A spherical projection of a complex Hilbert space is conformal iff it is the stereographic projection

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Abstract. We consider a family of nonlinear projections that map a complex Hilbert space onto a bounded "bowl" shaped subset of a sphere. Our main result states that a projection is conformal iff it is the stereographic projection and iff the projection renders all certain pairs of triangles induced by the projection to be similar. It follows that various so called "compactifications" that are given in the literature are special members of this family of nonlinear projections. These include the stereographic projection and the Poincare compactification. Background and motivation are discussed and several examples illustrating the results are provided.

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1 Introduction

The concept of a stereographic projection, where points of the "heavenly sphere" are matched in a one-to-one manner to the points of a plane passing through the center of the sphere, was known to Ptolemy about 200 CE (T. J. Heath [24], page 292). Ptolemy (Claudius Ptolemaeus, born c. 100 CE—died c. 170 CE) was motivated by astronomy. This is the earliest reference we could find where the unbounded set of points lying in a plane are mapped in a one-to-one manner onto the bounded set of points of a sphere (except the single fixed projection point that is either the north or south pole). In text books of complex function analysis, this sphere is called the *Riemann sphere* and the mapping is known as the *stereographic projection*. See e.g. L. V. Ahlfors [2] (page 19) and E. Hille [26] (pages 38-44) for derivations. In 1881, Poincare proposed a different mapping where the fixed projection point coincides with the center of the sphere. These two types of projection mappings are also referred to as "compactifications," as they map all points of the non-compact plane onto a bounded set of a sphere.

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Even though these projections have been known for approximately two thousand years, new applications have been found in the previous century and in this new milenia. The stereographic projection and Poincare's compactifications have been used to great advantage in $\mathbb{R}^n$ in theory and applications. For example, the Riemann sphere is utilized in $\mathbb{R}^2$ as a model for the extended complex plane. An induced metric is defined that provides a unified framework for the definition of convergence of bounded as well as unbounded sequences in $\mathbb{R}^2$. The stereographic projection has been applied to the study of systems of differential equations (I. Bendixon [7]). Several authors of ordinary differential equations textbooks, e.g. [4, 28, 33, 36, 40], favor the utilization of the Poincare compactification. Several researchers (C. Chicone and J. Sotomayor [10]; A. Cima and J. Lelibre [11]) utilized Poincaré's compactification to study solutions of systems of differential equations in the vicinity of infinity. The stereographic projection has also found relevance to the theory of quantum computation and quantum information, to conformal mappings, and to string theory in theoretical physics (J-w Lee et al. [32]).

The human visual system is based on the sensation of light in the 3D physical world projecting onto the inner surface of our eyes. 2D artwork depicts 3D shapes via various forms of projection. Artists and computer graphics researchers study the projection methods employed in the past [1, 27]. Artists and researchers employ non-linear projection methods to create 2D images that match an artist's intent, such as to depict a scene as viewed from multiple points-of-view simultaneously or to better match human perceptual expectations [1, 9, 46, 49]. The stereographic projection has been used for artistic purposes to create photographic effects [37]. Optical lenses produce non-linear projections [47]. These projections are investigated for the creation of affordable head-mounted displays for virtual and augmented reality [42, 48]. The behavior of angles under projection (conformality) is of key importance in these applications. It was known to Ptolemy that the stereographic projection is conformal. A proof of the converse theorem, that if a projection of a two dimensional plane on a surface is conformal then the surface must necessarily be a sphere, was undertaken by E. Kasner and J. De Cicco [29].

New nonlinear projections-compactifications and new applications were found as well. See H. Gingold [20], where a parabolic projection-compactification was introduced and applied to approximation theory. See U. Elias and H. Gingold [13] for the application of the parabolic compactification to the "blow up" of solutions of systems of differential equations. Applications of a "parabolic projection" to nonlinear systems of finite differences were taken up by H. Gingold in [15, 16, 17]. The parabolic and other well known projections-compactifications can be shown to be part of a larger framework of "radial compactifications" in $\mathbb{R}^n$ proposed in [13]. The blow up of solutions of systems of differential equations is an imporatnt criteria in Combustion theory. We defer to the references in J. Hell [25] and in A. Takayasu et al. [43]. Numerical validation of blow-up solutions of ordinary differential equations were obtained in A. Takayasu et al. [43]. New insights into the celebrated Lorenz system were found by H. Gingold and D. Solomon [18, 19]. For comparisons among the stereographic projection, the Poincare compactification, and the parabolic compactification, see Hell J. [25]. See also A. Takayasu et al. [43]. The definition of a critical point "infinity" for a nonlinear dynamical system $y' = f(y)$, $y, f(y) \in \mathbb{R}^n$ or the definition of a fixed point "infinity" for a nonlinear discrete system $y_{n+1} = f(y_n)$, $y_n, f(y_n) \in \mathbb{R}^n$ is of paramount importance. To this end, nonlinear projections of $\mathbb{R}^n$ on bounded sets have become an indispensible tool.

It is noteworthy that the new metric $\frac{d(a,b)}{1+d(a,b)}$ derived from the metric $d(a,b)$ of a given metric space is related to the mapping $\frac{x}{1+\|x\|}$ where $\|\|$ is a suitable norm in a Banach space or a Hilbert space. It is a well-known powerful and favored tool in functional analysis. See e.g. E. Kreyszig [30] p.17 and H. L. Royden [39] page p. 186 and p.203. It can be shown that $\frac{x}{1+\|x\|}$ is a projection of a Hilbert space into a bounded subset of a "cone".

A comment on the term "compactification" is necessary. Whilst it is proper to call a mapping of $\mathbb{R}^n$ into a bounded set a "compactification", it could be misleading and confusing to continue to call a mapping of an infinite dimensional space into a bounded set a "compactification". It is well known that the unit ball in a Hilbert space is not a compact set, and the image of the unit ball may not be compact either. This echoes J. Hell [25] who expressed concern about the usage of the word "compactification". Therefore, we add

Definition 1.1. A mapping of an unbounded set into a bounded set is called a boundization.

In 2007, Y. Gingold and H. Gingold [21] proposed and studied a family of nonlinear projections of the points of $\mathbb{R}^2$ onto subsets of a sphere in $\mathbb{R}^3$ that were "bowl" shaped. This family of projections depended on one real valued parameter. It turns out that the Poincare projection and the radically different stereographic projection are obtained from two different particular values of this one parameter family by varying this parameter continuously in an interval. The relevance of these projections to perspective were discussed in [22]. In 2009 J. Hell [25] published her Ph.D. dissertation entitled "Conley Index at Infinity". Her dissertation suggested studying the stereographic projection and the Poincare projection in the setting of a real Hilbert space.

It is also noteworthy that the extension of Euclidean and non-Euclidean geometry to complex Hilbert spaces attracted a substantial amount of attention. See Hahn [23] for Trigonometry on the unit ball of a complex Hilbert Space. This and other works mentioned in here should come as no surprise, given the special role of the sphere in the mathematical sciences.

The importance of angles in Euclidean geometry, in Hilbert spaces and in the mathematical sciences, can hardly be over estimated. While the definition of an angle and its measure in Euclidean geometry and in a real Hilbert space is well established and accepted, the same does not hold true in a real Banach space and in a complex Hilbert space. The latter were subject to numerous investigations. A partial list includes J. E. Valentine and S. G. Wayment, [45], A. Galantai and Cs. J. Hegedus [14], and more recently V. W. Thurey [44], V. Balestro, H. Martini, and R. Teixeira, [5, 6] and their references. Throughout this article we adopt a definition of the measure of an angle $\mathcal{X}$ in a complex Hilbert space as follows.

Definition 1.2. Let $<,>$ be an inner product in a given complex Hilbert space H. Let $\mathcal{V}, \hat{\mathcal{V}} \in H$ be two unit vectors . Then, the measure of the angle $\mathcal{X}$ between any two arrows representing them is given by

$$(1.1) \qquad \mathcal{X} = cos^{-1}[\frac{\langle \mathcal{V}, \hat{\mathcal{V}} \rangle + \langle \hat{\mathcal{V}}, \mathcal{V} \rangle}{2}], \; where \, cos(\mathcal{X}) = \frac{\langle \mathcal{V}, \hat{\mathcal{V}} \rangle + \langle \hat{\mathcal{V}}, \mathcal{V} \rangle}{2}, \, 0 \leq \mathcal{X} \leq \pi.$$

Notice that $cos(\mathcal{X})$ is a monotone decreasing function on the interval $0 \leq \mathcal{X} \leq \pi$ and hence invertible. It is readily observed that $cos(\mathcal{X}) = Real \, \langle \mathcal{V}, \hat{\mathcal{V}} \rangle$ and is real even in a complex Hilbert space. Moreover, $cos(\mathcal{X}) = \langle \mathcal{V}, \hat{\mathcal{V}} \rangle$ is compatible with the definition of an angle in a real Hilbert space. Furthermore, in $\mathbb{R}^3$, (1.1) is consistent with the definition of an angle in Euclidean geometry.

This article is concerned with the construction of a multi parameter family of nonlinear "spherical" projections of a complex Hilbert space H on bounded "bowl" shaped subsets of spheres in an extended complex Hilber space H_N. The article focuses on the properties of the images of angles under these nonlinear projections. We consider in a complex Hilbert space a family of nonlinear projections that depend on two complex valued parameters and on one real parameter. Two complex parameters θ_s, γ_s help define the center of the sphere and the fixed "projection point" from which projection takes place. The real parameter $R \geq 0$ denotes the radius of a sphere. Under the yet-to-be defined "spherical" projections, each point Q in a complex Hilbert space H is matched to a point Z in an "extended" complex Hilbert space H_N on a sphere.

Definition 1.3. Let $\mathcal{X}$ be an arbitrary angle in H and let $\mathcal{Y}$ be its image under the projection on the sphere. We say that the projection mapping is conformal if $\forall \mathcal{X}, \, \mathcal{Y} = \mathcal{X}$.

In order to formulate the main result of this article we first need to recall a result of Y. Gingold and H. Gingold [21] for $\mathbb{R}^2$ that provides the converse to a well known theorem in complex analysis. It states the following. Let Q and $\hat{Q}$ be any two distinct points in $\mathbb{R}^2$and let Z and $\hat{Z}$ be the images of Q and $\hat{Q}$ respectively under a spherical projection with fixed projection point P. Then, the triangles $PQ\hat{Q}$ and $P\hat{Z}Z$ are similar iff the spherical projection is the stereographic projection. In the sequel we allow Q and $\hat{Q}$ to be any two distinct points in a complex Hilbert space and we let Z and $\hat{Z}$ be the images of Q and $\hat{Q}$, respectively, under a "spherical projection" with fixed projection point P. The main result of this article is formulated below.

Theorem 1.1. *The following conditions are equivalent in a complex Hilbert space . i) A spherical projection is the stereographic projection. ii) The spherical projection is conformal. iii) Any two triangles $PQ\hat{Q}$ and $P\hat{Z}Z$ are similar under the spherical projection.*

This work raises additional questions related to Ptolemy's work about conformality under nonlinear projections. Does it hold true that if the measure of all angles under a nonlinear projection of H on a general surface in $\mathbb{R}^n$ is preserved, must then the surface be a sphere? Does it hold true that if $PQ\hat{Q}$ and $P\hat{Z}Z$ generated by a nonlinear projection on a general surface in $\mathbb{R}^n$ preserves their similarity then the general surface must be a sphere? Can the latter two questions be answered in $\mathbb{C}^n$ or in a Complex Hilbert space in the affirmative? Answers to these questions will also constitute a generalization and extension in several new directions to the work of E. Kasner and

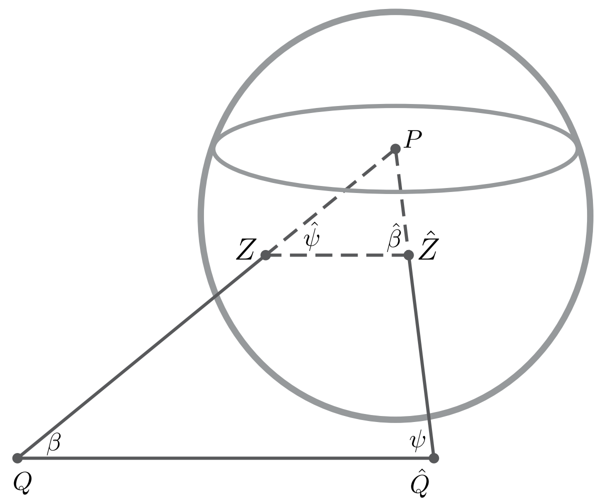

Figure 1: The projection mapping on a sphere

J. De Cicco [29]. This work indicates that the answers to these questions should be yes.

In the sequel we employ three related linear vector spaces H, H_N, S_N. We denote the zero element of H by $\vec{0}$.

We start with a given infinite dimensional Hilbert space H over the complex numbers $\mathbb{C}$. Let $u \neq \vec{0}$ be a unit element of H, namely $< u, u >= 1$. Denote by H_N a new set of elements $Z = (cu, z) \in H_N$ for all $c \in \mathbb{C}$ and $z \in H$. We define a new additional operation $\boxplus$ in H_N as follows

$$Z_1 \boxplus Z_2 = (c_1u + c_2u, z_1 + z_2). \tag{1.2}$$

We also add a new multiplication operation $\odot$ in H_N defined as follows

$$A \odot Z_1 = (Ac_1u, Az_1) \ where \ A \in \mathbb{C}. \tag{1.3}$$

Henceforth we replace the operations $\boxplus, \odot$ with the symbols $+$,
$;cdot$ respectively. We continue to denote by $+$ the addition operation in H and we denote by $\vec{0}$ its identity element in H with respect to the addition operation $+$. This, with the understanding that it is not to be confused with the $+$ the addition operation in $\mathbb{C}$ and with its 0 identity element in $\mathbb{C}$.

Now we define a new inner product in H_N as follows.

$$\langle Z_1, Z_2 \rangle_{H_N} = \langle z_1, z_2 \rangle + \langle c_1u, c_2u \rangle = \langle z_1, z_2 \rangle + c_1\bar{c}_2. \tag{1.4}$$

It can be verified that the new set H_N equipped with the new operations described above is a new and extended Hilbert space over $\mathbb{C}$.

Remark 1.4. Let $Z_1 = (cu, \vec{0})$ and $Z_2 = (\vec{0}, z)$ be two vectors in H_N then $\langle Z_1, Z_2 \rangle_{H_N} = 0$ for all $c \in \mathbb{C}$ and $z \in H$. It is evident that Z_1 and Z_2 are orthogonal. Moreover, It is easily recognized that the subset of elements of the form $(\vec{0}, q) \in H_N$, $q \in H$, to be denoted by S_N, forms a subspace of H_N. Furthermore, S_N is isomorphic to H_N. We may refer to the one dimensional set Axis:= $Z\Big|Z = (cu, \vec{0}), c \in \mathbb{C}$ as the "axis" subspace of H_N, and to S_N as the "plane" S_N in H_N. Evidently $H_N = Axis \oplus S_N$, $Axis \perp S_N$.

The order of presentation in this article is as follows. In sections 2, 3 and 4 we introduce nomenclature and prove certain lemmas that serve the proof of the main theorem in section 5. Specifically, in section 2, we construct a multi parameter family of projections and study its properties. In section 3 we study the projection image of the angle between curves meeting in a complex Hilbert space and derive a formula for the measure of the projected image angle. In section 4 we study the similarity of the triangles created by the family of projections and we derive a formula from which the criterion for the similarity of these triangles can be deduced. In section 5 we reformulate the main theorem that is given in this introduction and we prove it. In section 6 we discuss various particular examples of Hilbert spaces.

2 A family of spherical projections

We consider a sphere with center at a point $O = (\theta_s u, \vec{0})$, $\theta_s \in \mathbb{C}$ and radius $R \geq 0$. Let $Z \in H_N$ be a point on the sphere $SP(\theta_s, R) := \{Z \in H_N : \|Z - O\|_{H_N} = R\}$. Fix a "projection point" $P = (\gamma_s u, \vec{0})$, $0 \neq \gamma_s \in \mathbb{C}$. We make additional notations and assumptions, the motivation and purpose of which will be clarified by subsequent proceedings. We assume

I) $\alpha := \gamma_s\overline{\gamma}_s - \gamma_s\overline{\theta}_s - \theta_s\overline{\gamma}_s + \theta_s\overline{\theta}_s - R^2 = |\gamma_s - \theta_s|^2 - R^2 \leq 0$.

II) $|\theta_s| \leq |\gamma_s|$.

We add the notation and identity

$$\mathcal{A} := -2\gamma_s\overline{\gamma}_s + \gamma_s\overline{\theta}_s + \theta_s\overline{\gamma}_s = -[|\gamma_s|^2 - |\theta_s|^2 + |\gamma_s - \theta_s|^2]. \tag{2.1}$$

The assumptions I) and II) above may be replaced by the assumptions III) and IV) below.

III) $\alpha = \gamma_s\overline{\gamma}_s - \gamma_s\overline{\theta}_s - \theta_s\overline{\gamma}_s + \theta_s\overline{\theta}_s - R^2 = |\gamma_s - \theta_s|^2 - R^2 \leq 0$.

IV) $\mathcal{A} \leq 0$.

Condition IV) is recognized to hold iff the values of γ_s, θ_s satisfy II). II) or IV) hold without loss of generality. If $\mathcal{A} > 0$ the formulas to be derived in the sequel need to change form. I) or III) mean that the distance between the center of the sphere $(\theta_s u, \vec{0})$ and the location of the fixed projection point $(\gamma_s u, \vec{0})$ should not exceed R. Then

$$\alpha = 0 \Longleftrightarrow |\gamma_s - \theta_s|^2 = R^2 \Longleftrightarrow |\gamma_s - \theta_s| = R. \tag{2.2}$$

Namely, the distance between the center of the sphere and the projection point is precisely R. We formally recognize this in the important .

Definition 2.1. We say that our projection is stereographic if $\alpha = 0$. We then call the point $(\gamma_s u, \vec{0})$ the north (or south) pole of the sphere.

Condition I) (or III)) may be interpreted geometrically as follows. Let $H = \mathbb{R}^2$ be defined over the field $\mathbb{R}$. Then, $|\gamma_s - \theta_s| \leq R \Longleftrightarrow -R \leq \gamma_s - \theta_s \leq R$ so if $0 \leq \theta_s$ then $0 \leq \theta_s \leq \gamma_s \leq \theta_s + R$ and if $\theta_s \leq 0$ then $\theta_s - R \leq \gamma_s \leq \theta_s \leq 0$.

Namely, the projection point is situated between the center of the sphere and the north (south) pole. In order to visualize the above geometrical interpretation in a complex Hilbert space we could add the assumption that $\theta_s = m\gamma_s$, $0 \le m \le 1$. It should come without surprise that "collinearity" in a complex vector space does not lend itself to intuition as "collinearity" in a real vector space. In what follows we construct a spherical projection mapping $SPM(\theta_s, R, \gamma_s)$ that takes an arbitrary point $Q = (\vec{0}, q) \in S_N$ into a point $Z = (cu, z) \in H_N$ that lies on the surface of the sphere $SP(\theta_s, R) = \{Z \in H_N : \|Z - O\|_{H_N} = R\}$ and we describe its properties. We require the points P, Z and Q to "lie on the same straight line" and to be such that the "arrows" $\overrightarrow{PZ} := Z - P$ and $\overrightarrow{PQ} := Q - P$ possess the same direction. We have then

Lemma 2.1. *i) There exists a unique non negative scalar $0 \le t_s \in \mathbb{R}$ and a unique point Z on the sphere $SP(\theta_s, R) := \{Z \in H_N : \|Z - O\|_{H_N} = R\}$ such that*

$$Z - P = t_s(Q - P) \text{ where } 0 \le t_s \in \mathbb{R}.. \tag{2.3}$$

The mapping from S_N into the sphere is a bijection.

ii) "Circles" $\left\|(\vec{0}, q)\right\| = \rho$ in S_N map onto circles

$$\|z\| = \hat{\rho} = \frac{-\frac{\mathcal{A}}{2} + \sqrt{\frac{\mathcal{A}^2}{4} - (\rho^2 + \gamma_s\overline{\gamma}_s)\alpha}}{\rho^2 + \gamma_s\overline{\gamma}_s}\rho \quad \text{in } H_N.$$

iii) If $\alpha < 0$ the image of the complex Hilbert space H on the sphere has the shape of an "open bowl". iv) If $\alpha = 0$ the image of the complex Hilbert space H is the entire sphere save one point which is the "north pole".

Proof. The relation (2.3) requires that, $((c - \gamma_s)u, z) = t_s(-\gamma_s u, q)$ so that

$$z = t_s q \text{ and } c = (1 - t_s)\gamma_s. \tag{2.4}$$

The equation of the sphere $\|Z - O\| = R$ in combination with (2.4) yields

$$\langle z, z\rangle + \langle (c - \theta_s)u, (c - \theta_s)u\rangle = R^2, \tag{2.5}$$

that could be cast in the form

$$R^2 = \langle t_s q, t_s q\rangle + \langle ((1 - t_s)\gamma_s - \theta_s)u, ((1 - t_s)\gamma_s - \theta_s)u\rangle. \tag{2.6}$$

Recall that $\langle u, u\rangle = 1$, $t_s = \overline{t}_s$, so we have

$$R^2 = t_s^2\langle q, q\rangle + (1 - t_s)^2\gamma_s\overline{\gamma_s} - (1 - t_s)\gamma_s\overline{\theta}_s - (1 - t_s)\theta_s\overline{\gamma_s} + \theta_s\overline{\theta}_s$$

and then

$$t_s^2\{\langle q, q\rangle + \gamma_s\overline{\gamma}_s\}\} + \{(-2)\gamma_s\overline{\gamma}_s + \gamma_s\overline{\theta}_s + \theta_s\overline{\gamma}_s\}t_s - \theta_s\overline{\gamma}_s - \gamma_s\overline{\theta}_s + \gamma_s\overline{\gamma}_s + \theta_s\overline{\theta}_s - R^2 = 0. \tag{2.7}$$

Collection and rearranging terms lead to

$$t_s^2\{\langle q, q\rangle + \gamma_s\overline{\gamma}_s\}\} + \mathcal{A}t_s + \alpha = 0. \tag{2.8}$$

The quadratic equation (2.8) has two real roots

$$t_s = \frac{-\frac{\mathcal{A}}{2} \pm \sqrt{F}}{\|q\|^2 + \gamma_s \overline{\gamma}_s},\ F := \frac{\mathcal{A}^2}{4} - (\|q\|^2 + \gamma_s\overline{\gamma}_s)\alpha. \tag{2.9}$$

By virtue of the requirement $0 \le t_s$ we must choose the root in (2.9) to be

$$t_s = t_s(\|q\|) = \frac{-\frac{\mathcal{A}}{2} + \sqrt{F}}{\|q\|^2 + \gamma_s \overline{\gamma}_s}.. \tag{2.10}$$

In sum the point $Q = (\vec{0}, q) \in S_N$ is matched to the point $Z = ((1 - t_s)\gamma_s u,\ t_s q)$ by a spherical projection mapping constructed above. Formally we have

$$SPM(\theta_s, R, \gamma_s)[(\vec{0}, q)] = ((1 - t_s)\gamma_s u,\ t_s q). \tag{2.11}$$

$SPM(\theta_s, R, \gamma_s)$ is a bijection from S_N onto a subset of the sphere

$$SP(\theta_s, R) := \{Z \in H_N \Big| \|Z - O\|_{H_N} = R\}.. \tag{2.12}$$

In order to complete the proof of i) we need to prove that the mapping (2.11) is one to one. To that end we notice that

$$q = t_s^{-1} z,\ \|q\| = t_s^{-1}\|z\|,\ and\, t_s \ne 0 \Longleftrightarrow \|q\| \ne \infty$$

and we attempt to express t_s in terms of $\|z\|$. The quadratic equation (2.8) can be modified to become

$$(\|z\|^2 + \alpha)t_s^{-2} + \mathcal{A}t_s^{-1} + \gamma_s\overline{\gamma}_s = 0. \tag{2.13}$$

It is beneficial to discuss separately the stereographic projection case where $\alpha = 0$ as its properties differs radically from the projections when $\alpha < 0$. We notice that with $\alpha < 0$ the product of roots of the quadratic (2.13) must be negative because the product of roots of (2.8) is negative. Hence, for all $\|z\|^2$ in the image $z = t_s q$ we must have $(\|z\|^2 + \alpha) < 0 \Longleftrightarrow \|z\|^2 < -\alpha$ which implies that the positive root of (2.13) is given by

$$t_s^{-1} = \frac{-\mathcal{A} - \sqrt{\mathcal{A}^2 - 4\gamma_s\overline{\gamma}_s(\|z\|^2 + \alpha)}}{2(\|z\|^2 + \alpha)} = \frac{\frac{\mathcal{A}}{2} + \sqrt{\frac{\mathcal{A}^2}{4} - \gamma_s\overline{\gamma}_s(\|z\|^2 + \alpha)}}{-(\|z\|^2 + \alpha)}. \tag{2.14}$$

Thus $q = t_s^{-1} z = \frac{-\frac{\mathcal{A}}{2} - \sqrt{\frac{\mathcal{A}^2}{4} - \gamma_s\overline{\gamma}_s(\|z\|^2+\alpha)}}{(\|z\|^2+\alpha)} z$ and i) is concluded. For $\alpha = 0$ the quadratic equation is

$$t_s^2\{\langle q, q\rangle + \gamma_s\overline{\gamma}_s\}\} + \mathcal{A}t_s \equiv t_s[\{\langle q, q\rangle + \gamma_s\overline{\gamma}_s\}t_s + \mathcal{A}] \Rightarrow t_s = \frac{-\mathcal{A}}{\{\|q\|^2 + \gamma_s\overline{\gamma}_s\}} > 0, \tag{2.15}$$

and it has a unique non zero (positive) desired root. In order to obtain the inverse mapping we put

$$\langle z, z\rangle + t_s^2\{\gamma_s\overline{\gamma}_s\} + \mathcal{A}t_s = 0 \Longleftrightarrow \langle z, z\rangle t_s^{-2} + \mathcal{A}t_s^{-1} + \gamma_s\overline{\gamma}_s = 0. \tag{2.16}$$

(2.17) $$t_s = \frac{-\frac{\mathcal{A}}{2} + \sqrt{F}}{\|q\|^2 + \gamma_s \overline{\gamma}_s} = \frac{-\alpha}{\sqrt{F} + \frac{\mathcal{A}}{2}} \; if \; \alpha < 0.$$

Whilst (2.10) is a valid formula for all $\alpha \leq 0$ the simpler formula (2.17) holds for α strictly negative as for $\alpha = 0$ the expression $t_s = \frac{-\alpha}{\sqrt{F}+\frac{\mathcal{A}}{2}}$ is indeterminate as the denominator is also zero.

ii) A property of the stereographic projection that was known to Ptolemy in $\mathbb{R}^3$ can be shown to hold in complex Hilbert spaces for any projection $SPM(\theta_s, R, \gamma_s)$. It claims that "circles" $\left\|(\vec{0}, q)\right\|_{H_N} = \left\|q\right\|_H = \rho$ in S_N are mapped under $SPM(\theta_s, R, \gamma_s)$ onto "circles" in H_N. Let us verify that this is the case. Indeed, the "plane" $Z = ((1 - t_s(\rho))\gamma_s u, \, v), \, v \in H,$ "is parallel" to S_N. Its intersection with the sphere $\{Z \in H_N \big| \|Z - O\|_{H_N} = R\}$ is a "circle" in H_N with center at $\hat{O} := ((1 - t_s(\rho))\gamma_s u, \, \vec{0})$ and radius $r(\rho)$

$$r^2(\rho) =< Z - \hat{O}, Z - \hat{O} >_{H_N} =< (\vec{0}, t_s(\rho)q), \, (\vec{0}, t_s(\rho)q) >_{H_N} = t_s^2(\rho)\left\|q\right\|^2 = \left\|z\right\|^2.$$

The constant value of $r(\rho)$ is readily obtained from the relation

(2.18) $$z = t_s(\rho)q \Rightarrow r(\rho) = \left\|z\right\| = t_s(\rho)\rho = \frac{-\frac{\mathcal{A}}{2} + \sqrt{\frac{\mathcal{A}^2}{4} - (\rho^2 + \gamma_s\overline{\gamma}_s)\alpha}}{\rho^2 + \gamma_s\overline{\gamma}_s}\rho.$$

This concludes the proof of ii). The "open bowl" shaped image of S_N is obtained by varying ρ, $0 \leq \rho < \infty$. If $\alpha < 0$ we have from the above that $r(0) = 0$ and that $lim_{\rho\to\infty} r(\rho) = \sqrt{-\alpha} > 0$ which proves that the "bowl" shaped set is not the entire sphere. In the case that $\alpha = 0$ it is evident that $r(0) = lim_{\rho\to\infty} r(\rho) = \sqrt{-\alpha} = 0$ and that the entire sphere is the image of S_N save the north pole. ☐

Remark 2.2. We point out that (2.10) and (2.11) for all $\alpha \leq 0$. However, the simpler formula for t_s prescribed by (2.17) is not valid for $\alpha = 0$.

We also note

Remark 2.3. The "collinearity" of three points in a real vector space or the "collinearity" of two vectors in a real vector space is compatible with our geometric interpretation and with our intuition. However, in a vector space over the field of complex numbers $\mathbb{C}$, "collinearity" is not to be interpreted as it is in a real vector space $\mathbb{R}^n$ over the field $\mathbb{R}$. Multiplying a vector by a complex number may result in rotation. In a vector space over the Field $\mathbb{C}$ it would be more appropriate to use the language of linear dependence and linear independence.

3 The angle measure between the images of two curves

Our next task is to determine the measure of the image of an angle $\mathcal{X}$ under a spherical projection $SPM(\theta_s, R, \gamma_s)$. Let $\mathcal{V}$ and $\hat{\mathcal{V}}$ be two unit vectors in H such that $\|\mathcal{V}\| = \|\hat{\mathcal{V}}\| = 1$. Then $(\vec{0}, \mathcal{V})$ and $(\vec{0}, \mathcal{V})$ are unit vectors in S_N according to the definition $\langle Z_1, Z_2\rangle_{H_N} = \langle z_1, z_2\rangle + c_1\bar{c}_2$ in (1.4). Let $\mathcal{X}$ be the angle between $\mathcal{V}$ and $\hat{\mathcal{V}}$ in H as defined in (1.1). Then it is easily verified that

$$cos(\mathcal{X}) = \frac{\langle \mathcal{V}, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, \mathcal{V}\rangle}{2} = \frac{\langle (\vec{0}, \mathcal{V}), (\vec{0}.\hat{\mathcal{V}}\rangle + \langle (\vec{0}, \hat{\mathcal{V}}), (\vec{0}, \mathcal{V})\rangle}{2}. \tag{3.1}$$

Namely, the measure of the angle between $\mathcal{V}$ and $\hat{\mathcal{V}}$ in H is the same as the measure of the angle between $(\vec{0}, \mathcal{V})$ and $(\vec{0}, \mathcal{V})$ in S_N. Let $\mathcal{W}(q)$ and $\hat{\mathcal{W}}(q)$ be two curves in H and let $\mathcal{H}(\delta) = q_0 + \delta\mathcal{V}$ and $\hat{\mathcal{H}}(\delta) = q_0 + \delta\hat{\mathcal{V}}$ be their tangent lines at q_0 respectively, such that $\mathcal{H}(0) = \hat{\mathcal{H}}(0) = q_0$, where $0 \le \delta < \infty$. The three vectors q_0, $\mathcal{V}$ and $\hat{\mathcal{V}}$ create a configuration, where the arrows representing q_0, $\mathcal{V}$ and $\hat{\mathcal{V}}$ respectively emanate from the head of the arrow representing the vector q_0. In this configuration, we have three angles. The angle $\mathcal{X}$ between arrows representing $\mathcal{V}$ and $\hat{\mathcal{V}}$, the angle η between arrows representing q_0 and $\mathcal{V}$ and the angle $\hat{\eta}$ between arrows representing q_0 and $\hat{\mathcal{V}}$. We put for $\|q_0\| \neq 0$

$$\begin{aligned} cos(\eta) &:= \frac{\langle \|q_0\|^{-1} q_0, \mathcal{V}\rangle + \langle \mathcal{V}, \|q_0\|^{-1} q_0\rangle}{2}, \\ cos(\hat{\eta}) &:= \frac{\langle \|q_0\|^{-1} q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, \|q_0\|^{-1} q_0\rangle}{2}, \ \|q_0\| \neq 0. \end{aligned} \tag{3.2}$$

Evidently, the angle between a zero vector q_0 and another vector is not well defined. However, with U a unit vector in H and with $q_0 = \rho U, \rho > 0$ the radial limits

$$lim_{q_0 \to \hat{0}} \|q_0\|^{-1} q_0 = lim_{\rho \to 0^+} U = U$$

do exist.

$\|\mathcal{H}(\delta)\|$ is actually a function of $q_0 \in H$ and δ, $\delta \ge 0$. We need information about the existence and continuity of the derivative $\frac{\partial \|\mathcal{H}(\delta)\|}{\partial\delta}$ as a function of $q_0 \in H$ and δ, $\delta \ge 0$. The derivative $\frac{\partial \|\mathcal{H}(\delta)\|}{\partial\delta}$ evaluated at $\delta = 0$ is denoted by $\left.\frac{\partial \|\mathcal{H}(\delta)\|}{\partial\delta}\right|_{\delta=0}$. We also need information about the limit of the quotient $\frac{\|\mathcal{H}(\delta)\| - \|\mathcal{H}(0)\|}{\delta}$. This is given in the next Lemma.

Lemma 3.1. *i) The derivative* $\frac{\partial \|\mathcal{H}(\delta)\|}{\partial\delta}$ *exists for each fixed* $q_0 \in H$ *and it is a continuous function of* q_0 and δ for every fixed $q_0 \neq \vec{0}$, $\delta \ge 0$ and δ small enough *and*

$$lim_{\delta \to 0^+} \frac{\|\mathcal{H}(\delta)\| - \|\mathcal{H}(0)\|}{\delta} = lim_{\delta \to 0^+} \frac{\partial \|\mathcal{H}(\delta)\|}{\partial\delta} = \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2\sqrt{\langle q_0, q_0\rangle}} = cos(\eta). \tag{3.3}$$

ii) For $q_0 = \vec{0}$ *we have*

$$lim_{\delta \to 0^+} \frac{\|\mathcal{H}(\delta)\| - \|\mathcal{H}(0)\|}{\delta} = 1 = \frac{\partial \|\mathcal{H}(\delta)\|}{\partial\delta}. \tag{3.4}$$

Proof. By definition we have for $q_0 \neq \vec{0}$

$$lim_{\delta \to 0^+} \frac{\|\mathcal{H}(\delta)\| - \|\mathcal{H}(0)\|}{\delta} = lim_{\delta \to 0^+} \frac{\sqrt{\langle q_0 + \delta\mathcal{V}, q_0 + \delta\mathcal{V}\rangle} - \sqrt{\langle q_0, q_0\rangle}}{\delta} =$$

$$lim_{\delta\to 0^+}\frac{\langle q_0+\delta\mathcal{V},q_0+\delta\mathcal{V}\rangle-\langle q_0,q_0\rangle}{\delta[\sqrt{\langle q_0+\delta\mathcal{V},q_0+\delta\mathcal{V}\rangle}+\sqrt{\langle q_0,q_0\rangle}]}$$
$$=lim_{\delta\to 0^+}\frac{\langle q_0+\delta\mathcal{V},q_0+\delta\mathcal{V}\rangle-\langle q_0,q_0\rangle}{\delta[\sqrt{\langle q_0+\delta\mathcal{V},q_0+\delta\mathcal{V}\rangle}+\sqrt{\langle q_0,q_0\rangle}]}$$

$$(3.5)\quad =lim_{\delta\to 0^+}\frac{\langle q_0,\delta\mathcal{V}\rangle+\langle\delta\mathcal{V},q_0\rangle+\langle\delta\mathcal{V},\delta\mathcal{V}\rangle}{\delta[\sqrt{\langle q_0+\delta\mathcal{V},q_0+\delta\mathcal{V}\rangle}+\sqrt{\langle q_0,q_0\rangle}]}=\frac{\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle}{2\sqrt{\langle q_0,q_0\rangle}}=cos(\eta).$$

On the other hand for δ positive and δ small enough we have

$$\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}=\frac{\partial\sqrt{\langle q_0+\delta\mathcal{V},q_0+\delta\mathcal{V}\rangle}}{\partial\delta}=$$
$$=\frac{1}{2\sqrt{\langle q_0+\delta\mathcal{V},q_0+\delta\mathcal{V}\rangle}}\frac{\partial\langle q_0+\delta\mathcal{V},q_0+\delta\mathcal{V}\rangle}{\partial\delta}=$$
$$\frac{\langle q_0+\delta\mathcal{V},\mathcal{V}\rangle+\langle\mathcal{V},q_0+\delta\mathcal{V}\rangle}{2\|q_0+\delta\mathcal{V}\|}=\frac{\langle q_0,\mathcal{V}\rangle+\langle\delta\mathcal{V},\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle+\langle\mathcal{V},\delta\mathcal{V}\rangle}{2\|q_0+\delta\mathcal{V}\|}=$$

$$(3.6)\quad =\frac{2\delta+\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle}{2\|q_0+\delta\mathcal{V}\|}.$$

Notice that the denominator $\langle q_0+\delta\mathcal{V},q_0+\delta\mathcal{V}\rangle^{\frac{1}{2}}\neq 0$ for $0<\|q_0\|<\infty$ and q_0 fixed and for δ positive and δ small enough. This is so because then

$$(3.7)\quad \begin{aligned}\langle q_0+\delta\mathcal{V},q_0+\delta\mathcal{V}\rangle&=\left|\|q_0\|^2+\langle q_0,\delta\mathcal{V}\rangle+\langle\delta\mathcal{V},q_0\rangle+\langle\delta\mathcal{V},\delta\mathcal{V}\rangle\right|\\&\geq\|q_0\|^2-\delta[2\left|\langle q_0,\delta\mathcal{V}\rangle\right|+\delta]>0.\end{aligned}$$

Thus, Evidently, $\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}$ is a continuous function of $q_0\neq\vec{0}$ and the conclusion for $q_0\neq\vec{0}$ follows from the comparison of (3.6) with (3.5).

In case that $q_0=\hat{0}$ the derivative $\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}=\frac{\partial\sqrt{\langle\delta\mathcal{V},\delta\mathcal{V}\rangle}}{\partial\delta}=\frac{\partial\delta}{\partial\delta}\equiv 1$ exists. It is also consistent with the definition of the quotient $lim_{\delta\to 0^+}\frac{\sqrt{\langle\delta\mathcal{V},\delta\mathcal{V}\rangle-0}}{\delta}=1$ and

(3.4) follows. However, it is easily seen that the value 1 is not a continuous limit of $\frac{\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle}{2\sqrt{\langle q_0,q_0\rangle}}$ as $q_0\to\vec{0}$. □

We proceed to determine $\mathcal{Y}$, the image of the angle $\mathcal{X}$ on the sphere under the spherical projection $SPM(\theta_s,R,\gamma_s)$. It is given in the next Theorem.

Theorem 3.2. *Let*

$$(3.8)\quad Z(\delta)=((1-t_s(\delta))\gamma_s u,\, t_s(\delta)\mathcal{H}(\delta)),\ \hat{Z}(\delta)=\left(\left(1-\hat{t}_s(\delta)\right)\gamma_s u,\, \hat{t}_s(\delta)\hat{\mathcal{H}}(\delta)\right)$$

be the images of $(0,\mathcal{H}(\delta))$ *and* $(0,\hat{\mathcal{H}}(\delta))$ *under* $SPM(\theta_s,R,\gamma_s)$ *respectively.*

Then, the angle $\mathcal{Y}$ *between the curves* $Z(\delta)$ *and* $\hat{Z}(\delta)$ *that meet at* $Z(0)=\hat{Z}(0)$ *on the sphere* $SP(\theta_s,R)=\{Z\in H_N\Big|\|Z-O\|_{H_N}=R\}$ *is given by*

$$(3.9)\quad cos(\mathcal{Y})=\frac{cos(\mathcal{X})+\mathcal{J}}{\sqrt{1+\mathcal{J}^2+\Lambda}}.$$

$Cos(\mathcal{X})$ is the cosine of the angle between $\mathcal{H}(0)$ and $\hat{\mathcal{H}}(0)$, $\mathcal{J}$ is given by

$$\mathcal{J} := \frac{\alpha}{4\mathcal{F}(0)}\left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right]\left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right] = \frac{\alpha\left\|q_0\right\|^2}{\mathcal{F}(0)}cos(\eta)cos(\hat{\eta}) \tag{3.10}$$

and Λ is given by

$$\begin{aligned}\Lambda &:= \frac{\alpha}{4\mathcal{F}(0)}\left(\left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right]^2 + \left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right]^2\right)\\ &= \frac{\alpha\left\|q_0\right\|^2}{\mathcal{F}(0)}[cos^2(\eta) + cos^2(\hat{\eta})].\end{aligned} \tag{3.11}$$

Proof. We first elaborate on the notation needed for the images (3.8) of the rays $(0, q_0{+}\delta\mathcal{V})$ and $(0, q_0{+}\delta\hat{\mathcal{V}})$ in S_N on the sphere $SP(\theta_s, R) := \{Z \in H_N : \|Z{-}O\|_{H_N} = R\}$ according to section 2. Analogous to (2.17) we have

$$t_s(\delta) = \frac{\frac{-\mathcal{A}}{2} + \sqrt{\mathcal{F}(\delta)}}{\|\mathcal{H}(\delta)\|^2 + \gamma_s\overline{\gamma}_s}, \quad \hat{t}_s(\delta) = \frac{\frac{-\mathcal{A}}{2} + \sqrt{\hat{\mathcal{F}}(\delta)}}{\|\hat{\mathcal{H}}(\delta)\|^2 + \gamma_s\overline{\gamma}_s}, \tag{3.12}$$

where

$$\mathcal{F}(\delta) = \frac{\mathcal{A}^2}{4} - \alpha\left(\|\mathcal{H}(\delta)\|^2 + \gamma_s\overline{\gamma}_s\right), \hat{\mathcal{F}}(\delta) = \frac{\mathcal{A}^2}{4} - \alpha\left(\|\hat{\mathcal{H}}(\delta)\|^2 + \gamma_s\overline{\gamma}_s\right). \tag{3.13}$$

For $\delta = 0$ we have

$$\mathcal{F}(0) = \frac{\mathcal{A}^2}{4} - \alpha\left(\|q_0\|^2 + \gamma_s\overline{\gamma}_s\right) = \hat{\mathcal{F}}(0) \Rightarrow t_s(0) = \frac{\frac{-\mathcal{A}}{2} + \sqrt{\mathcal{F}(0)}}{\|q_0\|^2 + \gamma_s\overline{\gamma}_s} = \hat{t}_s(0). \tag{3.14}$$

Next we evaluate the inner products of the tangent vectors to $Z(\delta)$ and $\hat{Z}(\delta)$ expressed by

$$\langle\frac{\partial Z(\delta)}{\partial\delta}, \frac{\partial\hat{Z}(\delta)}{\partial\delta}\rangle_{H_N} \quad \textit{and} \ \langle\frac{\partial\hat{Z}(\delta)}{\partial\delta}, \frac{\partial Z(\delta)}{\partial\delta}\rangle_{H_N}. \tag{3.15}$$

To find the derivative of $Z(\delta)$ and $\hat{Z}(\delta)$, we must first find the derivatives of $t_s(\delta)$ and $\hat{t}_s(\delta)$

$$\begin{aligned}\frac{\partial t_s(\delta)}{\partial\delta} &= \frac{\left(\|\mathcal{H}(\delta)\|^2 + \gamma_s\overline{\gamma}_s\right)\frac{-\alpha\|\mathcal{H}(\delta)\|\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}}{\sqrt{\mathcal{F}(\delta)}}}{\left(\|\mathcal{H}(\delta)\|^2 + \gamma_s\overline{\gamma}_s\right)^2}\\ &- \frac{\left(\frac{-\mathcal{A}}{2} + \sqrt{\mathcal{F}(\delta)}\right)\left(2\|\mathcal{H}(\delta)\|\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}\right)}{\left(\|\mathcal{H}(\delta)\|^2 + \gamma_s\overline{\gamma}_s\right)^2}\end{aligned} \tag{3.16}$$

$$\begin{aligned}\frac{\partial\hat{t}_s(\delta)}{\partial\delta} &= \frac{\left(\|\hat{\mathcal{H}}(\delta)\|^2 + \gamma_s\overline{\gamma}_s\right)\frac{-\alpha\|\hat{\mathcal{H}}(\delta)\|\frac{\partial\|\hat{\mathcal{H}}(\delta)\|}{\partial\delta}}{\sqrt{\hat{\mathcal{F}}(\delta)}}}{\left(\|\hat{\mathcal{H}}(\delta)\|^2 + \gamma_s\overline{\gamma}_s\right)^2}\\ &- \frac{\left(\frac{-\mathcal{A}}{2} + \sqrt{\hat{\mathcal{F}}(\delta)}\right)\left(2\|\hat{\mathcal{H}}(\delta)\|\frac{\partial\|\hat{\mathcal{H}}(\delta)\|}{\partial\delta}\right)}{\left(\|\hat{\mathcal{H}}(\delta)\|^2 + \gamma_s\overline{\gamma}_s\right)^2},\end{aligned} \tag{3.17}$$

where $\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}$ and $\frac{\partial\|\hat{\mathcal{H}}(\delta)\|}{\partial\delta}$ are given by (3.3).

Factoring $\|\mathcal{H}(\delta)\|\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}$ from equation (3.16) and $\|\hat{\mathcal{H}}(\delta)\|\frac{\partial\|\hat{\mathcal{H}}(\delta)\|}{\partial\delta}$ from equation (3.17) gives us

$$\begin{aligned}&\frac{\partial t_s(\delta)}{\partial\delta}=\\ &-\|\mathcal{H}(\delta)\|\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}\left[\frac{2\left(\frac{-\mathcal{A}}{2}+\sqrt{\mathcal{F}(\delta)}\right)}{\left(\|\mathcal{H}(\delta)\|^2+\gamma_s\overline{\gamma}_s\right)^2}+\frac{\alpha}{\sqrt{\mathcal{F}}(\delta)\left(\|\mathcal{H}(\delta)\|^2+\gamma_s\overline{\gamma}_s\right)}\right].\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\frac{\partial\hat{t}_s(\delta)}{\partial\delta}=\\ &-\|\hat{\mathcal{H}}(\delta)\|\frac{\partial\|\hat{\mathcal{H}}(\delta)\|}{\partial\delta}\left[\frac{2\left(\frac{-\mathcal{A}}{2}+\sqrt{\hat{\mathcal{F}}(\delta)}\right)}{\left(\|\hat{\mathcal{H}}(\delta)\|^2+\gamma_s\overline{\gamma}_s\right)^2}+\frac{\alpha}{\sqrt{\hat{\mathcal{F}}(\delta)}\left(\|\hat{\mathcal{H}}(\delta)\|^2+\gamma_s\overline{\gamma}_s\right)}\right].\end{aligned} \tag{3.19}$$

We create the common denominator $\sqrt{\mathcal{F}}(\delta)\left(\|\mathcal{H}(\delta)\|^2+\gamma_s\overline{\gamma}_s\right)^2$ in (3.18) to yield

$$\frac{\partial t_s(\delta)}{\partial\delta}=-\|\mathcal{H}(\delta)\|\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}\left[\frac{-\mathcal{A}\sqrt{\mathcal{F}(\delta)}+2\mathcal{F}(\delta)+\alpha\left(\|\mathcal{H}(\delta)\|^2+\gamma_s\overline{\gamma}_s\right)}{\sqrt{\mathcal{F}(\delta)}\left(\|\mathcal{H}(\delta)\|^2+\gamma_s\overline{\gamma}_s\right)^2}\right]. \tag{3.20}$$

Note that

$$\alpha\left(\|\mathcal{H}(\delta)\|^2+\gamma_s\overline{\gamma}_s\right)=\frac{\mathcal{A}^2}{4}-\mathcal{F}(\delta). \tag{3.21}$$

Substituting the expression in equation (3.21) into equation (3.20) leaves us with

$$\begin{aligned}\frac{\partial t_s(\delta)}{\partial\delta}&=-\|\mathcal{H}(\delta)\|\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}\left[\frac{-\mathcal{A}\sqrt{\mathcal{F}(\delta)}+2\mathcal{F}(\delta)+\frac{\mathcal{A}^2}{4}-\mathcal{F}(\delta)}{\sqrt{\mathcal{F}(\delta)}\left(\|\mathcal{H}(\delta)\|^2+\gamma_s\overline{\gamma}_s\right)^2}\right]\\ &=-\|\mathcal{H}(\delta)\|\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}\left[\frac{\frac{\mathcal{A}^2}{4}-\mathcal{A}\sqrt{\mathcal{F}(\delta)}+\mathcal{F}(\delta)}{\sqrt{\mathcal{F}(\delta)}\left(\|\mathcal{H}(\delta)\|^2+\gamma_s\overline{\gamma}_s\right)^2}\right].\end{aligned} \tag{3.22}$$

Noticing the perfect square in the numerator of equation (3.22), we obtain

$$\begin{aligned}\frac{\partial t_s(\delta)}{\partial\delta}&=-\|\mathcal{H}(\delta)\|\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}\left[\frac{\left(\frac{-\mathcal{A}}{2}+\sqrt{\mathcal{F}(\delta)}\right)^2}{\sqrt{\mathcal{F}(\delta)}\left(\|\mathcal{H}(\delta)\|^2+\gamma_s\overline{\gamma}_s\right)^2}\right]\\ &=-\|\mathcal{H}(\delta)\|\frac{\partial\|\mathcal{H}(\delta)\|}{\partial\delta}\frac{\left(t_s(\delta)\right)^2}{\sqrt{\mathcal{F}(\delta)}}.\end{aligned}$$

Returning to the derivative of $Z(\delta)$, we see that

$$\begin{aligned}\frac{\partial Z(\delta)}{\partial\delta}&=\left(-\gamma_s\frac{\partial t_s(\delta)}{\partial\delta}u,\frac{\partial t_s(\delta)}{\partial\delta}\mathcal{H}(\delta)+\frac{\partial\mathcal{H}(\delta)}{\partial\delta}t_s(\delta)\right)\\ &=\left(-\gamma_s\frac{\partial t_s(\delta)}{\partial\delta}u,\frac{\partial t_s(\delta)}{\partial\delta}(q_0+\delta\mathcal{V})+\mathcal{V}t_s(\delta)\right).\end{aligned}$$

Evaluate $\frac{\partial Z(\delta)}{\partial \delta}$ at $\delta = 0$ and get

$$\left.\frac{\partial Z(\delta)}{\partial \delta}\right|_{\delta=0} = \left(-\gamma_s \left.\frac{\partial t_s(\delta)}{\partial \delta}\right|_{\delta=0} u, \left.\frac{\partial t_s(\delta)}{\partial \delta}\right|_{\delta=0} q_0 + \mathcal{V} t_s(0)\right), \tag{3.23}$$

where

$$\left.\frac{\partial t_s(\delta)}{\partial \delta}\right|_{\delta=0} = -\|\mathcal{H}(0)\| \left.\frac{\partial \|\mathcal{H}(\delta)\|}{\partial \delta}\right|_{\delta=0} \frac{\left(t_s(0)\right)^2}{\sqrt{\mathcal{F}(0)}},$$

with

$$\left.\frac{\partial \|\mathcal{H}(\delta)\|}{\partial \delta}\right|_{\delta=0} = \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2\|q_0\|} \quad \textit{and } \|\mathcal{H}(0)\| = \|q_0\|.$$

Therefore

$$\left.\frac{\partial Z(\delta)}{\partial \delta}\right|_{\delta=0} = \left(\gamma_s \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \frac{\left(t_s(0)\right)^2}{\sqrt{\mathcal{F}(0)}} u, -\frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \frac{\left(t_s(0)\right)^2}{\sqrt{\mathcal{F}(0)}} q_0 + \mathcal{V} t_s(0)\right). \tag{3.24}$$

Similarly,

$$\left.\frac{\partial \hat{Z}(\delta)}{\partial \delta}\right|_{\delta=0} = \left(\gamma_s \frac{\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle}{2} \frac{\left(\hat{t}_s(0)\right)^2}{\sqrt{\mathcal{F}(0)}} u, -\frac{\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle}{2} \frac{\left(\hat{t}_s(0)\right)^2}{\sqrt{\mathcal{F}(0)}} q_0 + \hat{\mathcal{V}} \hat{t}_s(0)\right). \tag{3.25}$$

Note that

$$\left.\frac{\partial \hat{t}_s(\delta)}{\partial \delta}\right|_{\delta=0} = -\|\hat{\mathcal{H}}(\delta)\| \left.\frac{\partial \|\hat{\mathcal{H}}(\delta)\|}{\partial \delta}\right|_{\delta=0} \left[\frac{\left(\hat{t}_s(0)\right)^2}{\sqrt{\mathcal{F}(0)}}\right],$$

with

$$\left.\frac{\partial \hat{\mathcal{H}}(\delta)}{\partial \delta}\right|_{\delta=0} = \frac{\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle}{2\|q_0\|} \quad \textit{and } \|\hat{\mathcal{H}}(0)\| = \|q_0\|.$$

In order to determine the image of the angle $\mathcal{X}$ we calculate

$$cos(\mathcal{Y}) = \left.\frac{\langle \frac{\partial Z(\delta)}{\partial \delta}, \frac{\partial \hat{Z}(\delta)}{\partial \delta}\rangle_{H_N} + \langle \frac{\partial \hat{Z}(\delta)}{\partial \delta}, \frac{\partial Z(\delta)}{\partial \delta}\rangle_{H_N}}{2\|\frac{\partial Z(\delta)}{\partial \delta}\|_{H_N}\|\frac{\partial \hat{Z}(\delta)}{\partial \delta}\|_{H_N}}\right|_{\delta=0}. \tag{3.26}$$

To this end we first calculate the numerator of (3.26) that requires the evaluation of $\left.\frac{\partial Z(\delta)}{\partial \delta}\right|_{\delta=0}$ and $\left.\frac{\partial \hat{Z}(\delta)}{\partial \delta}\right|_{\delta=0}$ given by equations (3.24) and (3.25). Therefore we have

$$
\begin{aligned}
&\langle \frac{\partial Z(\delta)}{\partial \delta}, \frac{\partial \hat{Z}(\delta)}{\partial \delta} \rangle_{H_N} \Big|_{\delta=0} \\
&= \langle \gamma_s \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \ \frac{(t_s(0))^2}{\sqrt{\mathcal{F}(0)}} u, \gamma_s \frac{\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle}{2} \ \frac{\left(\hat{t}_s(0)\right)^2}{\sqrt{\mathcal{F}(0)}} u\rangle + \\
&\langle \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \frac{(t_s(0))^2}{\sqrt{\mathcal{F}(0)}} q_0 + \mathcal{V} t_s(0), \frac{\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle}{2} \frac{\left(\hat{t}_s(0)\right)^2}{\sqrt{\mathcal{F}(0)}} q_0 + \hat{\mathcal{V}} \hat{t}_s(0)\rangle \\
&= \gamma_s \overline{\gamma}_s \left(t_s(0)\right)^2 \left(\hat{t}_s(0)\right)^2 \frac{\left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right] \left[\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}\right]}{4\mathcal{F}(0)} \langle u, u\rangle \\
&+ \langle - \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \ \frac{(t_s(0))^2}{\sqrt{\mathcal{F}(0)}} q_0, - \frac{\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle}{2} \ \frac{\left(\hat{t}_s(0)\right)^2}{\sqrt{\mathcal{F}(0)}} q_0\rangle \\
&+ \langle - \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \ \frac{(t_s(0))^2}{\sqrt{\mathcal{F}(0)}} q_0, \hat{\mathcal{V}} \hat{t}_s(0)\rangle \\
&+ \langle \mathcal{V} t_s(0), - \frac{\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle}{2} \ \frac{\left(\hat{t}_s(0)\right)^2}{\sqrt{\mathcal{F}(0)}} q_0\rangle + \langle \mathcal{V} t_s(0), \hat{\mathcal{V}} \hat{t}_s(0)\rangle \\
&= \gamma_s \overline{\gamma}_s \left(t_s(0)\right)^2 \left(\hat{t}_s(0)\right)^2 \frac{\left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right] \left[\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}\right]}{4\mathcal{F}(0)} \\
&+ \left(t_s(0)\right)^2 \left(\hat{t}_s(0)\right)^2 \frac{\left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right] \left[\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}\right]}{4\mathcal{F}(0)} \langle q_0, q_0\rangle \\
&- \left(t_s(0)\right)^2 \hat{t}_s(0) \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2\sqrt{\mathcal{F}(0)}} \langle q_0, \hat{\mathcal{V}}\rangle \\
&- t_s(0) \left(\hat{t}_s(0)\right)^2 \frac{\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}}{2\sqrt{\mathcal{F}(0)}} \langle \mathcal{V}, q_0\rangle + t_s(0) \hat{t}_s(0) \langle \mathcal{V}, \hat{\mathcal{V}}\rangle \\
&= \gamma_s \overline{\gamma}_s \left(t_s(0)\right)^4 \frac{\left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right] \left[\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}\right]}{4\mathcal{F}(0)} \\
&+ \left(t_s(0)\right)^4 \frac{\left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right] \left[\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}\right]}{4\mathcal{F}(0)} \langle q_0, q_0\rangle \\
&- \left(t_s(0)\right)^3 \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2\sqrt{\mathcal{F}(0)}} \langle q_0, \hat{\mathcal{V}}\rangle \\
&- \left(t_s(0)\right)^3 \frac{\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}}{2\sqrt{\mathcal{F}(0)}} \langle \mathcal{V}, q_0\rangle + \left(t_s(0)\right)^2 \langle \mathcal{V}, \hat{\mathcal{V}}\rangle.
\end{aligned}
$$

$$
\begin{aligned}
&\langle \frac{\partial Z(\delta)}{\partial \delta}, \frac{\partial \hat{Z}(\delta)}{\partial \delta} \rangle_{H_N} \Big|_{\delta=0} \\
&= (t_s(0))^4 \, \frac{[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle] \left[\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}\right]}{4\mathcal{F}(0)} \left(\gamma_s \overline{\gamma}_s + \|q_0\|\right) \\
&- (t_s(0))^3 \, \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2\sqrt{\mathcal{F}(0)}} \langle q_0, \hat{\mathcal{V}}\rangle \\
&- (t_s(0))^3 \, \frac{\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}}{2\sqrt{\mathcal{F}(0)}} \langle \mathcal{V}, q_0\rangle + (t_s(0))^2 \, \langle \mathcal{V}, \hat{\mathcal{V}}\rangle. \qquad (3.27)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\langle \frac{\partial \hat{Z}(\delta)}{\partial \delta}, \frac{\partial Z(\delta)}{\partial \delta} \rangle_{H_N} \Big|_{\delta=0} \\
&= (t_s(0))^4 \, \frac{\left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right] \left[\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}\right]}{4\mathcal{F}(0)} \left(\gamma_s \overline{\gamma}_s + \|q_0\|\right) \\
&- (t_s(0))^3 \, \frac{\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle}{2\sqrt{\mathcal{F}(0)}} \langle q_0, \mathcal{V}\rangle \\
&- (t_s(0))^3 \, \frac{\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}}{2\sqrt{\mathcal{F}(0)}} \langle \hat{\mathcal{V}}, q_0\rangle + (t_s(0))^2 \, \langle \hat{\mathcal{V}}, \mathcal{V}\rangle. \qquad (3.28)
\end{aligned}
$$

Add equations (3.27) and (3.28) to get

$$\begin{aligned}
&\langle \frac{\partial Z(\delta)}{\partial \delta}, \frac{\partial \hat{Z}(\delta)}{\partial \delta} \rangle_{H_N}\Big|_{\delta=0} + \langle \frac{\partial \hat{Z}(\delta)}{\partial \delta}, \frac{\partial Z(\delta)}{\partial \delta} \rangle_{H_N}\Big|_{\delta=0} \\
&= (t_s(0))^4 \, \frac{[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle]\left[\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}\right]}{4\mathcal{F}(0)} \, (\gamma_s \overline{\gamma}_s + \|q_0\|) \\
&+ (t_s(0))^4 \, \frac{\left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right]\left[\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}\right]}{4\mathcal{F}(0)} \, (\gamma_s \overline{\gamma}_s + \|q_0\|) \\
&- (t_s(0))^3 \, \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2\sqrt{\mathcal{F}(0)}} \langle q_0, \hat{\mathcal{V}}\rangle - (t_s(0))^3 \, \frac{\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}}{2\sqrt{\mathcal{F}(0)}} \langle \hat{\mathcal{V}}, q_0\rangle \\
&- (t_s(0))^3 \, \frac{\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}}{2\sqrt{\mathcal{F}(0)}} \langle \mathcal{V}, q_0\rangle - (t_s(0))^3 \, \frac{\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle}{2\sqrt{\mathcal{F}(0)}} \langle q_0, \mathcal{V}\rangle \\
&+ (t_s(0))^2 \, \langle \mathcal{V}, \hat{\mathcal{V}}\rangle + (t_s(0))^2 \, \langle \hat{\mathcal{V}}, \mathcal{V}\rangle \\
&= (t_s(0))^4 \, \frac{[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle]\left[\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}\right]}{4\mathcal{F}(0)} \, (\gamma_s \overline{\gamma}_s + \|q_0\|) \\
&+ (t_s(0))^4 \, \frac{\left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right]\left[\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}\right]}{4\mathcal{F}(0)} \, (\gamma_s \overline{\gamma}_s + \|q_0\|) \\
&- (t_s(0))^3 \, \frac{\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}}{2\sqrt{\mathcal{F}(0)}} \langle q_0, \hat{\mathcal{V}}\rangle - (t_s(0))^3 \, \frac{\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}}{2\sqrt{\mathcal{F}(0)}} \langle \hat{\mathcal{V}}, q_0\rangle \\
&- (t_s(0))^3 \, \frac{\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}}{2\sqrt{\mathcal{F}(0)}} \langle \mathcal{V}, q_0\rangle - (t_s(0))^3 \, \frac{\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}}{2\sqrt{\mathcal{F}(0)}} \langle q_0, \mathcal{V}\rangle \\
&+ (t_s(0))^2 \, \langle \mathcal{V}, \hat{\mathcal{V}}\rangle + (t_s(0))^2 \, \langle \hat{\mathcal{V}}, \mathcal{V}\rangle \\
&= (t_s(0))^4 \, \frac{[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle]\left[\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}\right]}{4\mathcal{F}(0)} \, (\gamma_s \overline{\gamma}_s + \|q_0\|) \\
&+ (t_s(0))^4 \, \frac{\left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right]\left[\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}\right]}{4\mathcal{F}(0)} \, (\gamma_s \overline{\gamma}_s + \|q_0\|) \\
&- \frac{(t_s(0))^3}{2\sqrt{\mathcal{F}(0)}} \left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right]\left[\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}\right] \\
&- \frac{(t_s(0))^3}{2\sqrt{\mathcal{F}(0)}} [\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle]\left[\overline{\langle q_0, \hat{\mathcal{V}}\rangle} + \overline{\langle \hat{\mathcal{V}}, q_0\rangle}\right] \\
&+ (t_s(0))^2 \, \langle \mathcal{V}, \hat{\mathcal{V}}\rangle + (t_s(0))^2 \, \langle \hat{\mathcal{V}}, \mathcal{V}\rangle.
\end{aligned}$$

Note that

$$
\begin{aligned}
&[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle]\left[\overline{\langle q_0,\hat{\mathcal{V}}\rangle}+\overline{\langle\hat{\mathcal{V}},q_0\rangle}\right]+\left[\langle q_0,\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},q_0\rangle\right]\left[\overline{\langle q_0,\mathcal{V}\rangle}+\overline{\langle\mathcal{V},q_0\rangle}\right]\\
&=\\
&[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle]\left[\langle q_0,\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},q_0\rangle\right]+\left[\langle q_0,\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},q_0\rangle\right][\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle]\\
&=2\,[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle]\left[\langle q_0,\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},q_0\rangle\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\langle\frac{\partial Z(\delta)}{\partial\delta},\frac{\partial\hat{Z}(\delta)}{\partial\delta}\rangle_{H_N}\Big|_{\delta=0}+\langle\frac{\partial\hat{Z}(\delta)}{\partial\delta},\frac{\partial Z(\delta)}{\partial\delta}\rangle_{H_N}\Big|_{\delta=0}\\
&=\frac{(t_s(0))^4}{2\mathcal{F}(0)}\left(\|q_0\|+\gamma_s\overline{\gamma}_s\right)[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle]\left[\langle q_0,\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},q_0\rangle\right]\\
&-\frac{(t_s(0))^3}{\sqrt{\mathcal{F}(0)}}[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle]\left[\langle q_0,\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},q_0\rangle\right]+(t_s(0))^2\left[\langle\mathcal{V},\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},\mathcal{V}\rangle\right].
\end{aligned}
\tag{3.29}
$$

Factoring $(t_s(0))^2\,[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle]\left[\langle q_0,\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},q_0\rangle\right]$ from first and second terms in the equation (3.29) leaves us with

$$
\begin{aligned}
&\langle\frac{\partial Z(\delta)}{\partial\delta},\frac{\partial\hat{Z}(\delta)}{\partial\delta}\rangle_{H_N}\Big|_{\delta=0}+\langle\frac{\partial\hat{Z}(\delta)}{\partial\delta},\frac{\partial Z(\delta)}{\partial\delta}\rangle_{H_N}\Big|_{\delta=0}=(t_s(0))^2\,\{\\
&\frac{[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle]\left[\langle q_0,\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},q_0\rangle\right]}{\sqrt{\mathcal{F}(0)}}\left[\frac{(t_s(0))^2}{2\sqrt{\mathcal{F}(0)}}\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)-t_s(0)\right]\}\\
&+(t_s(0))^2\left[\langle\mathcal{V},\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},\mathcal{V}\rangle\right].
\end{aligned}
\tag{3.30}
$$

We examine $\frac{(t_s(0))^2}{2\sqrt{\mathcal{F}(0)}}\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)-t_s(0)$. Expanding $(t_s(0))^2$ we have

$$
\frac{(t_s(0))^2}{2\sqrt{\mathcal{F}(0)}}\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)-t_s(0)=\frac{t_s(0)\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)}{2\sqrt{\mathcal{F}(0)}}t_s(0)-t_s(0)
$$

Recalling the definition of $t_s(0)$ from equation (4.6), we are left with

$$
\begin{aligned}
\frac{t_s(0)\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)}{\sqrt{\mathcal{F}(0)}}t_s(0)-t_s(0)&=\frac{\frac{-\mathcal{A}}{2}+\sqrt{\mathcal{F}(0)}}{2\sqrt{\mathcal{F}(0)}}t_s(0)-t_s(0)\\
&=t_s(0)\left(\frac{\frac{-\mathcal{A}}{2}+\sqrt{\mathcal{F}(0)}}{2\sqrt{\mathcal{F}(0)}}-1\right)
\end{aligned}
\tag{3.31}
$$

Setting a common denominator in equation (3.31) gives us

$$
\begin{aligned}
t_s(0)\left(\frac{\frac{-\mathcal{A}}{2}+\sqrt{\mathcal{F}(0)}}{2\sqrt{\mathcal{F}(0)}}-1\right)&=t_s(0)\left(\frac{\frac{-\mathcal{A}}{2}+\sqrt{\mathcal{F}(0)}-2\sqrt{\mathcal{F}(0)}}{2\sqrt{\mathcal{F}(0)}}\right)\\
&=t_s(0)\left(\frac{\frac{-\mathcal{A}}{2}-\sqrt{\mathcal{F}(0)}}{2\sqrt{\mathcal{F}(0)}}\right)
\end{aligned}
\tag{3.32}
$$

Substituting $t_s(0)$ as expressed in equation (3.14) into equation (3.32), we have

$$
\begin{aligned}
t_s(0)\left(\frac{\frac{-\mathcal{A}}{2}-\sqrt{\mathcal{F}(0)}}{2\sqrt{\mathcal{F}(0)}}\right) &= \frac{\frac{-\mathcal{A}}{2}+\sqrt{\mathcal{F}(0)}}{\|q_0\|^2+\gamma_s\overline{\gamma}_s}\left(\frac{\frac{-\mathcal{A}}{2}-\sqrt{\mathcal{F}(0)}}{2\sqrt{\mathcal{F}(0)}}\right)\\
&= \frac{\frac{\mathcal{A}^2}{4}-\mathcal{F}(0)}{2\sqrt{\mathcal{F}(0)}\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)}.
\end{aligned}
\tag{3.33}
$$

We substitute the expression given in equation (3.14) into equation (3.33) to obtain

$$
\frac{\frac{\mathcal{A}^2}{4}-\mathcal{F}(0)}{2\sqrt{\mathcal{F}(0)}\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)} = \frac{\frac{\mathcal{A}^2}{4}-\left(\frac{\mathcal{A}^2}{4}-\alpha\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)\right)}{2\sqrt{\mathcal{F}(0)}\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)} = \frac{\alpha}{2\sqrt{\mathcal{F}(0)}}.
$$

Therefore

$$
\frac{(t_s(0))^2}{2\sqrt{\mathcal{F}(0)}}\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)-t_s(0)=\frac{\alpha}{2\sqrt{\mathcal{F}(0)}}.
\tag{3.34}
$$

Substituting the expression given in equation (3.34) into (3.30) we get

$$
\begin{aligned}
&\langle\frac{\partial Z(\delta)}{\partial\delta},\frac{\partial\hat{Z}(\delta)}{\partial\delta}\rangle_{H_N}\Big|_{\delta=0}+\langle\frac{\partial\hat{Z}(\delta)}{\partial\delta},\frac{\partial Z(\delta)}{\partial\delta}\rangle_{H_N}\Big|_{\delta=0}\\
&=(t_s(0))^2\,\frac{\left[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle\right]\left[\langle q_0,\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},q_0\rangle\right]}{\sqrt{\mathcal{F}(0)}}\left[\frac{\alpha}{2\sqrt{\mathcal{F}(0)}}\right]\\
&+(t_s(0))^2\left[\langle\mathcal{V},\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},\mathcal{V}\rangle\right]\\
&=\frac{(t_s(0))^2}{2\mathcal{F}(0)}\alpha\left[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle\right]\left[\langle q_0,\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},q_0\rangle\right]+(t_s(0))^2\left[\langle\mathcal{V},\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},\mathcal{V}\rangle\right].
\end{aligned}
\tag{3.35}
$$

Factoring $(t_s(0))^2$ from equation (3.35) and then substituting $cos(\mathcal{X})$ and $\mathcal{J}$ expressed in equations (1.1) and (3.10) respectively, we see that

$$
\langle\frac{\partial Z(\delta)}{\partial\delta},\frac{\partial\hat{Z}(\delta)}{\partial\delta}\rangle_{H_N}\Big|_{\delta=0}+\langle\frac{\partial\hat{Z}(\delta)}{\partial\delta},\frac{\partial Z(\delta)}{\partial\delta}\rangle_{H_N}\Big|_{\delta=0}=2\,(t_s(0))^2\left(cos(\mathcal{X})+\mathcal{J}\right).
\tag{3.36}
$$

We now return to the denominator of $cos(\mathcal{Y})$ in equation (3.26). Expanding the squared quantities and substituting as in equation (3.29), we have

$$\left\|\frac{\partial Z(\delta)}{\partial \delta}\right\|_{\delta=0}^{2}$$

$$= \langle \gamma_s \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \; \frac{(t_s(0))^2}{\sqrt{\mathcal{F}(0)}} u, \gamma_s \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \; \frac{(t_s(0))^2}{\sqrt{\mathcal{F}(0)}} u\rangle$$

$$+ \langle -\frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \; \frac{(t_s(0))^2}{\sqrt{\mathcal{F}(0)}} q_0 + \mathcal{V} t_s(0), -\frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \; \frac{(t_s(0))^2}{\sqrt{\mathcal{F}(0)}} q_0$$

$$+ \mathcal{V} t_s(0)\rangle$$

$$= \gamma_s \overline{\gamma}_s \frac{(t_s(0))^4}{4\mathcal{F}(0)} \left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right] \left[\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}\right] \langle u, u\rangle$$

$$+ \langle -\frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \; \frac{(t_s(0))^2}{\sqrt{\mathcal{F}(0)}} q_0, -\frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \; \frac{(t_s(0))^2}{\sqrt{\mathcal{F}(0)}} q_0\rangle$$

$$+ \langle -\frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \; \frac{(t_s(0))^2}{\sqrt{\mathcal{F}(0)}} q_0, \mathcal{V} t_s(0)\rangle$$

$$+ \langle \mathcal{V} t_s(0), -\frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \; \frac{(t_s(0))^2}{\sqrt{\mathcal{F}(0)}} q_0\rangle + \langle \mathcal{V} t_s(0), \mathcal{V} t_s(0)\rangle$$

$$= \gamma_s \overline{\gamma}_s \frac{(t_s(0))^4}{4\mathcal{F}(0)} \left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right] \left[\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}\right]$$

$$+ \frac{(t_s(0))^4}{4\mathcal{F}(0)} \left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right] \left[\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}\right] \langle q_0, q_0\rangle$$

$$- \frac{(t_s(0))^3}{\sqrt{\mathcal{F}(0)}} \; \frac{\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle}{2} \; \langle q_0, \mathcal{V}\rangle$$

$$- \frac{(t_s(0))^3}{\sqrt{\mathcal{F}(0)}} \; \frac{\overline{\langle q_0, \mathcal{V}\rangle} + \overline{\langle \mathcal{V}, q_0\rangle}}{2} \langle \mathcal{V}, q_0\rangle + (t_s(0))^2 \langle \mathcal{V}, \mathcal{V}\rangle.$$

Next we get

$$
\begin{aligned}
&\left\|\frac{\partial Z(\delta)}{\partial\delta}\right\|^2_{\delta=0}\\
&= \frac{(t_s(0))^4}{4\mathcal{F}(0)}\left[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle\right]\left[\overline{\langle q_0,\mathcal{V}\rangle}+\overline{\langle\mathcal{V},q_0\rangle}\right]\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)\\
&-\frac{(t_s(0))^3}{\sqrt{\mathcal{F}(0)}}\ \frac{\overline{\langle q_0,\mathcal{V}\rangle}+\overline{\langle\mathcal{V},q_0\rangle}}{2}\ \langle q_0,\mathcal{V}\rangle\\
&-\frac{(t_s(0))^3}{\sqrt{\mathcal{F}(0)}}\ \frac{\overline{\langle q_0,\mathcal{V}\rangle}+\overline{\langle\mathcal{V},q_0\rangle}}{2}\langle\mathcal{V},q_0\rangle+(t_s(0))^2\langle\mathcal{V},\mathcal{V}\rangle\\
&= \frac{(t_s(0))^4}{4\mathcal{F}(0)}\left[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle\right]\left[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle\right]\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)\\
&-\frac{(t_s(0))^3}{2\sqrt{\mathcal{F}(0)}}\ \left[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle\right]\left[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle\right]+(t_s(0))^2\langle\mathcal{V},\mathcal{V}\rangle\\
&= \frac{(t_s(0))^4}{4\mathcal{F}(0)}\left[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle\right]^2\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)\\
&-\frac{(t_s(0))^3}{2\sqrt{\mathcal{F}(0)}}\ \left[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle\right]^2+(t_s(0))^2\langle\mathcal{V},\mathcal{V}\rangle.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\left\|\frac{\partial Z(\delta)}{\partial\delta}\right\|^2_{\delta=0}=\\
&\frac{(t_s(0))^2}{2\sqrt{\mathcal{F}(0)}}\ \left[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle\right]^2\left(\frac{(t_s(0))^2}{2\sqrt{\mathcal{F}(0)}}\left(\|q_0\|^2+\gamma_s\overline{\gamma}_s\right)-t_s(0)\right)\\
&+(t_s(0))^2\langle\mathcal{V},\mathcal{V}\rangle.
\end{aligned}
\tag{3.37}
$$

Substituting the expression given in equation (3.34) into (3.37) yields

$$
\begin{aligned}
&\left\|\frac{\partial Z(\delta)}{\partial\delta}\right\|^2_{\delta=0}\\
&=\frac{(t_s(0))^2}{2\sqrt{\mathcal{F}(0)}}\ \left[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle\right]^2\frac{\alpha}{2\sqrt{\mathcal{F}(0)}}+(t_s(0))^2\\
&=\frac{\alpha\,(t_s(0))^2}{4\mathcal{F}(0)}\ \left[\langle q_0,\mathcal{V}\rangle+\langle\mathcal{V},q_0\rangle\right]^2+(t_s(0))^2.
\end{aligned}
\tag{3.38}
$$

Similarly,

$$
\begin{aligned}
&\left\|\frac{\partial \hat{Z}(\delta)}{\partial\delta}\right\|^2_{\delta=0}\\
&=\frac{\alpha\,(t_s(0))^2}{4\mathcal{F}(0)}\ \left[\langle q_0,\hat{\mathcal{V}}\rangle+\langle\hat{\mathcal{V}},q_0\rangle\right]^2+(t_s(0))^2.
\end{aligned}
\tag{3.39}
$$

Next we calculate the product

$$\left|\left|\frac{\partial Z(\delta)}{\partial \delta}\right|\right|^2_{\delta=0} . \left|\left|\frac{\partial \hat{Z}(\delta)}{\partial \delta}\right|\right|^2_{\delta=0} .$$

We have

$$\begin{aligned}
&\left|\left|\frac{\partial Z(\delta)}{\partial \delta}\right|\right|^2_{\delta=0} . \left|\left|\frac{\partial \hat{Z}(\delta)}{\partial \delta}\right|\right|^2_{\delta=0} \\
&= \left[(t_s(0))^2 + \frac{\alpha\,(t_s(0))^2}{4\mathcal{F}(0)} \ \left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right]^2\right] \\
&\quad \left[(t_s(0))^2 + \frac{\alpha\,(t_s(0))^2}{4\mathcal{F}(0)} \ \left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right]^2\right] \\
&= (t_s(0))^4 + \frac{\alpha\,(t_s(0))^4}{4\mathcal{F}(0)} \ \left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right]^2 \\
&+ \frac{\alpha\,(t_s(0))^4}{4\mathcal{F}(0)} \ \left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right]^2 \\
&+ \left(\frac{\alpha\,(t_s(0))^2}{4\mathcal{F}(0)}\right)^2 \left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right]^2 \left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right]^2 .
\end{aligned}$$

Thus

$$\begin{aligned}
&\left|\left|\frac{\partial Z(\delta)}{\partial \delta}\right|\right|^2_{\delta=0} . \left|\left|\frac{\partial \hat{Z}(\delta)}{\partial \delta}\right|\right|^2_{\delta=0} \\
&= (t_s(0))^4 + \frac{\alpha\,(t_s(0))^4}{4\mathcal{F}(0)} \left(\left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right]^2 + \left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right]^2\right) \\
&+ \left(\frac{\alpha\,(t_s(0))^2}{4\mathcal{F}(0)}\right)^2 \left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right]^2 \left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right]^2 .
\end{aligned} \tag{3.40}$$

Factoring $(t_s(0))^4$ from equation (3.40) and substituting $\mathcal{J}$ and Λ expressed in equations (3.10) and (3.11), respectively, we see that

$$\left|\left|\frac{\partial Z(\delta)}{\partial \delta}\right|\right|^2_{\delta=0} . \left|\left|\frac{\partial \hat{Z}(\delta)}{\partial \delta}\right|\right|^2_{\delta=0} = (t_s(0))^4 \left(1 + \mathcal{J}^2 + \Lambda\right). \tag{3.41}$$

Recall that $cos(\mathcal{Y})$ is the quotient of $\langle \frac{\partial Z(\delta)}{\partial \delta}, \frac{\partial \hat{Z}(\delta)}{\partial \delta}\rangle_{H_N}\Big|_{\delta=0} + \langle \frac{\partial \hat{Z}(\delta)}{\partial \delta}, \frac{\partial Z(\delta)}{\partial \delta}\rangle_{H_N}\Big|_{\delta=0}$ and $2\left|\left|\frac{\partial Z(\delta)}{\partial \delta}\right|\right|_{\delta=0} . \left|\left|\frac{\partial \hat{Z}(\delta)}{\partial \delta}\right|\right|_{\delta=0}$. Therefore, the division of (3.36) by 2 times the square root of the right hand side of equation (3.41) yields the desired formula (3.9). □

4 Similarity of triangles

Prior to stating the main result of this section we note the following. Let $Q = (\vec{0}, \hat{q}) \in S_N$ and let $\hat{Q} = (\vec{0}, \hat{q}) \in S_n$ be another arbitrary point in S_N and let Z, $\hat{Z}$ be the projected points of Q, $\hat{Q}$ on the sphere respectively, such that in analogy to (2.11) and (2.10), we have

$$SPM(\theta_s, R, \gamma_s)[(0, \hat{q})] = ((1-\hat{t}_s)\gamma_s u,\, \hat{t}_s \hat{q}) \tag{4.1}$$

with

$$\hat{t}_s = \frac{-\frac{\mathcal{A}}{2} + \sqrt{\hat{F}}}{\|\hat{q}\|^2 + \gamma_s \overline{\gamma}_s},\ \hat{F} := \frac{\mathcal{A}^2}{4} - \left(\|\hat{q}\|^2 + \gamma_s \overline{\gamma}_s\right)\alpha. \tag{4.2}$$

We are now ready to prove

Lemma 4.1. *i)* $\forall Q \in S_N, \forall \hat{Q} \in S_N,\ PQ\hat{Q} \approx P\hat{Z}Z \Longleftrightarrow$

$$I := -\alpha\left(\|\hat{q}\|^2 - \|q\|^2\right)\left[1 + \frac{-\mathcal{A}}{\sqrt{F} + \sqrt{\hat{F}}}\right] = 0. \tag{4.3}$$

ii) Moreover, if $\alpha = 0$ *then* $I = 0\ \forall Q \in S_N, \forall \hat{Q} \in S_N$ *so* $PQ\hat{Q} \approx P\hat{Z}Z$. *If* $\alpha < 0$ *then* $PQ\hat{Q} \approx P\hat{Z}Z$ *iff*
$\|\hat{q}\| = \|q\|$.

Proof. We observe that $PQ\hat{Q} \approx P\hat{Z}Z$ iff

$$\frac{\|\overrightarrow{PZ}\|^2_{H_N}}{\|\overrightarrow{P\hat{Q}}\|^2_{H_N}} = \frac{\|\overrightarrow{P\hat{Z}}\|^2_{H_N}}{\|\overrightarrow{PQ}\|^2_{H_N}} \Longleftrightarrow \tag{4.4}$$

$$I = \|\overrightarrow{P\hat{Z}}\|^2_{H_N}\|\overrightarrow{P\hat{Q}}\|^2_{H_N} - \|\overrightarrow{PZ}\|^2_{H_N}\|\overrightarrow{PQ}\|^2_{H_N} = 0. \tag{4.5}$$

We now proceed to calculate I in ((4.5)) by calculating separately each term in ((4.5)) as follows. We have

$$\|\overrightarrow{PQ}\|^2_{H_N} = \langle Q - P, Q - P\rangle_{H_N} = \|q\|^2 + \gamma_s\overline{\gamma}_s. \tag{4.6}$$

$$\|\overrightarrow{P\hat{Q}}\|^2_{H_N} = \langle \hat{Q} - P, -\hat{Q}P\rangle_{H_N} = \|\hat{q}\|^2 + \gamma_s\overline{\gamma}_s$$

$$\|\overrightarrow{PZ}\|^2_{H_N} = \langle Z - P, Z - P\rangle_{H_N} = \|z\|^2 + (c - \gamma_s)\overline{(c-\gamma_s)}$$

$$\|\overrightarrow{P\hat{Z}}\|^2_{H_N} = \langle \hat{Z} - P, \hat{Z} - P\rangle_{H_N} = \|\hat{z}\|^2 + (\hat{c} - \gamma_s)\overline{(\hat{c}-\gamma_s)}$$

since $c - \gamma_s = (1 - t_s)\gamma_s - \gamma_s = -t_s\gamma_s$ and since $\hat{c} - \gamma_s = -\hat{t}_s\gamma_s$ then

$$\|\overrightarrow{PZ}\|^2_{H_N} = t_s^2(\|q\|^2 + \gamma_s\bar{\gamma}_s), \quad \|\overrightarrow{P\hat{Z}}\|^2_{H_N} = \hat{t}_s^2(\|\hat{q}\|^2 + \gamma_s\bar{\gamma}_s). \tag{4.7}$$

Substituting the expressions for $\|\overrightarrow{PQ}\|^2_{H_N}$, $\|\overrightarrow{P\hat{Q}}\|^2_{H_N}$, $\|\overrightarrow{PZ}\|^2_{H_N}$ and $\|\overrightarrow{P\hat{Z}}\|^2_{H_N}$ into the expression for I defined by ((4.5)) we obtain

$$
\begin{aligned}
I &= [\hat{t}_s^2\|\hat{q}\|^2 + \hat{t}_s^2\gamma_s\bar{\gamma}_s][\|\hat{q}\|^2 + \gamma_s\overline{\gamma}_s] - [t_s^2\|q\|^2 + t_s^2\gamma_s\bar{\gamma}_s][\|q\|^2 + \gamma_s\overline{\gamma}_s] \\
&= \hat{t}_s^2[\|\hat{q}\|^2 + \gamma_s\bar{\gamma}_s]^2 - t_s^2[\|q\|^2 + \gamma_s\bar{\gamma}_s]^2 \\
&= \left(-\frac{\mathcal{A}}{2} + \sqrt{\hat{F}}\right)^2 - \left(-\frac{\mathcal{A}}{2} + \sqrt{F}\right)^2 \\
&= \left[\left(-\frac{\mathcal{A}}{2} + \sqrt{\hat{F}}\right) - \left(-\frac{\mathcal{A}}{2} + \sqrt{F}\right)\right]\left[\left(-\frac{\mathcal{A}}{2} + \sqrt{\hat{F}}\right) + \left(-\frac{\mathcal{A}}{2} + \sqrt{F}\right)\right] \qquad (4.8) \\
&= \left[\sqrt{\hat{F}} - \sqrt{F}\right]\left[-\mathcal{A} + \sqrt{F} + \sqrt{\hat{F}}\right] = \frac{\hat{F} - F}{\sqrt{F} + \sqrt{\hat{F}}}\left[-\mathcal{A} + \sqrt{F} + \sqrt{\hat{F}}\right] \\
&= (\hat{F} - F)\left[1 + \frac{-\mathcal{A}}{\sqrt{F} + \sqrt{\hat{F}}}\right].
\end{aligned}
$$

Substitute in the above the relation $(\hat{F} - F) = -\alpha\left(\|\hat{q}\|^2 - \|q\|^2\right)$ to obtain ((4.3)). In order to prove ii) notice that

$$
\frac{-\mathcal{A}}{\sqrt{F} + \sqrt{\hat{F}}} \geq 0 \Longrightarrow \left[1 + \frac{-\mathcal{A}}{\sqrt{F} + \sqrt{\hat{F}}}\right] \geq 1, \qquad (4.9)
$$

and the conclusions follow. □

Remark 4.1. The points $P, Q, \hat{Q}, Z, \hat{Z}$ all lie in one plane. The points $Q, \hat{Q}, Z, \hat{Z}$ lie on one circle with radius $\sqrt{\|\overrightarrow{PZ}\|_{H_N}\|\overrightarrow{PQ}\|_{H_N}}$ and the quantity $\|\overrightarrow{PZ}\|_{H_N}\|\overrightarrow{PQ}\|_{H_N}$ may be defined in analogy to Euclidean geometry as the power of the points Q, Z, with respect to the point P so that $\mathrm{POWER}(Q, Z, P) := \|\overrightarrow{PZ}\|_{H_N}\|\overrightarrow{PQ}\|_{H_N}$. Recall from (4.6) and (4.7) that we have in a complex Hilbert space

$$
\mathrm{POWER}(Q, Z, P) = t_s[\|q\|^2 + \gamma_s\overline{\gamma}_s] = [-\frac{\mathcal{A}}{2} + \sqrt{\frac{\mathcal{A}^2}{4} - (\|q\|^2 + \gamma_s\overline{\gamma}_s)\alpha}]. \qquad (4.10)
$$

Thus we have proven

Proof. The $\mathrm{POWER}(Q, Z, P)$ of the points Q, Z under a projection is a sole functions of $\|q\|$. It is a constant independent of $\|q\|$ iff the projection is the stereographic one. □

5 Reformulation of the main theorem and its proof

We are ready to reformulate the main theorem of this article that is announced in the introduction section and to prove it utilizing the notations and the lemmas of the previous sections.

Theorem 5.1. *The following conditions are equivalent. i) $\alpha = 0$. ii) $\forall \mathcal{X}$, $\mathcal{Y} = \mathcal{X}$. iii) $\forall Q\, \forall \hat{Q}$, $PQ\hat{Q} \approx P\hat{Z}Z$.*

Proof. Let $\alpha = 0$. Then by (3.9) we have for all choices of sphere parameters $0 \neq \gamma_s, \theta_s, R$ subject to assumptions III) and IV) and for any $cos(\mathcal{X})$, $cos(\eta)$, $cos(\hat{\eta})$ and $\|q_0\|$ we have $\mathcal{J} = \Lambda = 0$ and by (3.9) we have $cos(\mathcal{Y}) = cos(\mathcal{X}) \Longrightarrow \mathcal{Y} = \mathcal{X}$.

It remains to show that if $\alpha \neq 0$ there exist angles $\mathcal{X}$ such that $cos(\mathcal{Y}) \neq cos(\mathcal{X})$. We prove this by contradiction. Recall (3.2) and fix $U_0 := \|q_0\|^{-1} q_0$ but allow $\|q_0\| > 0$ to be a free variable and let

$$cos(\eta) = \frac{\langle U_0, \mathcal{V}\rangle + \langle \mathcal{V}, U_0\rangle}{2},\; cos(\hat{\eta}) = \frac{\langle U_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, U_0\rangle}{2}. \tag{5.1}$$

Now choose $\mathcal{V}, \hat{\mathcal{V}}$ such that $-\frac{1}{4} \leq cos(\mathcal{X}) = \frac{\langle \mathcal{V}, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, \mathcal{V}\rangle}{2} \leq \frac{1}{4}$. Next choose U_0 such that $cos(\eta)cos(\hat{\eta}) \neq 0$ and observe that thanks to $\alpha < 0$ that

$$lim_{\|q_0\|\to\infty}|\mathcal{J}| = \infty \Longrightarrow lim_{\|q_0\|\to\infty}\frac{\Lambda}{|\mathcal{J}^2|} = 0 \Longrightarrow$$

$$lim_{\|q_0\|\to\infty}\frac{\mathcal{J}}{\sqrt{1+\mathcal{J}^2+\Lambda} - 1\}} = lim_{\|q_0\|\to\infty}\frac{\mathcal{J}}{|\mathcal{J}|} = sgn(\alpha cos(\eta)cos(\hat{\eta})). \tag{5.2}$$

Notice that if $\forall \mathcal{X}$, $\mathcal{Y} = \mathcal{X} \Longrightarrow cos(\mathcal{Y}) = cos(\mathcal{X})$ and then by (3.9) we have

$$\sqrt{1+\mathcal{J}^2+\Lambda}cos(\mathcal{X}) = cos(\mathcal{X}) + \mathcal{J} \Longrightarrow cos(\mathcal{X}) = \frac{\mathcal{J}}{\sqrt{1+\mathcal{J}^2+\Lambda} - 1}. \tag{5.3}$$

Now let $\|q_0\|$ to be large enough to make $|cos(\mathcal{X})| = \left|\frac{\mathcal{J}}{\sqrt{1+\mathcal{J}^2+\Lambda-1}}\right| > \frac{1}{4}$ and the contradiction follows.

It is noteworthy that even if $\alpha \neq 0$, certain special angles $\mathcal{X}$ are preserved under all spherical boundizations. Most conspicuous are all the angles with vertex at the origin, namely those where $q_0 = 0$. Then, it is easily observed that $\mathcal{J} = \Lambda = 0$ and by (3.9) we have $cos(\mathcal{Y}) = cos(\mathcal{X}) \Longrightarrow \mathcal{Y} = \mathcal{X}$. Next we observe that with $\|q_0\| \neq 0$ all the angles with $\mathcal{V}, \hat{\mathcal{V}}$ such that

$$\langle q_0, \mathcal{V}\rangle =< \mathcal{V}, q_0 >= \langle q_0, \hat{\mathcal{V}}\rangle = \langle \hat{\mathcal{V}}, q_0 \geq 0.$$

We have again $\mathcal{J} = \Lambda = 0 \Longrightarrow \mathcal{Y} = \mathcal{X}$. □

6 Examples and utility of the multi parameter family

The advantages of having a multi-parameter family of projections rather than one “canonical” form are several. It provides an efficient unified framework. If $\theta_s = 0$ and $\gamma_s = 1 = R$, we obtain the stereographic projection for $H = \mathbb{R}^2$ employed in L. V. Ahlfors [2] (chapter one, pages 19–20). If $\theta_s = \frac{1}{2} = R$ and $\gamma_s = 1$ we obtain for $H = \mathbb{R}^2$ the stereographic projection obtained in E. Hille [26] (part one, chapter two, pages 42–43). If $\gamma_s = \theta_s = \frac{1}{2} = R$ then we obtain for $H = \mathbb{R}^2$ the Poincare “compactification”. All three derivations follow from the one set of formulas given

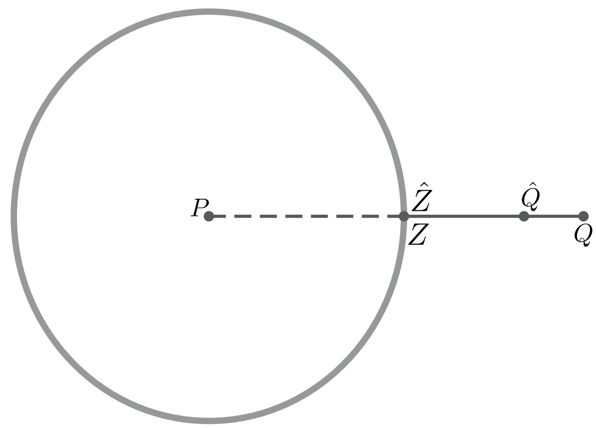

Figure 2: Degeneracy of a nonlinear projection.

in here. Notice that in both derivations of the stereographic projections we compute and find $\alpha = 0$ which is consistent with our generalized definition of a stereographic projection in a complex Hilbert space.

It goes without saying that a multi parameter family of projections provides mathematical means to view objects in different perspectives. It is noteworthy though that if the projection point $(\gamma_s u, 0)$ is chosen to coincide with the coordinate center $(\gamma_s u, 0) = (0, 0)$, then by II) we also have then that $\theta_s = 0$, and a degeneracy and catastrophe besets the nonlinear projection.

The relevant quantities become

$$\begin{aligned} &\mathcal{A} = 0,\ \alpha = -R^2,\ F = R^2\|q\|^2,\ t_s = \frac{R}{\|q\|}, \\ &z = t_s q = \frac{R}{\|q\|} q \ and\ cu = 0\ if\ \|q\| \neq 0. \end{aligned} \tag{6.1}$$

It is readily observed that t_s becomes an unbounded quantity as $\|q\| \to 0$ and z becomes a discontinuous mapping as $\|q\| \to 0$. Moreover, all the points $(\vec{0}, q) \in H_N$, $q \in H$ that lie on the fixed "ray" $U = \frac{1}{\|q\|} q = constant$, map onto one point $(\vec{0}, z) \in H_N$, $z = RU \in H$. This mathematical aspect could serve as a guidance to those who need to locate a spherical lens in a proper location to obtain low distortion. Images are going to be highly distorted if the projection point is placed too close to the projected plane $\mathbb{R}^2$.

The several parameters on which the family of boundizations depends satisfies additional purposes. It brings out the fact that the celebrated stereographic projection is a degenerate member of a family of non linear projections that preserve all directions at infinity. This can easily be seen in $\mathbb{R}^3$ in [21]. For $\alpha = 0$ all directions arrows $\infty(cos(\theta), sin(\theta))$ in $\mathbb{R}^2$ are mapped on the north pole (or the south pole) and shrink to a point. This is in contrast to the case $\alpha < 0$ where all directions $\infty(cos(\theta), sin(\theta))$ in $\mathbb{R}^2$ are preserved and $\mathbb{R}^2$ is mapped onto a subset of a sphere that is "bowl shaped". The rich family of boundizations shows also the special place that Poincare's compactification fits within our larger family.

In the following examples we compute the measure of angles between vectors in various specialized Hilbert spaces and the measure of their images for the purpose of making the scope of this article more tangible. Recall that $\mathcal{A} = -2\gamma_s\overline{\gamma}_s + \gamma_s\overline{\theta}_s + \theta_s\overline{\gamma}_s$, $\alpha = |\gamma_s - \theta_s|^2 - R^2$ and that

$$cos(\mathcal{X}) = \frac{\langle \mathcal{V}, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, \mathcal{V}\rangle}{2},\ cos(\mathcal{Y}) = \frac{cos(\mathcal{X}) + \mathcal{J}}{\sqrt{1 + \mathcal{J}^2 + \varLambda}}. \tag{6.2}$$

Example 6.1. The space of all continuous complex-valued functions on $[a, b]$ with inner product defined by $< f, g >= \int_a^b f(s)\bar{g}(s)ds$, where $f, g \ \in C[a, b]$ and a, b are real numbers. Assume

$$[\int_a^b f(s)\bar{f}(s)ds] > 0 \int_a^b g(s)\bar{g}(s)ds]^{-\frac{1}{2}} > 0.$$

Then,

$$\mathcal{V} =[\int_a^b f(s)\bar{f}(s)ds]^{-\frac{1}{2}} f(s),\ \hat{\mathcal{V}} = \int_a^b g(s)\bar{g}(s)ds]^{-\frac{1}{2}} g(s),$$

$$\|\mathcal{V}\|^2 = 1, \quad \|\hat{\mathcal{V}}\|^2 = 1.$$

and

$$\begin{aligned}\mathcal{F}(0) &= \frac{\mathcal{A}^2}{4} - \alpha\left(\|q_0\|^2 + \gamma_s\overline{\gamma}_s\right)\\ &= \frac{\mathcal{A}^2}{4} - \alpha\left[\int_a^b q_0(s)\bar{q}_0(s)ds + \gamma_s\overline{\gamma}_s\right] = \hat{\mathcal{F}}(0).\end{aligned}$$

Also we have

$$\begin{aligned}&\mathcal{J} = \frac{\alpha}{4\mathcal{F}(0)}\{\\ &\left[\int_a^b q_0(s)\overline{\mathcal{V}}(s)ds + \int_a^b \mathcal{V}(s)\overline{q}_0(s)ds\right]\left[\int_a^b q_0(s)\overline{\hat{\mathcal{V}}}(s)ds + \int_a^b \hat{\mathcal{V}}(s)\overline{q}_0(s)ds\right]\}\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}\varLambda = \frac{\alpha}{4\mathcal{F}(0)}\{&\left[\int_a^b q_0(s)\overline{\mathcal{V}}(s)ds + \int_a^b \mathcal{V}(s)\overline{q}_0(s)ds\right]^2\\ &+\left[\int_a^b q_0(s)\overline{\hat{\mathcal{V}}}(s)ds + \int_a^b \hat{\mathcal{V}}(s)\overline{q}_0(s)ds\right]^2\}.\end{aligned} \tag{6.4}$$

Thus,

$$\begin{aligned}cos(\mathcal{X}) &= \frac{\langle \mathcal{V}, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, \mathcal{V}\rangle}{2} = Re\langle \mathcal{V}, \hat{\mathcal{V}}\rangle = Re\int_a^b \mathcal{V}(s)\overline{\hat{\mathcal{V}}}(s)ds\\ &= [\int_a^b f(s)\bar{f}(s)ds]^{-\frac{1}{2}}[\int_a^b g(s)\bar{g}(s)ds]^{-\frac{1}{2}} Re\int_a^b f(s)\bar{g}(s)ds\end{aligned}$$

and

$$cos(\mathcal{Y}) = \frac{[\int_a^b f(s)\bar{f}(s)ds]^{-\frac{1}{2}}[\int_a^b g(s)\bar{g}(s)ds]^{-\frac{1}{2}} Re \int_a^b f(s)\bar{g}(s)ds + \mathcal{J}}{\sqrt{1+\mathcal{J}^2+\Lambda}},$$

with $\mathcal{J}$ and Λ subject to (6.3) and (6.4) respectively.

Example 6.2. The space of all square matrices M^{nxn} $A, B \in M^{nxn}$ with inner product defined by $< A, B >:= Trace(A^*B)$. Assume $Trace(A^*A) > 0$ and $Trace(B^*B) > 0$ then we have $\mathcal{V} = (Trace(A^*A))^{-\frac{1}{2}} A$, and $\hat{\mathcal{V}} = (Trace(B^*B))^{-\frac{1}{2}} B$, $\|\mathcal{V}\|^2 = \|\hat{\mathcal{V}}\|^2 = 1$, and

$$\mathcal{F}(0) = \frac{\mathcal{A}^2}{4} - \alpha\left(\|q_0\|^2 + \gamma_s\overline{\gamma}_s\right) = \frac{\mathcal{A}^2}{4} - \alpha\left[Trace(q_0^*q_0) + \gamma_s\overline{\gamma}_s\right] = \hat{\mathcal{F}}(0).$$

Also we have

$$\mathcal{J} = \frac{\alpha}{4\mathcal{F}(0)}\left[Trace(q_0^*\mathcal{V}) + Trace(\mathcal{V}^*q_0)\right]\left[Trace(q_0^*\hat{\mathcal{V}}) + Trace(\hat{\mathcal{V}}^*q_0)\right] \tag{6.5}$$

and

$$\Lambda = \frac{\alpha}{4\mathcal{F}(0)}\{\left[Trace(q_0^*\hat{\mathcal{V}}) + Trace(\hat{\mathcal{V}}^*q_0)\right]^2 + \left[Trace(q_0^*\mathcal{V}) + Trace(\mathcal{V}^*q_0)\right]^2\}. \tag{6.6}$$

Thus,

$$cos(\mathcal{X}) = (Trace(A^*A))^{-\frac{1}{2}}(Trace(B^*B))^{-\frac{1}{2}} Trace(A^*B)$$

and

$$cos(\mathcal{Y}) = \frac{(Trace(A^*A))^{-\frac{1}{2}}(Trace(B^*B))^{-\frac{1}{2}} Trace(A^*B) + \mathcal{J}}{\sqrt{1+\mathcal{J}^2+\Lambda}}$$

with $\mathcal{J}$ and Λ subject to (6.5) and (6.6) respectively.

Example 6.3. Take $H = \mathbb{C}^n$ as a Hilbert space over $\mathbb{C}$ and define

$$\langle q, \hat{q}\rangle = x_1\overline{\hat{x}}_1 + \cdots + x_n\overline{\hat{x}}_n,$$

where $q = (x_1, \ldots, x_n)$ and $\hat{q} = (\hat{x}_1, \ldots \hat{x}_n)$. Assume $q_0 = (x_{0_1}, \ldots, x_{0_n})$, $(x_1\overline{x}_1 + \cdots + x_n\overline{x}_n) > 0$ and $(\hat{x}_1\overline{\hat{x}}_1 + \cdots + \hat{x}_n\overline{\hat{x}}_n) > 0$, then we have $\mathcal{V} = (x_1\overline{x}_1 + \cdots + x_n\overline{x}_n)^{-\frac{1}{2}}(x_1 \ldots x_n)$ and $\hat{\mathcal{V}} = (\hat{x}_1\overline{\hat{x}}_1 + \cdots + \hat{x}_n\overline{\hat{x}}_n)^{-\frac{1}{2}}(\hat{x}_1 \ldots \hat{x}_n)$, $\|\mathcal{V}\|^2 = \|\hat{\mathcal{V}}\|^2 = 1$. Thus we have $\mathcal{A} = -2\gamma_s\overline{\gamma}_s + \gamma_s\overline{\theta}_s + \theta_s\overline{\gamma}_s$, $\alpha = \gamma_s\overline{\gamma}_s - \gamma_s\overline{\theta}_s - \theta_s\overline{\gamma}_s + \theta_s\overline{\theta}_s - R^2$ and

$$\begin{aligned}\mathcal{F}(0) &= \frac{\mathcal{A}^2}{4} - \alpha\left(\|q_0\|^2 + \gamma_s\overline{\gamma}_s\right)\\ &= \frac{\mathcal{A}^2}{4} - \alpha\left[(x_{0_1}\overline{x}_{0_1} + \cdots + x_{0_n}\overline{x}_{0_n}) + \gamma_s\overline{\gamma}_s\right] = \hat{\mathcal{F}}(0)\end{aligned}$$

Also we have

$$\mathcal{J} = \frac{\alpha}{4\mathcal{F}(0)}\left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right]\left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right] \tag{6.7}$$

and

$$\Lambda = \frac{\alpha}{4\mathcal{F}(0)}\left(\left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right]^2 + \left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right]^2\right), \tag{6.8}$$

where

$$\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle = (x_{0_1}\overline{v}_1 + \cdots + x_{0_n}\overline{v}_n) + (v_1\overline{x}_{0_1} + \cdots + v_n\overline{x}_{0_n})$$

$$\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle = \left(x_{0_1}\overline{\hat{v}}_1 + \cdots + x_{0_n}\overline{\hat{v}}_n\right) + (\hat{v}_1\overline{x}_{0_1} + \cdots + \hat{v}_n\overline{x}_{0_n}).$$

Thus,

$$cos(\mathcal{X}) = (x_1\overline{x}_1 + \cdots + x_n\overline{x}_n)^{-\frac{1}{2}}(\hat{x}_1\overline{\hat{x}}_1 + \cdots + \hat{x}_n\overline{\hat{x}}_n)^{-\frac{1}{2}}(x_1\overline{\hat{x}}_1 + \cdots + x_n\overline{\hat{x}}_n)$$

and

$$cos(\mathcal{Y}) = \frac{(x_1\overline{x}_1 + \cdots + x_n\overline{x}_n)^{-\frac{1}{2}}(\hat{x}_1\overline{\hat{x}}_1 + \cdots + \hat{x}_n\overline{\hat{x}}_n)^{-\frac{1}{2}}(x_1\overline{\hat{x}}_1 + \cdots + x_n\overline{\hat{x}}_n) + \mathcal{J}}{\sqrt{1 + \mathcal{J}^2 + \Lambda}}$$

with $\mathcal{J}$ and Λ subject to (6.7) and (6.8) respectively.

Example 6.4. We will show that Y. Gingold and H. Gingold [22] family of projections is a special case of our methodology. Take $H = \mathbb{R}^2$ as a Hilbert space over $\mathbb{R}$ and define

$$\langle q, \hat{q}\rangle = x\hat{x} + y\hat{y},$$

where $q = (x, y)$ and $\hat{q} = (\hat{x}, \hat{y})$. Consider $\theta = 0$ and $R = 1$, and assume $q_0 = (x_0, y_0)$, $(x_1^2 + x_2^2) > 0$ and $(\hat{x}_1^2 + \hat{x}_2^2) > 0$ then $\mathcal{V} = (x_1^2 + x_2^2)^{-\frac{1}{2}}(x_1, x_2)$ $\hat{\mathcal{V}} = (\hat{x}_1^2 + \hat{x}_2^2)^{-\frac{1}{2}}(\hat{x}_1, \hat{x}_2)$, $\|\mathcal{V}\|^2 = \|\hat{\mathcal{V}}\|^2 = 1$. Thus we have $\mathcal{A} = -2\gamma^2$, $\alpha = \gamma^2 - 1$ and

$$\begin{aligned}\mathcal{F}(0) &= \frac{\mathcal{A}^2}{4} - \alpha\left(\|q_0\|^2 + \gamma_s^2\right) = \gamma_s^4 - \left(\gamma_s^2 - 1\right)\left(\|q_0\|^2 + \gamma_s^2\right) \\ &= \gamma_s^4 - \left(\gamma_s^2 - 1\right)\|q_0\|^2 - \left(\gamma_s^2 - 1\right)\gamma_s^2 = \gamma_s^4 - \left(\gamma_s^2 - 1\right)\|q_0\|^2 - \gamma_s^4 + \gamma_s^2 \\ &= \gamma_s^2 - \left(\gamma_s^2 - 1\right)\|q_0\|^2 = \gamma_s^2 + \left(1 - \gamma_s^2\right)\|q_0\|^2 \\ &= \gamma_s^2 + \left(1 - \gamma_s^2\right)\left(x_0^2 + y_0^2\right) = \hat{\mathcal{F}}(0).\end{aligned}$$

Since $H = \mathbb{R}^2$ over $\mathbb{R}$ then $\langle q_0, \mathcal{V}\rangle = \langle \mathcal{V}, q_0\rangle$ and $\langle q_0, \hat{\mathcal{V}}\rangle = \langle \hat{\mathcal{V}}, q_0\rangle$ therefore we have

$$\begin{aligned}\mathcal{J} &= \frac{\alpha}{4\mathcal{F}(0)}\left[\langle q_0, \mathcal{V}\rangle + \langle \mathcal{V}, q_0\rangle\right]\left[\langle q_0, \hat{\mathcal{V}}\rangle + \langle \hat{\mathcal{V}}, q_0\rangle\right] = \frac{\alpha}{4\mathcal{F}(0)}\left[2\langle q_0, \mathcal{V}\rangle\right]\left[2\langle q_0, \hat{\mathcal{V}}\rangle\right] \\ &= \frac{\alpha}{4\mathcal{F}(0)}4\langle q_0, \mathcal{V}\rangle\langle q_0, \hat{\mathcal{V}}\rangle = \frac{\alpha}{\mathcal{F}(0)}\langle q_0, \mathcal{V}\rangle\langle q_0, \hat{\mathcal{V}}\rangle \\ &= \frac{\alpha}{\mathcal{F}(0)}\left(x_0v_1 + y_0v_2\right)\left(x_0\hat{v}_1 + y_0\hat{v}_2\right) \\ &= \frac{-\left(1 - \gamma_s^2\right)}{\gamma_s^2 + (1 - \gamma_s^2)(x_0^2 + y_0^2)}\left(x_0v_1 + y_0v_2\right)\left(x_0\hat{v}_1 + y_0\hat{v}_2\right)\end{aligned} \tag{6.9}$$

and

$$
\begin{aligned}
\Lambda = & \frac{\alpha}{4\mathcal{F}(0)} \left(\left[\langle q_0, \hat{\mathcal{V}} \rangle + \langle \hat{\mathcal{V}}, q_0 \rangle \right]^2 + \left[\langle q_0, \mathcal{V} \rangle + \langle \mathcal{V}, q_0 \rangle \right]^2 \right) \\
= & \frac{\alpha}{4\mathcal{F}(0)} \left(\left[2\langle q_0, \hat{\mathcal{V}} \rangle \right]^2 + \left[2\langle q_0, \mathcal{V} \rangle \right]^2 \right) \\
= & \frac{\alpha}{4\mathcal{F}(0)} 4 \left(\left[\langle q_0, \hat{\mathcal{V}} \rangle \right]^2 + \left[\langle q_0, \mathcal{V} \rangle \right]^2 \right) \\
= & \frac{\alpha}{\mathcal{F}(0)} \left(\left[x_0 \hat{v}_1 + y_0 \hat{v}_2 \right]^2 + \left[x_0 v_1 + y_0 v_2 \right]^2 \right) \\
= & \frac{-\left(1-\gamma_s^2\right)}{\gamma_s^2 + \left(1-\gamma_s^2\right)\left(x_0^2+y_0^2\right)} \left(\left[x_0 \hat{v}_1 + y_0 \hat{v}_2 \right]^2 + \left[x_0 v_1 + y_0 v_2 \right]^2 \right).
\end{aligned} \tag{6.10}
$$

Thus,

$$cos(\mathcal{X}) = (x_1^2 + x_2^2)^{-\frac{1}{2}} (\hat{x}_1^2 + \hat{x}_2^2)^{-\frac{1}{2}} (x_1 \hat{x}_1 + x_2 \hat{x}_2)$$

and

$$cos(\mathcal{Y}) = \frac{(x_1^2 + x_2^2)^{-\frac{1}{2}} (\hat{x}_1^2 + \hat{x}_2^2)^{-\frac{1}{2}} (x_1 \hat{x}_1 + x_2 \hat{x}_2) + \mathcal{J}}{\sqrt{1 + \mathcal{J}^2 + \Lambda}}$$

with $\mathcal{J}$ and Λ subject to (6.9) and (6.10) respectively.

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