$M$-projective curvature tensor over cosymplectic manifolds

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Abstract. In this paper, properties of the $\alpha$-cosymplectic manifolds with $M$-projective curvature tensor are studied. Meanwhile, we obtain some connections between different curvature tensors.

Key words: cosymplectic manifolds, $M$-projective curvature tensor, projective curvature tensor, Riemannian manifolds.

1 Introduction and preliminaries

Although $M$-projective curvature tensor studies are based on old times, the properties of the $M$-projective curvature tensor has recently come popular and has been studied by many mathematicians. In particular studies on contact manifolds with $M$-projective curvature tensor have contributed significantly to the literature. To mention some of these; at the beginning, in 1958, Boothby and Wong [18] introduced odd dimensional manifolds with contact and almost contact structures from topological point of view. Next, they were re-investigated by Sasaki and Hatakeyama in 1961 [19] using tensor calculus. In 1985, R.H Ojha showed some properties of $M$-projective curvature tensor in a Sasakian manifold [12]. An important study on this subject in 2010 by Chaubey and Ojha was about the properties of $M$-projective curvature tensor in Riemannian manifolds and also in Kenmotsu manifolds [17]. They obtained the relation between different curvature tensors. In addition in 2012, O. F. Zengin studied $M$-projectively flat spacetimes and showed that $M$-projectively flat Riemannian manifold is an Einstein manifold [20].

If we take an $m$-dimensional differentiable manifold $M^m$ with differentiability class $C^\infty$ on board, the elements on the manifold are as a $(1, 1)$ tensor field $\varphi$, a vector field $\xi$, a contact form $\eta$ and the associated Riemannian metric $g$. In 1971 on an $m$-dimensional Riemannian manifold, ones [10] defined a tensor field $W^*$ as

\[
W^*(X, Y)Z = R(X, Y)Z - \frac{1}{2(m - 1)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]
\]

so that

\[
\left. W^*(X, Y, Z, U) \right| = g(W^*(X, Y)Z, U) = \left. W^*(Z, U, X, Y) \right|
\]

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and

\[ \omega^*_{ijkl} w^{ij} = \omega^*_{ijkl} w^{ij} \]

where \( \omega^*_{ijkl} \) and \( \omega^*_{ijkl} \) are components of \( \omega^* \) and \( \omega^* \) respectively and \( w^{ij} \) is a skew-symmetric tensor [11, 15]. They called this tensor \( \omega^* \) as \( \omega^* \)-projective curvature tensor. In [11, 12], authors have focused on Sasakian and Kähler manifolds admitting \( \omega^* \)-projective curvature tensor. Some connections between conformal, con-harmonic, con-circular and \( H^* \)-projective curvature tensors have also been studied by them.

In addition to these definitions, the Weyl projective curvature tensor \( W \), con-circular curvature tensor \( C \) and conformal curvature tensor \( V \) are given as follows [13]

\[ W(X, Y)Z = R(X, Y)Z - \frac{1}{m - 1} \{ S(Y, Z)X - S(X, Z)Y \}, \]

\[ C(X, Y)Z = R(X, Y)Z - \frac{r}{m(m - 1)} \{ g(Y, Z)X - g(X, Z)Y \} \]

and

\[ V(X, Y)Z = R(X, Y)Z - \frac{1}{m - 2} \{ S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \}
- g(X, Z)QY \} + \frac{r}{(m - 1)(m - 2)} \{ g(Y, Z)X - g(X, Z)Y \}. \]

Now we will give some important theorems and corollary that we use in some of our results, without proofs.

**Theorem 1.1.** [17] The \( \omega^* \)-projective and Weyl projective curvature tensors of the Riemannian manifold \( M \) are linearly dependent if and only if \( M \) is an Einstein manifold.

**Theorem 1.2.** [17] The necessary and sufficient condition for a Riemannian manifold to be an Einstein manifold is that the \( \omega^* \)-projective curvature tensor \( W^* \) and con-circular curvature tensor \( C \) are linearly dependent.

**Theorem 1.3.** [17] A Riemannian manifold becomes an Einstein manifold if and only if conformal and \( \omega^* \)-projective curvature tensors of the manifold are linearly dependent.

**Corollary 1.4.** [17] In an Riemannian manifold \( M \), the following are equivalent

i) \( M \) is an Einstein manifold.

ii) \( \omega^* \)-projective and Weyl projective curvature tensors are linearly dependent.

iii) \( \omega^* \)-projective and con-circular curvature tensors are linearly dependent.

iv) \( \omega^* \)-projective curvature and conformal curvature tensors are linearly dependent.
Again in this passage to introduce an almost contact manifold, we repeat the relevant material from Blair [7] without proofs.

Let \( M \) be a \((2n+1)\)-dimensional differentiable manifold equipped with a triplet \((\varphi, \xi, \eta)\), where \( \varphi \) is a type of \((1,1)\) tensor field, \( \xi \) is a vector field, \( \eta \) is a 1-form on \( M \) such that

\[
\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi,
\]

which implies

\[
\varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad \text{rank}(\varphi) = 2n.
\]

\( M \) is said to have an almost contact metric structure \((\varphi, \xi, \eta, g)\) when it admits a Riemannian metric \( g \), such that

\[
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X) \eta(Y),
\]

\[
\eta(X) = g(X, \xi),
\]

On this type of manifold, \( \Phi \) is the fundamental 2-form on \( M^{2n+1} \) and defined by

\[
\Phi(X, Y) = g(\varphi X, Y),
\]

for vector fields \( X, Y \) on \( M^{2n+1} \).

An almost contact manifold \((M^{2n+1}, \varphi, \xi, \eta)\) is said to be normal if the Nijenhuis torsion

\[
N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y] + 2d\eta(X, Y)\xi,
\]

vanishes for any vector fields \( X, Y \) on \( M^{2n+1} \). With the normality condition, an almost cosymplectic manifold is called a cosymplectic manifold. Equipping with almost contact metric structure, if both \( \nabla \eta \) and \( \nabla \Phi \) vanish then we can say that the manifold is cosymplectic. Meanwhile, an almost contact metric structure is Kenmotsu if and only if

\[
(\nabla_X \varphi) Y = g(\varphi X, Y) \xi - \eta(Y) \varphi X.
\]

A normal almost cosymplectic and almost Kenmotsu manifolds are called a cosymplectic manifold and Kenmotsu manifold, respectively.

An almost contact metric manifold \( M^{2n+1} \) is said to be almost \( \alpha \)-Kenmotsu if \( d\eta = 0 \) and \( d\Phi = 2\alpha \eta \wedge \Phi \), \( \alpha \) is a non-zero real constant. Geometrical properties and examples of almost \( \alpha \)-Kenmotsu manifolds are studied by many mathematicians in [1, 5, 6, 8]. Giving an almost Kenmotsu metric structure \((\varphi, \xi, \eta, g)\), considering the deformed structure

\[
\eta' = \frac{1}{\alpha} \eta, \quad \xi' = \alpha \xi, \quad \varphi' = \varphi, \quad g' = \frac{1}{\alpha} g, \quad \alpha \neq 0, \quad \alpha \in \mathbb{R},
\]

where \( \alpha \) is a non-zero real constant and we get an almost \( \alpha \)-Kenmotsu structure \((\varphi', \xi', \eta', g')\). We can call this deformation as a homothetic deformation. It should be noted that almost \( \alpha \)-Kenmotsu structures are associated with some of the specific local conformal deformations of almost cosymplectic structures(see [8]).
If we examine these in two classes, we get a new expression about an $\alpha$-cosymplectic manifold as defined by the formula

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi,$$

for any real number $\alpha$ (see [1]). It is obvious that a normal almost $\alpha$-cosymplectic manifold is an $\alpha$-cosymplectic manifold. In addition, an $\alpha$-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or $\alpha$-Kenmotsu ($\alpha \neq 0$) for $\alpha \in \mathbb{R}$. Almost $\alpha$-cosymplectic manifolds have been studied by several authors [2, 3, 4].

Suppose that $M^{2n+1}$ is an $\alpha$-cosymplectic manifold. Denoted by $A$ the $(1,1)$-tensor field on $M^{2n+1}$ defined by

$$A = -\nabla \xi,$$

and given by the following relations where $\mathcal{L}$ is the Lie derivative of $g$. It is obvious that, $A(\xi) = 0$. Furthermore, the tensor fields $A$ is a symmetric operators and satisfies the following relations

$$AX = -\alpha \varphi^2 X,$$

$$\nabla X \eta) Y = \alpha [g(X, Y) - \eta(X)\eta(Y)],$$

$$\delta \eta = -2\alpha n,$$

$$tr(A) = -2\alpha n,$$

$$tr(\varphi A) = 0,$$

$$A\varphi + \varphi A = -2\alpha \varphi,$$

$$A\xi = 0,$$

$$\nabla X A) \xi = A^2 X,$$

for any vector fields $X, Y$ on $M^{2n+1}$.

In this paper, properties of $\alpha$-cosymplectic manifolds with $M$-projective curvature tensor are studied. While Section 2 includes the curvature properties of $\alpha$-cosymplectic manifolds, in Section 3 we show that an $M$-projectively flat $2n + 1$-dimensional $\alpha$-cosymplectic manifold $M^{2n+1}$ is locally isometric to the hyperbolic space $H^{2n+1}(-\alpha^2)$. In Section 4 the zero state of $W^*$ is examined and interpreted. In this part we give an $\alpha$-cosymplectic manifold satisfying this condition and we obtain some results. And finally in Section 5, we proved that in an $\alpha$-cosymplectic manifold $M^{2n+1}$ with $M$-projective curvature tensor is irrotational if and only if it is locally isometric to the hyperbolic space $H^{2n+1}(-\alpha^2)$. 
2 Basic curvature relations

In this section, basic curvature relations will be briefly given. Let \((M^{2n+1}, \phi, \xi, \eta, g)\) be an \(\alpha\)-cosymplectic manifold. We denote the curvature tensor \(g\) by \(R\) and Ricci tensor of \(g\) by \(S\). Meanwhile, we define a self adjoint operator \(l = R(\cdot, \xi)\) (the Jacobi operator with respect to \(\xi\)). One can easily see the followings

**Proposition 2.1.** [2] Let \(M^{2n+1}\) be an \(\alpha\)-cosymplectic manifold. Then we have

\[
\begin{align*}
R(X, Y)\xi &= \alpha^2 [\eta(X)Y - \eta(Y)X], \quad (2.1) \\
R(X, \xi)\xi &= \alpha^2 \varphi^2 X, \quad (2.2) \\
R(\xi, X)Y &= \alpha^2 [\eta(Y)X - g(X, Y)\xi], \quad (2.3) \\
R(X, \xi)\xi - \varphi R(\varphi X, \xi)\xi &= 2\alpha^2 \varphi^2 X, \quad (2.4) \\
S(X, \xi) &= -\alpha^2 2n\eta(X), \quad (2.5) \\
\eta(R(X, Y)Z) &= \alpha^2 [\eta(Y)g(X, Z) - \eta(X)g(Y, Z)], \quad (2.6) \\
tr(l) &= S(\xi, \xi) = -2n\alpha^2,
\end{align*}
\]

for any vector fields \(X, Y\) on \(M^{2n+1}\).

If a \((2n+1)\)-dimensional \(\alpha\)-cosymplectic manifold \((M^{2n+1}, g)\) is \(\eta\)-Einstein, then its non-vanishing Ricci-tensor \(S\) can be written as follows.

\[
S(X, Y) = a g(X, Y) + b \eta(X)\eta(Y)
\]

for arbitrary vector fields \(X, Y\) and \(a, b\) are smooth functions on \((M^{2n+1}, g)\). When \(b = 0\), then \(\eta\)-Einstein manifold becomes Einstein manifold. Kenmotsu [5] proved that an \(\eta\)-Einstein Kenmotsu manifold \((M^{2n+1}, g)\) satisfies the relation \(a + b = -2n\). With the same way, we can prove the following lemma.

**Lemma 2.2.** On an almost \(\alpha\)-cosymplectic manifold \((M^{2n+1}, g)\), \(a + b = -2n\alpha^2\).

**Proof.** In view of (1.7)-(1.9) and (2.7), we have
\[
QX = aX + b\eta(X)\xi,
\]

such that Ricci operator \(Q\) is defined by
\[
S(X, Y) \overset{\text{def}}{=} g(QX, Y).
\]

Again, contracting (8.8) with respect to \(X\) and using (1.7)-(1.9), we have
\[
r = (2n + 1)a + b.
\]

Now, putting \(\xi\) instead of \(X\) and \(Y\) in (2.7) and then using the equations in (1.7)-(1.9) and (2.5) we get
\[
a + b = -2n\alpha^2.
\]

Equations (2.10) and (2.11) give
\[
a = \left(\frac{r}{2n} + \alpha^2\right) \text{ and } b = -\left(\frac{r}{2n} + (2n + 1)\alpha^2\right).
\]

Equation (2.12) prove the statement of the Lemma 2.2.
3 $M$-projectively flat $\alpha$-cosymplectic manifolds

In view of $W^* = 0$, (1.1) becomes

$$R(X, Y)Z = \frac{1}{4n}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)gX - g(X, Z)gY].$$

Substituting $Z = \xi$ in (3.1) and then using (1.7)-(1.9) and (2.5), we obtain

$$2\alpha^2n(\eta(Y)X - \eta(X)Y) + 4nR(X, Y)\xi = \eta(Y)QX - \eta(X)QY.$$  

Again putting $Y = \xi$ in the above relation and using (1.7), (1.8), (1.9) and (2.5), we have

$$QX = -2\alpha^2nX \iff S(X, Y) = -2\alpha^2ng(X, Y)$$

and

$$r = -2\alpha^2n(2n + 1).$$

In consequence of (3.3), the equation (3.1) becomes

$$R(X, Y)Z = -\alpha^2\{g(Y, Z)X - g(X, Z)Y\}.$$  

A space form is said to be hyperbolic if and only if the sectional curvature tensor is negative [12, 14]. Thus, we can express the following theorem.

**Theorem 3.1.** An almost $\alpha$-cosymplectic manifold $M^{2n+1}$ is $M$-projectively flat if and only if it is locally isometric to the hyperbolic space $H^{2n+1}(\alpha^2)$.

In view of the equations (3.3) and (3.4), and the Theorem 3.1, we can give the following corollaries.

**Corollary 3.2.** Every $M$-projectively flat almost $\alpha$-cosymplectic manifold $M^{2n+1}$ is Ricci symmetric.

**Corollary 3.3.** An $M$-projectively flat almost $\alpha$-cosymplectic manifold $M^{2n+1}$ possesses a constant scalar curvature.

A triplet $(g, V, \lambda)$ defined on $M^{2n+1}$ is said to a Ricci soliton if it satisfies

$$\mathcal{L}_V g(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0$$

for vector fields $X$ and $Y$ on $M^{2n+1}$, where $\mathcal{L}_V g$ denotes the Lie derivative of the Riemannian metric $g$ along the complete vector field $V$ on $M^{2n+1}$ and $\lambda$ is a real constant. The Ricci soliton $(g, V, \lambda)$ is said to be shrinking, expanding or steady if $\lambda$ is $< 0$, $> 0$ or $= 0$, respectively.

If possible, we suppose that the $M$-projectively flat almost $\alpha$-cosymplectic manifolds admit a Ricci soliton $(g, V, \lambda)$. Replacing $X$, $Y$ and $V$ by the structure vector field $\xi$ in (3.6) and keeping in mind that $\nabla_\xi \xi = 0$ on $M^{2n+1}$ and the equation (3.3), we find

$$\lambda = -S(\xi, \xi) = 2n\alpha^2$$

and

$$\mathcal{L}_V g(X, Y) = 0.$$  

This shows that the complete vector field $V$ is Killing, that is $\mathcal{L}_V g = 0$, on $M^{2n+1}$. By the way we can specify the following theorem.
Theorem 3.4. If an $M$-projectively flat almost $\alpha$-cosymplectic manifold $M^{2n+1}$ admits a Ricci soliton $(g, V, \lambda)$, then the vector field $V$ is Killing and the Ricci soliton $(g, V, \lambda)$ to be expanding.

4 $M$-projectively semi-symmetric $\alpha$-cosymplectic manifolds

In view of (1.7)-(1.9), (2.3), (2.7), (2.8) and (2.12), the equation (1.1) becomes

\begin{equation}
(4.1) \quad W^*(\xi, X)Y = \left\{ \alpha^2 + \frac{1}{4n} \left( \frac{r}{2n} - \alpha^2 (2n - 1) \right) \right\} \left\{ \eta(Y)X - g(X, Y)\xi \right\}.
\end{equation}

Now, we have

\begin{equation}
(4.2) \quad (W^*(\xi, X) \cdot \mathbb{R})(Y, Z)U = W^*(\xi, X)R(Y, Z)U - R(W^*(\xi, X)Y, Z)U - R(Y, W^*(\xi, X)Z)U - R(Y, Z)W^*(\xi, X)U.
\end{equation}

Let us suppose that an $\alpha$-cosymplectic manifold $M^{2n+1}$ is $M$-projectively semi-symmetric, that is, $W^*(X, Y) \cdot \mathbb{R} = 0$ for all vector fields $X$ and $Y$ on $M^{2n+1}$. In consequence of $W^*(\xi, X) \cdot \mathbb{R} = 0$, the equation (4.2) becomes

\begin{equation}
\end{equation}

In view of (4.1), the last equation takes the form

\begin{equation}
\{ \alpha^2 + \frac{1}{4n} \left( \frac{r}{2n} - \alpha^2 (2n - 1) \right) \} \left\{ \eta(R(Y, Z)U)X - \mathcal{R}(Y, Z, U, X)\xi - \eta(Y)R(X, Z)U + g(X, Y)R(\xi, Z)U - \eta(\xi)R(Y, Z)U + g(Y, Z)R(\eta, X)U - \eta(U)R(Y, Z)U \right\} = 0,
\end{equation}

where

\begin{equation}
\mathcal{R}(X, Y, Z, U) = g(\mathcal{R}(X, Y)Z, U).
\end{equation}

Taking inner-product of the above equation according to the Riemannian metric $g$ and then using (1.7)-(1.9) and (2.6), we have

\begin{equation}
(4.3) \quad \left\{ \alpha^2 + \frac{1}{4n} \left( \frac{r}{2n} - \alpha^2 (2n - 1) \right) \right\} \left\{ \mathcal{R}(Y, Z, U, X) + \alpha^2 \left\{ g(X, Y)g(Z, U) - g(X, Z)g(Y, U) \right\} \right\} = 0,
\end{equation}

which implies that either $r = -2n(2n + 1)\alpha^2$, that is, the scalar curvature of $M^{2n+1}$ is constant or

\begin{equation}
\mathcal{R}(Y, Z, U, X) = \alpha^2 \left\{ g(X, Z)g(Y, U) - g(Y, Z)g(X, U) \right\},
\end{equation}

equivalent to

\begin{equation}
(4.4) \quad R(Y, Z)U = \alpha^2 \left\{ g(Y, U)Z - g(Z, U)Y \right\}.
\end{equation}

This reflects that the $M$-projectively semi-symmetric $\alpha$-cosymplectic manifold $M^{2n+1}$ is a space form. Contracting (4.4) according to the vector field $Y$, we find

\begin{equation}
(4.5) \quad S(Z, U) = -2\alpha^2 ng(Z, U),
\end{equation}

where $S$ is the scalar curvature.
which gives

\[(4.6) \quad QZ = -2\alpha^2 nZ\]

and

\[(4.7) \quad r = -2\alpha^2 n(2n + 1).\]

Conversely, the equations (1.7)-(1.9), (4.1), (4.4)-(4.6) and (4.2) give \(W^*(\xi, X) \cdot R = 0\). Thus, consequently we state:

**Theorem 4.1.** Every \(\alpha\)-cosymplectic manifold \(M^{2n+1}\) is \(M\)-projectively semi-symmetric if and only if either \(M^{2n+1}\) is locally isometric to the hyperbolic space \(H^{2n+1}(-\alpha^2)\) or \(M^{2n+1}\) has constant scalar curvature \(-2\alpha^2 n(2n+1)\).

In presence of the Theorem 3.4 and the Theorem 4.1, we can state the following corollary.

**Corollary 4.2.** If an \(M\)-projectively semi-symmetric \(\alpha\)-cosymplectic manifold \(M^{2n+1}\) confesses a Ricci soliton \((g, V, \lambda)\), then the vector field \(V\) is a Killing vector field and \((g, V, \lambda)\) is expanding.

Also, in the light of the Corollary 1.4, Theorem 3.1 and the Theorem (4.1), we have

**Corollary 4.3.** A \((2n+1)\)-dimensional \(\alpha\)-cosymplectic manifold \(M^{2n+1}\) is \(M\)-projectively semi-symmetric if and only if it is conformally flat.

## 5 \(\eta\)-Einstein \(\alpha\)-cosymplectic manifolds with the irrotational \(M\)-projective curvature tensor

In this section, we study the properties of the irrotational \(M\)-projective curvature tensor on an \(\eta\)-Einstein \(\alpha\)-cosymplectic manifold \(M^{2n+1}\). Before going to prove our results, we give the following definition.

**Definition 5.1.** Let \(\nabla\) be a Riemannian connection with respect to the Riemannian metric \(g\), then the rotation (Curl) of \(M\)-projective curvature tensor \(W^*\) on a \((2n+1)\)-dimensional \(\alpha\)-cosymplectic manifold \(M^{2n+1}\) is defined as

\[
(5.1) \quad \text{Rot} W^* = (\nabla_V W^*)(X, Y)Z + (\nabla_X W^*)(U, Y)Z + (\nabla_Y W^*)(X, U)Z - (\nabla_Z W^*)(X, Y)U
\]

for all vector fields \(X, Y, Z\) and \(U\) on \(M^{2n+1}\) (see [17]). If \(\text{Rot} W^* = 0\) on \(M^{2n+1}\), then the \(M\)-projective curvature tensor is irrotational.

In consequence of the Bianchi’s second identity for the Riemannian connection \(\nabla\), the equation (5.1) becomes

\[
(5.2) \quad \text{Rot} W^* = - (\nabla_Z W^*)(X, Y)U.
\]
If the $M$-projective curvature tensor is irrotational, then curl $W^* = 0$ and therefore $(\nabla_Z W^*)(X,Y)U = 0$, which gives

\begin{equation}
\end{equation}

Replacing $U = \xi$ in (5.3), we have

\begin{equation}
(\nabla_Z W^*)(X,Y)\xi = W^*(\nabla_Z X,Y)\xi + W^*(X,\nabla_Y Z)\xi + W^*(X,Y)\nabla_Z \xi.
\end{equation}

Let us suppose that $M^{2n+1}$ is a $(2n+1)$-dimensional $\eta$-Einstein $\alpha$-cosymplectic manifold. Then substituting $Z = \xi$ in (1.1) and then using (1.7)-(1.9), (2.1), (2.5) and (2.12), we obtain

\begin{equation}
W^*(X,Y)\xi = k[\eta(X)Y - \eta(Y)X],
\end{equation}

where

\begin{equation}
k = \left\{ \frac{\alpha^2}{2} + \frac{1}{4n} \left( \frac{r}{2n} + \alpha^2 \right) \right\}.
\end{equation}

Using (1.7)-(1.9), (1.14), (5.5) and (5.6) in (5.4), we obtain

\begin{equation}
W^*(X,Y)Z = \frac{dr(Z)}{8n^2} \{ \eta(X)Y - \eta(Y)X \} + k\alpha[g(X,Z)Y - g(Y,Z)X],
\end{equation}

where $d$ is an exterior derivative. Contracting (5.7) along the vector field $X$ and then using the equations (1.1), (1.7), (2.5) and (5.6) together with $Y = \xi$, we have

\begin{equation}
dr(Z) = 0,
\end{equation}

which shows that the scalar curvature of $M^{2n+1}$ is constant. In consequence of (5.8), the equation (5.7) shows that either $\alpha = 0$ or

\begin{equation}
W^*(X,Y)Z = k[g(X,Z)Y - g(Y,Z)X], \quad \alpha \neq 0.
\end{equation}

An $\eta$-Einstein $\alpha$-cosymplectic manifold $M^{2n+1}$ is cosymplectic if $\alpha = 0$. If possible, we suppose that $\alpha \neq 0$, then the equations (1.1) and (5.9) give

\begin{equation}
k[g(X,Z)Y - g(Y,Z)X] = R(X,Y)Z - \frac{1}{4n} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].
\end{equation}

Contracting the above equation with respect to the vector field $X$ and then using (5.6), we find

\begin{equation}
S(Y,Z) = -2n\alpha^2 g(Y,Z) \iff QY = -2n\alpha^2 Y,
\end{equation}

which gives

\begin{equation}
r = -2n(2n + 1)\alpha^2,
\end{equation}

provided $\alpha \neq 0$. In consequence of (1.1), (5.6), (5.7), (5.10) and (5.11), we can find

\begin{equation}
R(X,Y)Z = -\alpha^2[g(Y,Z)X - g(X,Z)Y].
\end{equation}

Thus, we can state:
Theorem 5.1. The $M$-projective curvature tensor in an $\eta$-Einstein $\alpha-$cosymplectic manifold $M^{2n+1}$ is irrotational if and only if it is locally isometric to the hyperbolic space $H^{2n+1}(-\alpha^2)$ or locally Riemannian product of an almost Kaehler manifold with the real line.

Theorem 4.1 together with the Theorem 5.1 lead to the following corollaries.

Corollary 5.2. A $2n + 1$-dimensional $\eta$-Einstein $\alpha$-cosymplectic manifold $M^{(2n+1)}$ to be $M$-projectively semi-symmetric if and only if the $M$-projective curvature tensor of $M^{(2n+1)}$ is irrotational.

Corollary 5.3. The $M$-projective curvature tensor in an $\eta$-Einstein $\alpha$-cosymplectic manifold $M^{2n+1}$ is irrotational if and only if the manifold is conformally flat.

Corollary 5.4. If a $(2n + 1)$-dimensional $\eta$-Einstein $\alpha$-cosymplectic manifold $M^{2n+1}$ is irrotational, then $M^{2n+1}$ possesses a constant scalar curvature.

Corollary 5.5. If the $M$-projective curvature tensor of an $\eta$-Einstein $\alpha$-cosymplectic manifold $M^{2n+1}$ is irrotational, then $M^{2n+1}$ is either cosymplectic or $\alpha$-Kenmotsu.

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