelles (IRSEN), Brazzaville (Congo).
E-mail: nguengueapepe@gmail.com

Olivier Mabiala Mikanou,
Department of Mathematics, Marien Ngouabi University, Brazzaville (Congo).
E-mail: stive.elg@gmail.com

Basile Guy Richard Bossoto
Department of Mathematics, Marien Ngouabi University,
and
Institut de Recherche en Sciences Exactes et Naurelles (IRSEN), Brazzaville (Congo).
E-mail: basile.bossoto@umng.cg