Some properties of pseudo conharmonically symmetric Riemannian manifolds

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Abstract. In the present paper, we consider pseudo conharmonically symmetric manifold denoted by $(PCHS)_n$. In the first section, we give the definition of this manifold. In the second section, we prove some theorems including some properties of this manifold. In the last section, we give an example for the existence of this manifold.

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1 Introduction

As we know, in differential geometry, symmetric spaces play an important role. In the late twenties, Cartan [4] initiated Riemannian symmetric spaces and obtained a classification of these spaces. Let (M, g) be an n-dimensional Riemannian manifold with the Riemannian metric g and the Levi-Civita connection ∇. If the Riemannian curvature tensor of a Riemannian manifold satisfies the condition $\nabla R = 0$ then this manifold is called locally symmetric [4]. For every point P of this manifold, this symmetry condition is equivalent to the fact that the local geodesic symmetry $F(P)$ is an isometry [12]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature. Many authors have been studied the notion of locally symmetric manifolds extending several manifolds such as conformally symmetric manifolds [5], recurrent manifolds [25], conformally recurrent manifolds [2], conformally symmetric Ricci-recurrent spaces [18], pseudo-Riemannian manifold with recurrent concircular curvature tensor [13], semi-symmetric manifolds [23], pseudo-symmetric manifolds [6, 14, 15], weakly symmetric manifolds [24], projective symmetric manifolds [22], almost pseudo-concircularly symmetric manifolds [7], decomposable almost pseudo-conharmonically symmetric manifolds [3], etc. A non-flat Riemannian manifold (M, g) $(n > 2)$ is said to be a pseudo-symmetric manifold

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[6] if its curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W =& 2A(X)R(Y,Z)W + A(Y)R(X,Z)W + A(Z)R(Y,X)W \\ &+ A(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho,\end{aligned} \tag{1.1}$$

where A is a non-zero 1-form, ρ is a vector field defined by

$$g(X,\rho) = A(X), \tag{1.2}$$

for all X and ∇ denotes the operator of the covariant differentiation with respect to the metric tensor g. The 1-form A is called the associated 1-form of the manifold. If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of E.Cartan. An n-dimensional pseudo-symmetric manifold is denoted by $(PS)_n$. This is to be noted that the notion of pseudo-symmetric manifold studied in particular by Deszcz [11] is different from that Chaki [6]. The notion of weakly symmetric manifolds was introduced by Tamassy and Binh [24]. If the curvature tensor of type (1,3) of an n-dimensional Riemannian manifold $(n > 2)$ satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W =& A(X)R(Y,Z)W + B(Y)R(X,Z)W + D(Z)R(Y,X)W \\ &+ E(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho,\end{aligned} \tag{1.3}$$

where ∇ denotes the Levi-Civita connection on (M,g) and A,B,D,E and ρ are 1-forms and a vector field respectively, which are non-zero simultaneously, then this manifold is denoted by $(WS)_n$. Many authors have been studied weakly symmetric manifolds [8, 9, 16, 17, 19], etc.

Conformal transformation of a Riemannian structure is an important object of study in differential geometry. The conharmonic transformation which is a special type of conformal transformations preserves the harmonicity of smooth functions. Such transformation has an invariant tensor which is called the conharmonic curvature tensor. It is easy to verify that this tensor is an algebraic curvature tensor, that is, it possesses the classical symmetry properties of the Riemannian curvature tensor.

Let M and N be two Riemannian manifolds with the metrics g and $\overline{g}$, respectively related by

$$\bar{g} = e^{2\sigma} g, \tag{1.4}$$

where σ is a real function. Then M and N are called conformally related manifolds, and the correspondence M and N is known as conformal transformation [21]. It is known that a harmonic function is defined as a function whose Laplacian vanishes. In generally, the harmonic function is not invariant. In 1957, Ishii obtained the conditions which a harmonic function remains invariant and he introduced the conharmonic transformation as a subgroup of the conformal transformation (1.4) satisfying the condition

$$\sigma^h_{,h} + \sigma^h_{,h}\sigma^h_{,} = 0, \tag{1.5}$$

where comma denotes the covariant differentiation with respect to the metric g.

A rank-four tensor H that remains invariant under conharmonic transformation of a Riemannian manifold (M, g) is given by

$$\begin{aligned} H(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{n-2}\big[&g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \\ &+ g(X,U)S(Y,Z) - g(Y,U)S(X,Z)\big], \end{aligned} \tag{1.6}$$

where R and S denote the Riemannian curvature tensor of type (0,4) defined by $R(X,Y,Z,U) = g(R(X,Y)Z,U)$ and the Ricci tensor of type (0,2), respectively. The curvature tensor defined by (1.6) is known as conharmonic curvature tensor. A manifold whose conharmonic curvature tensor vanishes at every point of the manifold is called conharmonically flat. Thus, this tensor represents the deviation of the manifold from conharmonic flatness. Many authors have been studied the conharmonic curvature tensor, [1, 21]. The present paper deals with an n-dimensional pseudo-conharmonically symmetric Riemannian manifold (M, g) (non-conharmonically flat) whose conharmonic curvature tensor H satisfies the condition

$$\begin{aligned} (\nabla_X H)(Y,Z,U,V) =& 2A(X)H(Y,Z,U,V) + A(Y)H(X,Z,U,V) \\ &+ A(Z)H(Y,X,U,V) + A(U)H(Y,Z,X,V) \\ &+ A(V)H(Y,Z,U,X), \end{aligned} \tag{1.7}$$

where A has the meaning already mentioned in (1.2). Such a manifold is called a pseudo-conharmonically symmetric manifold [6] and denoted by $(PCHS)_n$. Since the conformal curvature tensor vanishes identically for $n = 3$, we assume that $n > 3$ throughout the paper. This paper is organized as follows:

Section 2 deals with some properties of $(PCHS)_n$.

In this section, we find the conditions on the scalar curvature if the curvature tensor is generalized recurrent and the conharmonic curvature tensor is generalized recurrent. We also find that if the conharmonic curvature tensor of $(PCHS)_n$ is Codazzi type, then the scalar curvature of this manifold must be zero. We prove that, in a $(PCHS)_n$, r is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P of this manifold admits pseudo-symmetric curvature tensor whose the associated 1-form is the same with the 1-form of $(PCHS)_n$, where $g(X,P) = A(X)$.

In this section, we also show that if a $(PCHS)_n$ with non-constant scalar curvature admits torse-forming vector field obtained by the associated 1-form then the torse-forming vector field ϕ and the associated 1-form A are collinear.

In the last section, we give an example for the existence of these manifolds.

2 Some properties of pseudo-conharmonically symmetric Riemannian manifold

Let L denote the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S of type (0,2), that is

$$g(LX,Y) = S(X,Y). \tag{2.1}$$

Let e_i, $(1 \leq i \leq n)$ be an orthonormal basis of the tangent space at any point of the manifold. From (1.6), we have

$$\overline{H}(X,Y) = \sum_{i=1}^{n} H(X,e_i,e_i,Y) = \sum_{i=1}^{n} H(e_i,X,Y,e_i) = -\frac{r}{n-2} g(X,Y) \tag{2.2}$$

and

$$\sum_{i=1}^{n} H(e_i,e_i,X,Y) = \sum_{i=1}^{n} H(X,Y,e_i,e_i) = 0, \tag{2.3}$$

where r is the scalar curvature of the manifold. Also, from (1.6) it follows that [20]

$$\begin{aligned} &H(X,Y,Z,U) = -H(Y,X,Z,U)\\ &H(X,Y,Z,U) = -H(X,Y,U,Z)\\ &H(X,Y,Z,U) = H(Z,U,X,Y)\\ &H(X,Y,Z,U) + H(X,Z,U,Y) + H(X,U,Y,Z) = 0. \end{aligned} \tag{2.4}$$

We assume that our manifold is $(PCHS)_n$. Thus, the relation (1.7) holds.

Theorem 2.1. *If the Ricci tensor of a $(PCHS)_n$ with non-zero scalar curvature is recurrent then the recurrence vector field and the associated 1-form of this manifold are related by*

$$\lambda(X) = \left(\frac{2n+4}{n}\right) A(X).$$

Proof. If we assume that $(PCHS)_n$ admits recurrent Ricci tensor then we have

$$(\nabla_X S)(Y,Z) = \lambda(X) S(Y,Z). \tag{2.5}$$

By taking the covariant derivative of (1.6), we get

$$\begin{aligned} (\nabla_W H)(X,Y,Z,U) =& (\nabla_W R)(X,Y,Z,U)\\ &- \frac{1}{n-2}\big[g(Y,Z)(\nabla_W S)(X,U) - g(X,Z)(\nabla_W S)(Y,U)\\ &+ g(X,U)(\nabla_W S)(Y,Z) - g(Y,U)(\nabla_W S)(X,Z)\big]. \end{aligned} \tag{2.6}$$

By putting (2.5) in (2.6), we find

$$\begin{aligned} (\nabla_W H)(X,Y,Z,U) =& (\nabla_W R)(X,Y,Z,U)\\ &- \frac{\lambda(W)}{n-2}\big[g(Y,Z)S(X,U) - g(X,Z)S(Y,U)\\ &+ g(X,U)S(Y,Z) - g(Y,U)S(X,Z)\big]. \end{aligned} \tag{2.7}$$

Comparing (1.6) and (2.7), we obtain

$$(\nabla_W H)(X,Y,Z,U) = (\nabla_W R)(X,Y,Z,U) + \lambda(W)\big(H(X,Y,Z,U) - R(X,Y,Z,U)\big)$$

i.e.,

$$(\nabla_W H)(X,Y,Z,U) - \lambda(W)H(X,Y,Z,U) = (\nabla_W R)(X,Y,Z,U) - \lambda(W)R(X,Y,Z,U). \tag{2.8}$$

If we put (1.7) in (2.8), it can be found that

$$\begin{aligned} &2A(W)H(X,Y,Z,U) + A(X)H(W,Y,Z,U) + A(Y)H(X,W,Z,U) \\ &+ A(Z)H(X,Y,W,U) + A(U)H(X,Y,Z,W) - \lambda(W)H(X,Y,Z,U) \\ &= (\nabla_W R)(X,Y,Z,U) - \lambda(W)R(X,Y,Z,U). \end{aligned} \tag{2.9}$$

Firstly, contracting on X and U in (2.9), secondly contracting on Y and Z in the last equation and using (1.6), we get

$$\big((2n+4)A(X) - n\lambda(X)\big)r = 0. \tag{2.10}$$

Assuming that r is non-zero, we obtain

$$\lambda(X) = \left(\frac{2n+4}{n}\right)A(X). \tag{2.11}$$

This completes the proof. □

Definition 2.1 A non-flat n-dimensional Riemannian manifold (M,g),$(n > 2)$ is called a generalized recurrent manifold if its curvature tensor R of type (0,4) satisfies the condition

$$(\nabla_X R)(Y,Z,U,W) = \alpha(X)R(Y,Z,U,W) + \beta(X)G(Y,Z,U,W), \tag{2.12}$$

where $G(Y,Z,U,W) = (g(Y,W)g(Z,U) - g(Y,U)g(Z,W))$, α and β are non-zero 1-forms, [10].

Theorem 2.2. *If a $(PCHS)_n$ admits generalized recurrent curvature tensor then the scalar curvature of this manifold is in the following form*

$$r = \frac{n^2(n-1)\beta(X)}{(2n+4)A(X) - n\alpha(X)} \quad \left(A(X) \neq \frac{n}{2n+4}\alpha(X)\right),$$

where β and α are the recurrence vector fields and A is the associated 1-form of this manifold.

Proof. If we assume that $(PCHS)_n$ admits generalized recurrent curvature tensor then from (2.12), we also have

$$(\nabla_X S)(Z,U) = \alpha(X)S(Z,U) + (n-1)\beta(X)g(Z,U). \tag{2.13}$$

By taking the covariant derivative of (1.6) and putting (2.12) and (2.13) in the last equation, we find

$$\begin{aligned} (\nabla_X H)(Y,Z,U,W) = &\alpha(X)\big[R(Y,Z,U,W) \\ &- \frac{1}{n-2}(g(Z,U)S(Y,W) - g(Y,U)S(Z,W) \\ &+ g(Y,W)S(Z,U) - g(Z,W)S(Y,U))\big] \\ &- \frac{n}{n-2}\beta\ (X)(g(Z,U)g(Y,W) - g(Y,U)g(Z,W)). \end{aligned} \tag{2.14}$$

By putting (1.6) in (2.14), we obtain

$$(\nabla_X H)(Y,Z,U,W) = \alpha(X)H(Y,Z,U,W) - \frac{n}{n-2}\beta(X)G(Y,Z,U,W), \tag{2.15}$$

where $G(Y,Z,U,W) = (g(Y,W)g(Z,U) - g(Y,U)g(Z,W))$. Contracting on Y, W in (2.15), after that contracting on Z, U and using (1.6), we get

$$r\big[(2n+4)A(X) - n\alpha(X)\big] = n^2(n-1)\beta(X)$$

i.e.,

$$r = \frac{n^2(n-1)\beta(X)}{(2n+4)A(X) - n\alpha(X)},$$

where $A(X) \neq \frac{n}{2n+4}\alpha(X)$. Thus, the proof is completed. □

Theorem 2.3. *If the conharmonic curvature tensor of a $(PCHS)_n$ is generalized recurrent then the scalar curvature of this manifold is in the following form*

$$r = \frac{n(2-n)(n-1)\beta(X)}{2(n+2)A(X) - n\alpha(X)},$$

where $\alpha(X) \neq \frac{2(n+2)}{n}A(X)$. And if $\alpha(X) > \frac{2(n+2)}{n}A(X)$, then this manifold is of positive scalar curvature or, if $\alpha(X) < \frac{2(n+2)}{n}A(X)$, then it is of negative scalar curvature, where β have positive values.

Proof. If we assume that the conharmonic curvature tensor of $(PCHS)_n$ is generalized recurrent, from (2.12), we can write

$$(\nabla_X H)(Y,Z,U,W) = \alpha(X)H(Y,Z,U,W) + \beta(X)G(Y,Z,U,W), \tag{2.16}$$

where $G(Y,Z,U,W) = (g(Y,W)g(Z,U) - g(Y,U)g(Z,W))$.
Comparing the equation (2.16) with (1.7), we get

$$\begin{aligned} &\alpha(X)H(Y,Z,U,W) + \beta(X)(g(Z,U)g(Y,W) - g(Y,U)g(Z,W)) \\ &\quad = 2A(X)H(Y,Z,U,W) + A(Y)H(X,Z,U,W) + A(Z)H(Y,X,U,W) \\ &\quad + A(U)H(Y,Z,X,W) + A(W)H(Y,Z,U,X). \end{aligned} \tag{2.17}$$

Contracting on Y,W and again contracting on Z,U in (2.17) and using (1.6), we get

$$\big(2(n+2)A(X) - n\alpha(X)\big)r = n(2-n)(n-1)\beta(X)$$

i.e.,

$$r = \frac{n(2-n)(n-1)\beta(X)}{2(n+2)A(X) - n\alpha(X)}, \tag{2.18}$$

where $\alpha(X) \neq \frac{2(n+2)}{n}A(X)$. Assuming that $\beta(X)$ have positive values. Thus, if $\alpha(X) > \frac{2(n+2)}{n}A(X)$, from (2.18) then the scalar curvature is positive; if $\alpha(X) < \frac{2(n+2)}{n}A(X)$ then the scalar curvature is negative. Thus, the proof is completed. □

Corollary 2.1. *If the conharmonic curvature tensor of $(PCHS)_n$ with non-zero scalar curvature is recurrent then the recurrence vector field and the associated 1-form of this manifold are related by the following form*

$$\alpha(X) = \frac{2(n+2)}{n} A(X).$$

Proof. If we assume that the conharmonic curvature tensor of $(PCHS)_n$ is recurrent, from Theorem 2.3, for $\beta(X) = 0$, we get

$$r\big(2(n+2)A(X) - n\alpha(X)\big) = 0. \tag{2.19}$$

Considering that $(PCHS)_n$ admits non-zero scalar curvature, by the aid of (2.19), we find that

$$\alpha(X) = \frac{2(n+2)}{n} A(X).$$

Thus, the proof is completed. □

Theorem 2.4. *If the conharmonic curvature tensor of a $(PCHS)_n$ is Codazzi type then this manifold must be of zero scalar curvature.*

Proof. Let us assume that the conharmonic curvature tensor of $(PCHS)_n$ is Codazzi type then we have

$$(\nabla_X H)(Y,V) - (\nabla_V H)(Y,X) = 0. \tag{2.20}$$

Contracting (1.7) on Z, U and using (2.2) and (2.20) we finally get

$$(n+2)A(X)r = 0. \tag{2.21}$$

Since $A(X) \neq 0$, the scalar curvature of $(PCHS)_n$ must be zero. This completes the proof. □

Theorem 2.5. *In a $(PCHS)_n$, r is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P if this manifold admits pseudo-symmetric curvature tensor whose associated 1-form is the same with the 1-form of $(PCHS)_n$, where*

$$g(X,P) = A(X),$$

for every vector field P and non-zero 1-form A.

Proof. We assume that the curvature tensor of $(PCHS)_n$ is pseudo-symmetric then we have the equation (1.1).
By using (1.1) we get

$$\begin{aligned}(\nabla_Z S)(X,Y) =& 2A(Z)S(X,Y) + A(R(X,Y)Z) + A(X)S(Z,Y)\\ &+ A(Y)S(X,Z) + A(R(Z,Y)X).\end{aligned} \tag{2.22}$$

By changing X,Y and Z in (2.22) and putting these four equations in (1.7), we finally get

$$A(QX) = (r/n)A(X).$$

Thus, we can say that r is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P. The proof is completed. □

Definition 2.2. A vector field ξ in a Riemannian manifold M is called torse-forming if it satisfies the following condition

$$\nabla_X \xi = \rho X + \phi(X)\xi, \tag{2.23}$$

where $X \in TM$, $\phi(X)$ is a linear form and ρ is a function [26] .
In local transcription, this reads

$$\xi^h_{,i} = \rho\delta^h_i + \xi^h \phi_i, \tag{2.24}$$

where ξ^h and ϕ_i are the components of ξ and ϕ, and δ^h_i is the Kronecker symbol. A torse-forming vector field ξ is called
i) recurrent, if $\rho = 0$
ii) concircular, if the form ϕ_i is gradient covector, i.e., there is a function ψ such that $\phi = d\psi(X)$.
iii) convergent, if it is concircular and $\rho = const.exp(\phi)$ Therefore, recurrent vector fields are characterized by the following equation

$$\nabla_X \xi = \phi(X)\xi. \tag{2.25}$$

Also, from Definition2.2, for concircular vector field ξ, we get

$$(\nabla_Y \xi)X = \rho g(X, Y) \tag{2.26}$$

for all $X, Y \in \mathbb{TM}$.

Theorem 2.6. *If a $(PCHS)_n$ with non-constant scalar curvature admits torse-forming vector field obtained by the associated 1-form then the torse-forming vector field ϕ and the associated 1-form A must be collinear.*

Proof. Contracting on Y,V; Z,U in (1.7) and using (1.6), we get

$$\nabla_X r = \frac{2(n+2)}{n} A(X)r. \tag{2.27}$$

By taking the covariant derivative of (2.27), we find

$$\nabla_Y \nabla_X r = \frac{2(n+2)}{n}[(\nabla_Y A)(X)r + A(X)(\nabla_Y r)]. \tag{2.28}$$

Substituting (2.27) in (2.28), it can be easily seen that

$$\nabla_Y \nabla_X r = \frac{2(n+2)}{n}[(\nabla_Y A)(X) + \frac{2(n+2)}{n} A(X)A(Y)]r. \tag{2.29}$$

If we assume that the 1-form A is a torse-forming vector field, from (2.24), we have

$$(\nabla_Y A)(X) = \rho g(X,Y) + \phi(Y)A(X), \tag{2.30}$$

where ρ is a function, $\phi(Y)$ is a linear form. By putting (2.30) in (2.29), we obtain

$$\nabla_Y \nabla_X r = \frac{2(n+2)}{n}[\rho g(X,Y) + \phi(Y)A(X) + \frac{2(n+2)}{n} A(X)A(Y)]r. \tag{2.31}$$

By changing X and Y in (2.31) and subtracting these two equations, we get

$$\phi(Y)A(X) - \phi(X)A(Y) = 0.$$

Finally, we can say that ϕ and A are collinear. Thus, the proof is completed. □

3 An example of $(PCHS)_n$

In this section we will give an example for $(PCHS)_n$ satisfying the conditions (1.6) and (1.7).
We define a Riemannian metric on R^n $(n \geq 4)$ by the formula, [18]

$$ds^2 = \varphi(dx^1)^2 + k_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n, \tag{3.1}$$

where $[k_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constant and φ is a function of $x^1, x^2, ..., x^{n-1}$ and independent of x^n. Let each Latin index runs over $1, 2, .., n$ and each Greek index runs over $2, 3, .., (n-1)$.

In the metric considered, the only non-vanishing components of Christoffel symbols, the curvature tensor and the Ricci tensor are, according to [18]

$$\begin{gathered}\Gamma^\beta_{11} = -\frac{1}{2}k^{\alpha\beta}\varphi_{,\alpha}, \quad \Gamma^n_{11} = \frac{1}{2}\varphi_{,1} \quad \Gamma^n_{1\alpha} = \frac{1}{2}\varphi_{,\alpha} \\ R_{1\alpha\beta 1} = \frac{1}{2}\varphi_{,\alpha\beta}, \quad R_{11} = \frac{1}{2}k^{\alpha\beta}\varphi_{,\alpha\beta},\end{gathered} \tag{3.2}$$

where "," denotes the partial differentiation with respect to the coordinates and $k^{\alpha\beta}$ are the elements of the matrix inverse to $[k_{\alpha\beta}]$.
We consider $k_{\alpha\beta}$ as the Kronecker symbol $\delta_{\alpha\beta}$ and φ as, [10]

$$\varphi = M_{\alpha\beta}x^\alpha x^\beta e^{(x^1)^2}, \tag{3.3}$$

where $M_{\alpha\beta}$ are constants and satisfy the relations

$$M_{\alpha\beta} \neq 0 \quad for \quad \alpha, \beta = 2, ..., (n-1)$$

$$\sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0. \tag{3.4}$$

In this case, we have the following relations

$$\varphi_{,\alpha\beta} = 2M_{\alpha\beta}e^{(x^1)^2}$$

$$\delta^{\alpha\beta}M_{\alpha\beta} = \sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0. \tag{3.5}$$

Thus, from (3.3) and (3.5), we have

$$\delta^{\alpha\beta}\varphi_{,\alpha\beta} = 0. \tag{3.6}$$

By using (3.2), we find the only non-zero components for R_{hijk} and S_{ij} as

$$\begin{aligned} R_{1\alpha\beta 1} &= \frac{1}{2}\varphi_{,\alpha\beta} = M_{\alpha\beta}e^{(x^1)^2} \\ S_{11} &= \frac{1}{2}\varphi_{,\alpha\beta}\delta^{\alpha\beta} = 0. \end{aligned} \tag{3.7}$$

Hence, the only non-zero components of the conharmonic curvature tensor H_{hijk} are

$$\begin{aligned} H_{1\alpha\alpha 1} &= R_{1\alpha\alpha 1} - \frac{1}{n-2}(g_{\alpha\alpha}S_{11}) \\ &= M_{\alpha\beta}e^{(x^1)^2} \end{aligned} \tag{3.8}$$

which never vanish. From (3.8), the only non-zero components of the derivative of H_{hijk} are found as

$$\begin{aligned} H_{1\alpha\beta 1,1} &= 2x^1 M_{\alpha\beta}e^{(x^1)^2} \\ &= 2x^1 H_{1\alpha\beta 1}. \end{aligned} \tag{3.9}$$

Let us consider the associated 1-form as

$$A_i(x) = \begin{cases} \frac{x^1}{2}, & \text{for } i = 1 \\ 0, & \text{otherwise} \end{cases} \tag{3.10}$$

at any point $x \in V_n$.
To verify the relation (1.7) it is sufficient to prove that the equation

$$H_{1\alpha\beta 1,1} = 4A_1 H_{1\alpha\beta 1}. \tag{3.11}$$

By using (3.9) and (3.10), we can easily see that (3.11) is satisfied. The other components of each term of (1.7) vanish identically and the relation (1.7) holds trivially. Under our assumptions (3.1), (3.3) and (3.4), this manifold is a $(PCHS)_n$. □

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