Complex Finsler spaces with \((\gamma, |\beta|)\)-metric

Sweta Kumari and P. N. Pandey

Abstract. The present paper deals with the differential geometry of a complex Finsler space endowed with \((\gamma, |\beta|)\)-metric, where \(\gamma\) is a cubic-root metric and \(\beta\) is a differential \((1, 0)\)-form. Expressions for the fundamental metric tensor, complex angular metric tensor, their inverses, Chern-Finsler connection, holomorphic curvature and Euler-Lagrange equations are obtained.

Key words: Complex Finsler space; \((\gamma, |\beta|)\)-metric; Chern-Finsler connection coefficients; curvature; Euler-Lagrange equations.

1 Introduction

In 1979, M. Matsumoto [9], introduced the concept of cubic metric on a differentiable manifold with the local coordinates \(x^i\), defined by \(L(x, y) = (a_{ijk}(x)y^iy^jy^k)^{\frac{1}{3}}\), where \(a_{ijk}(x)\) are components of a symmetric tensor field of \((0, 3)\)-type depending upon the position \(x\) alone. The Finsler space with a cubic metric is called a cubic Finsler space. There are some papers related to the cubic Finsler space [6, 12, 13] etc. In 2011, T. N. Pandey and V. K. Chaubey [11] introduced the concept of \((\gamma, \beta)\)-metric, where \(\gamma\) is a cubic-root metric and \(\beta\) is a 1-form metric defined by \(\gamma = \sqrt[3]{a_{ijk}(x)y^iy^jy^k}\) and \(\beta = b_i(x)y^i\) respectively. N. Aldea and G. Munteanu [3] introduced a complex Finsler space with Randers metric following the ideas from real case [4, 5, 8, 14] in 2009. The authors of the current paper [7] studied a complex Randers space with metric \(L = \alpha + \epsilon|\beta| + k|\beta|^2\alpha, \epsilon, k \neq 0\).

The aim of the present paper is to introduce and study a complex Finsler space with the fundamental function, \(F(\gamma, |\beta|)\) on the lines of the Finsler space with \((\alpha, |\beta|)\) metric as studied by N. Aldea and G. Munteanu [3], such that

\[
F(z, \eta) = \gamma(z, \eta) + |\beta|(z, \eta),
\]

where

\[
\begin{align*}
\gamma &= \sqrt[3]{a_{ijk}y^iy^jy^k}; \\
|\beta(z, \eta)| &= \sqrt{\beta(z, \eta)\beta(z, \eta)} \text{ with } \beta(z, \eta) = b_i(z)y^i.
\end{align*}
\]

In this paper we determine the fundamental metric tensor (it’s inverse and determinant), the complex angular metric tensor (it’s inverse and determinant), Chern-Finsler connection coefficients, holomorphic curvature and Euler-Lagrange equations for the complex Finsler space \((M, F)\) with \((\gamma, |\beta|)\)-metric.

## 2 Preliminaries

Let \(M\) be a complex manifold of dimension \(n\) and \((z^k)_{k=1,...,n}\) be complex coordinates in a local chart. The complexified tangent bundle \(T_C M\) splits into holomorphic tangent bundle \(T'_C M\) and anti-holomorphic tangent bundle \(T''_C M\), i.e. \(T_C M = T'_C M \oplus T''_C M\). The holomorphic tangent bundle \(T'_C M\) is itself a complex manifold with local coordinates \(u = (z^k, \eta^k)\) in a chart, which changes by the following rules

\[
(z^k)' = z^k(z), \quad \eta^k = \frac{\partial z^k}{\partial z^j} \eta^j.
\]

Further, \(T_C(T'_C M)\) decomposes as a sum of holomorphic and anti-holomorphic tangent bundles \(T'_C u(T'_C M)\) and \(T''_C u(T'_C M)\) respectively. A natural local frame \(\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^j}\}\) for \(T'_C u(T'_C M)\) changes according to the rules obtained from Jacobi matrix of (3). Since the changing rule of \(\frac{\partial}{\partial z^j}\) contains the second order partial derivatives, the concept of complex non-linear connection \((c:n:c:\)\) was introduced.

Let \(V(T'_C M) \subset T'(T'_C M)\) be the vertical bundle spanned by \(\{\partial/\partial \eta^k\}\). The complex non-linear connection \((c:n:c:\)\) determines a supplementary complex subbundle to \(V(T'_C M)\) in \(T'(T'_C M)\), i.e. \(T'(T'_C M) = H(T'_C M) \oplus V(T'_C M)\), called the horizontal bundle. It determines an adapted frame \(\{\frac{\partial}{\partial z^k} = N^l_k \frac{\partial}{\partial \eta^l}\}\), where \(N^l_k(z, \eta)\) are the coefficients of the \((c:n:c:\)\) [1, 2, 10].

A complex Finsler metric \(F\) on complex manifold \(M\) is a continuous function \(F: T'_C M \rightarrow \mathbb{R}^+\) satisfying following conditions [10]

1. \(L = F^2\) is smooth on \(T'_C M \setminus \{0\}\);
2. \(F(z, \eta) \geq 0\), the equality holds if and only if \(\eta = 0\);
3. \(F(z, \lambda \eta) = |\lambda| F(z, \eta)\), for all \(\lambda \in \mathbb{C}\);
4. the Hermitian matrix \(\{g_{ij}(z, \eta)\}\) where \(g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}\), is positive definite on \(T'_C M \setminus \{0\}\).

Let us write \(L = F^2\). Then, the pair \((M, F)\) is called a complex Finsler space. A Hermitian connection of \((1,0)\) type named the Chern-Finsler Connection [1] has a special meaning in a complex Finsler space. Notationally, it is \(D^I N = (L^i_{jk}, 0, C^i_{jk}, 0)\), where

\[
C^F_{ijk} = g^{m \ell} \frac{\partial g_{m \ell}}{\partial z^j} \eta^i, \quad L^i_{jk} = g^{m \ell} \frac{\delta g_{m \ell}}{\delta z^k} = \frac{\partial N^l_k}{\partial \eta^j}, \quad C^i_{jk} = g^{m \ell} \frac{\partial g_{m \ell}}{\partial \eta^k}.
\]
The holomorphic curvature [1, 10] of the complex Finsler space \((M, F)\) in the direction \(\eta\) is

\[
\kappa_F(z, \eta) = \frac{2}{L^2(z, \eta)} G(R(\chi, \overline{\chi}) \chi, \overline{\chi}),
\]

where \(G\) is the \(N\)-lift of the complex Finsler metric tensor \(g_{\overline{\gamma}}\) defined by \(G = g_{\overline{\gamma}} dz^i \otimes d\overline{z}^j + g_{\overline{\gamma}} \delta \eta^i \otimes \delta \overline{\eta}^j\) and \(R\) is the curvature of Chern-Finsler connection. Locally,

\[
\kappa_F(z, \eta) = \frac{2}{L^2} R_{\overline{h}k} \overline{\eta}^i \eta^k,
\]

where

\[
R_{\overline{h}k} = -g_{\overline{h}l} \delta_{k}^{(CF)} \left\{ N_{l}^{j} \right\} \overline{\eta}^h [4].
\]

Consider a \(C^\infty\) curve \(c(t), t \in R\) on a complex manifold \(M\) and \((z^k(t), \eta^k(t) = dz^k/dt)\) be its extension on \(T'M\). The Euler-Lagrange equations with respect to a complex Lagrangian \(L = F^2\) [2, 10] are given by

\[
E_i(L) \equiv \frac{\partial L}{\partial z^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}^i} \right) = 0,
\]

where \(L\) is considered along the curve \(c(t)\) on \(T'M\). The solutions of the Euler-Lagrange equations are the extremal curves with respect to the arc length.

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Differentiating (1.2) partially with respect to \(\eta^j\) and \(\overline{\eta}^p\) and using the symmetry of \(a_{\overline{\eta}k}\) in its indices, we obtain

\[
\begin{align*}
\frac{\partial a_1}{\partial \eta^j} &= \frac{a_1}{3 \eta^j}, & \frac{\partial a_2}{\partial \eta^j} &= \frac{2 \eta^j}{3 \eta^2}, & \frac{\partial |\beta|}{\partial \eta^j} &= \frac{\eta^j}{2 |\beta|}, & \frac{\partial |\beta|}{\partial \overline{\eta}^j} &= \frac{\eta^j}{2 |\beta|},
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 a_1}{\partial \eta^j \partial \eta^p} &= \frac{a_1}{3 \eta^p} - \frac{4a_1 \eta^p}{3 \eta^2}, & \frac{\partial^2 |\beta|}{\partial \eta^j \partial \eta^p} &= \frac{b_1 b_2}{3 |\beta|},
\end{align*}
\]

where \(a_1 = a_{\overline{\eta}k} \eta^j \overline{\eta}^k\), \(a_2 = a_{\overline{\eta}k} \eta^j \overline{\eta}^k\), \(a_{\overline{\eta}p} = 2a_{\overline{\eta}k} \eta^k \overline{\eta}^p\).

The function \(L = F^2\) depends on \(z\) and \(\eta\) because of \(\gamma = \gamma(z, \eta)\) and \(|\beta| = |\beta(z, \eta)|\). Also, \(\gamma\) and \(\beta\) are homogeneous with respect to \(\eta\), i.e. \(\gamma(\lambda z, \lambda \eta) = |\lambda| \gamma(z, \eta)\) and \(\beta(z, \lambda \eta) = \lambda \beta(z, \eta), \forall \lambda \in \mathbb{C}\). Therefore, \(L(z, \lambda \eta) = \lambda \overline{\lambda} L(z, \eta), \forall \lambda \in \mathbb{C}\).

From the homogeneity property, we have the following

\[
\frac{\partial \gamma}{\partial \eta^j} \eta^j = \frac{1}{3} \gamma, \quad \frac{\partial |\beta|}{\partial \eta^j} \eta^j = \frac{1}{2} |\beta|.
\]

Now,

\[
L = F^2 = (\gamma + |\beta|)^2.
\]
On differentiating (3.3) partially with respect to \( \gamma \) and \(|\beta|\) respectively, we have

\[
(3.4) \quad L_\gamma = 2F = L_{|\beta|}, \quad L_{\gamma|\beta|} = 2 = L_{|\beta||\beta|}.
\]

Again differentiating (3.3) partially with respect to \( \eta^i \) and \( \overline{\eta}^j \) respectively, we get

\[
(3.5) \quad \eta_i = \frac{\partial L}{\partial \eta^i} = \frac{2F}{3\gamma^2} a_i + \frac{F\beta_i}{|\beta|} b_i; \quad \overline{\eta}_j = \frac{\partial L}{\partial \overline{\eta}^j} = \frac{4F}{3\gamma^2} a_j + \frac{F\beta_j}{|\beta|} b_j.
\]

Using (3.4), we conclude the following relations

\[
(3.6) \quad \left\{
\begin{align*}
\gamma L_\gamma + |\beta|L_{|\beta|} &= 2L, \quad \gamma L_{\gamma|\beta|} + |\beta|L_{|\beta||\beta|} = L, \\
\gamma^2 L_{\gamma \gamma} + 2|\beta|L_{|\beta|} + |\beta|^2 L_{|\beta||\beta|} &= 2L.
\end{align*}
\right.
\]

The fundamental metric tensor \( g_{ij} \) of the complex Randers space \((M, F)\) is given by

\[
(3.7) \quad g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \overline{\eta}^j} + \frac{\partial L}{\partial \eta^i} \frac{\partial L}{\partial \overline{\eta}^j} + L_{\gamma|\beta|} \left( \frac{\partial \gamma}{\partial \eta^i} \frac{\partial |\beta|}{\partial \overline{\eta}^j} + \frac{\partial |\beta|}{\partial \eta^i} \frac{\partial \gamma}{\partial \overline{\eta}^j} \right).
\]

where \( L_\gamma = \frac{\partial L}{\partial \gamma} \), \( L_{|\beta|} = \frac{\partial L}{\partial |\beta|} \), \( L_{\gamma|\beta|} = \frac{\partial^2 L}{\partial \gamma \partial |\beta|} \) and \( L_{|\beta||\beta|} = \frac{\partial^2 L}{\partial |\beta| \partial |\beta|} \).

Using (3.1) and (3.4) in (3.7), we have

\[
(3.8) \quad g_{ij} = \frac{2F}{3\gamma^2} a_i a_j - \frac{4}{9\gamma^5} (F + |\beta|) a_i a_j + \frac{1}{2|\beta|} (F + |\beta|) b_i b_j
\]

\[
+ \frac{1}{3\gamma^2 |\beta|} (\beta a_i b_j + 2\overline{\beta} b_i a_j).
\]

If we assume \( \rho_0 = \frac{L_{|\beta|}}{3\gamma^2} = \frac{2F}{3\gamma^2} \) and \( \mu_0 = \frac{L_{|\beta||\beta|}}{|\beta|} = \frac{F}{|\beta|} \), (3.5) gives

\[
(3.9) \quad (\beta a_i b_j + 2\overline{\beta} b_i a_j) = \frac{1}{\rho_0 \mu_0} \eta_i \overline{\eta}_j - \frac{\mu_0}{\rho_0} |\beta|^2 b_i b_j - \frac{2\rho_0}{\mu_0} a_i a_j.
\]

Substituting (3.9) in (3.8), we obtain

\[
(3.10) \quad g_{ij} = \frac{2F}{3\gamma^2} a_i a_j - \frac{8F}{9\gamma^5} a_i a_j + \frac{F}{2|\beta|} b_i b_j + \frac{1}{2L} \eta_i \overline{\eta}_j.
\]

This leads to

**Theorem 3.1.** The fundamental metric tensor of a complex Finsler space \((M, F)\) with \((\gamma, |\beta|)\)-metric is given by (3.10).

Next, we have the following proposition [4] given by D. Bao, S. S. Chern and Z. Shen:
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Proposition 3.2. Suppose
1. $(Q_{ij})$ is a nonsingular $n \times n$ complex matrix with inverse $(Q_{ij})^\dagger$.
2. $C_i$ and $C_i' = \overline{C_i}, i = 1, 2, 3, \ldots, n$ are complex numbers.
3. $C' = Q^\dagger C Q$ and its conjugates, $C'' = C' C' = \overline{C'} C' ; H_{ij} = Q_{ij}' \pm C_i C_j$.

Then,
(i) $\det(H_{ij}) = (1 \pm C^2) \det(Q_{ij})$.
(ii) whenever $1 \pm C^2 \neq 0$ the matrix $(H_{ij})$ is invertible and in this case its inverse is $H_{ij} = Q_{ij}^\dagger + \frac{1}{1 \pm C^2} C_i C_j$.

We use the Proposition 3.2 to find the inverse and determinant of the fundamental metric tensor. (3.10) may be written as

$$(3.11) \quad g_{ij} = \frac{2F}{3\gamma^2} \left\{ a_{ij} - \frac{4}{3\gamma^3} a_i a_j + \frac{3\gamma^2}{4[\beta]} b_i b_j + \frac{3\gamma^2}{4LF} \eta_i \eta_j \right\}.$$  

Assuming $Q_{ij} = a_{ij}$ and $C_i = \frac{2}{4\gamma^2} a_i$, and applying Proposition 3.2, we find $Q_{ij} = a_{ij}^\dagger, C_i = \frac{2}{\sqrt{4\gamma^2}} a_i, C'' = \frac{2}{\sqrt{4\gamma^2}}$, where $(a_{ij}^\dagger)$ is the Hermitian inverse of $(a_{ij})$. Since $1 - C^2 = \frac{1}{4} \neq 0$, the matrix $H_{ij} = a_{ij} - \frac{2}{3\gamma^3} a_i a_j$ is invertible with the inverse $H_{ij} = a_{ij}^\dagger + \frac{2\gamma^3}{3\gamma^2} b_i b_j$ and $\det(H_{ij}) = \frac{4}{9\gamma^6} \det(a_{ij})$.

Now, assuming $Q_{ij} = a_{ij} - \frac{4}{3\gamma^3} a_i a_j$ and $C_i = \sqrt{\frac{3}{\gamma^2}} \frac{2}{\gamma^2} b_i$ and applying Proposition 3.2, we obtain $Q_{ij} = a_{ij}^\dagger, C'' = \frac{3}{\gamma^2} \left\{ \left| b \right|^2 + \frac{\left| b \right|^2}{\gamma^2} \right\}$, where $b_i = a_{ij} b_j$. Therefore

$$(3.12) \quad C'' = \frac{3\gamma^2}{4LF} \left\{ \left| b \right|^2 + \frac{\left| b \right|^2}{\gamma^2} \right\}, \quad \left| b \right|^2 = \frac{2}{\gamma^2} b_i b_j.$$  

From (3.5), we get

$$(3.13) \quad \left\{ \begin{array}{l}
\eta_i \eta_j = \frac{2F}{3\gamma^2} \eta_i + \frac{2\beta}{\gamma} b_i b_j \eta_j = \frac{2F}{3\gamma^2} \beta + \frac{2\beta}{\gamma} \left| b \right|^2, b_i \eta_i = \frac{F}{\gamma^2} \beta + \frac{\beta}{\gamma} \left| b \right|^2, \\
\eta_i \eta_j = \frac{2F}{\gamma} + F[\beta], \eta_i \eta_j = \frac{4F}{\gamma} + F[\beta].
\end{array} \right.$$  

Using (3.13) in (3.12), we have

$$(3.14) \quad C'' = \frac{3}{LF} \left\{ \frac{F}{\gamma} \left( 1 - \frac{\left| b \right|^2}{\gamma^2} \right) - \frac{\beta \gamma b_i}{\gamma^2} \right\}.$$  

Therefore $C^2 = 1 + \frac{\gamma^2}{2F^2 \sigma}$. Since $1 + C^2 \neq 0$, $H_{ij} = a_{ij} - \frac{4}{3\gamma^3} a_i a_j + \frac{3\gamma^2}{4[\beta]} b_i b_j + \frac{3\gamma^2}{4LF} \eta_i \eta_j$ is invertible with the inverse

$$(3.15) \quad H_{ij} = a_{ij}^\dagger + A \eta_i \eta_j + B b_i b_j + C(\beta b_i \eta_j + \bar{\beta} \gamma \eta_j),$$
where
\begin{equation}
A = \frac{4\beta^2 - 2\gamma^2 - 12\beta^2 + 2\gamma^2 + \gamma^2 + \gamma^2}{\gamma^3}, \quad B = \frac{-3\gamma^3 (4F\sigma + 3\gamma^2)}{\gamma^3 (4F\sigma + 3\gamma^2)}, \quad C = \frac{3(4F\sigma + 3\gamma^2) + 6\gamma^2}{\gamma^3 (4F\sigma + 3\gamma^2)}.
\end{equation}

Also, the determinant of $H_{ij}$ is
\begin{equation}
det(H_{ij}) = \frac{(4F\sigma + \gamma^2)^2}{24F}\det(a_{ij}).
\end{equation}

From (3.11), $g_{ij} = \frac{2F}{3\gamma^2}H_{ij}$, the inverse of the fundamental metric tensor is given by
\begin{equation}
g^{ij} = \frac{3\gamma^2}{2F}H_{ji},
\end{equation}

where $H_{ji}$ is given by (3.15).

Also, the determinant of the fundamental metric tensor is given by
\begin{equation}
det(g_{ij}) = \left(\frac{2F}{3\gamma^2}\right)^n det(H_{ij}),
\end{equation}

where $det(H_{ij})$ is given by (3.17). Thus, we have

**Theorem 3.3.** The inverse and determinant of the fundamental metric tensor of a complex Finsler space $(M, F)$ with $(\gamma, |\beta|)$-metric are given by (3.18) and (3.19) respectively.

Next, we define the complex angular metric tensor of the complex Finsler space $(M, F)$ with $(\gamma, |\beta|)$-metric as

\begin{equation}
k_{ij} = \frac{\partial^2 F}{\partial \eta^i \partial \eta^j} \quad \quad (3.20)
\end{equation}

\begin{equation}
k_{ij} = F_{\gamma i} \frac{\partial}{\partial \eta^i} \frac{\partial}{\partial \eta^j} + F_{\gamma j} \left( \frac{\partial}{\partial \eta^i} \frac{\partial |\beta|}{\partial \eta^j} + \frac{\partial |\beta|}{\partial \eta^i} \frac{\partial}{\partial \eta^j} \right) \quad \quad (3.21)
\end{equation}

\begin{equation}
k_{ij} = \frac{\partial}{\partial \eta^i} \frac{\partial |\beta|}{\partial \eta^j} + F_{\beta i} \frac{\partial}{\partial \eta^i} \frac{\partial}{\partial \eta^j} + F_{\beta j} \frac{\partial^2 |\beta|}{\partial \eta^i \partial \eta^j},
\end{equation}

where $F_{\gamma i} = \frac{\partial F}{\partial \gamma^i}, F_{\beta i} = \frac{\partial F}{\partial |\beta|}, F_{\gamma j} = \frac{\partial^2 F}{\partial \gamma^i \partial \gamma^j}, F_{\beta |\beta|} = \frac{\partial^2 F}{\partial \gamma^i \partial \beta^j}$ and $F_{\gamma |\beta|} = \frac{\partial^2 F}{\partial \gamma^i \partial |\beta|} = F_{\beta |\gamma|}$.

On differentiating $F$ partially with respect to $\gamma$ and $|\beta|$ respectively, we have

\begin{equation}
F_{\gamma} = 1 = F_{|\beta|}, \quad F_{\gamma \gamma} = 0 = F_{\gamma |\beta|} = F_{|\beta| |\beta|.}
\end{equation}

On substituting (3.1) and (3.21) in (3.20), we obtain

\begin{equation}
k_{ij} = \frac{1}{3\gamma^2} h_{ij} + \frac{1}{4|\beta|} h_{i} b_{j},
\end{equation}

where

\begin{equation}
h_{ij} = a_{ij} = \frac{4}{3\gamma^2} a_{i} a_{j}.
\end{equation}
To find out the inverse of the complex angular metric tensor, we will apply Proposition 3.2. From (3.22), \( k_{ij} = \frac{1}{\gamma^2} \left\{ a_{ij} - \frac{4}{\sqrt{\gamma^2}} a_i a_j + \frac{3\gamma^2}{4|\beta|} b_i b_j \right\} \). Assuming \( Q_{ij} = a_{ij} \) and \( C_i = \frac{2}{\sqrt{\gamma^2}} \), and applying Proposition 3.2, we get \( Q^i_j = a^{ji}, C^i = \frac{1}{\gamma^2} a^i \) and \( C^2 = \frac{2}{\gamma^3} \). Since \( 1 - C^2 = \frac{1}{\gamma^2} \neq 0 \), the matrix \( H_{ij} = a_{ij} - \frac{4}{\sqrt{\gamma^2}} a_i a_j \) is invertible with the inverse \( H^{-1} = a^{ji} + \frac{a^i}{\gamma^2} \) and \( \det(H_{ij}) = \frac{1}{\gamma^2} \det(a_{ij}) \). Taking \( Q_{ij} = a_{ij} - \frac{4}{\gamma^2} a_i a_j \) and \( C_i = \sqrt{\frac{2}{|\beta|}} b_i \), Proposition 3.2 gives \( Q^i_j = a^{ji} + \frac{a^i}{\gamma^2} \) and \( C^i = \sqrt{\frac{2}{|\beta|}} \left( b^i + \frac{a^i}{\gamma^2} \right) \). Therefore \( C^2 = \frac{3\gamma^2}{2|\beta|} \left( ||b||^2 + \frac{|\beta|^2}{\gamma^2} \right) \). Since \( 1 + C^2 \neq 0 \), the inverse of \( H_{ij} = a_{ij} - \frac{4}{\gamma^2} a_i a_j + \frac{3\gamma^2}{4|\beta|} b_i b_j \) exists and is given by

\[
H^{-1} = a^{ji} + \frac{a^i}{\gamma^2} = \frac{3\gamma^3}{2} \left( b^i + \frac{a^i}{\gamma^2} \right) \left( b^j + \frac{a^j}{\gamma^2} \right)
\]

and

\[
\det(H_{ij}) = \frac{\sigma}{12|\beta|^3} \det(a_{ij}).
\]

Since \( k_{ij} = \frac{1}{\gamma^2} H_{ij} \), the inverse of the angular metric tensor \( k_{ij} \) is given by

\[
k^{-1} = 3\gamma^2 H^{-1},
\]

where \( H^{-1} \) is given by (3.24).

Also, the determinant of the angular metric tensor \( k_{ij} \) is

\[
\det(k_{ij}) = \frac{\sigma}{4(3\gamma^2)^{n+1}|\beta|^3} \det(a_{ij}).
\]

Thus, we have

**Theorem 3.4.** The inverse and determinant of the complex angular metric tensor of a complex Finsler space \((M, F)\) with \((\gamma, |\beta|)\)-metric are given by (3.26) and (3.27) respectively.

### 4 Connection coefficients and curvature

The Chern-Finsler connection coefficients (c.n.c.) of a complex Finsler space \((M, F)\) with \((\gamma, |\beta|)\)-metric is defined by

\[
^C_F N^i_j = g^{m\ell} \frac{\partial g_{m\ell}}{\partial z^j} \eta^i = g^{m\ell} \frac{\partial g_{m\ell}}{\partial z^j}.
\]

Differentiating (3.5) with respect to \( z^j \), we have

\[
\frac{\partial \eta^m}{\partial z^j} = \frac{1}{2|\beta|} \left( \frac{2|\beta|}{3\gamma^2} \frac{\partial a_{m\ell}}{\partial z^j} \eta^i \eta^k + \beta \frac{\partial b_m}{\partial z^j} \eta^i + \beta \frac{\partial b_i}{\partial z^j} \eta^m \right)
\]

\[
\times \left( \frac{4a_{m\ell}}{3\gamma^2} + \beta \frac{b_m}{|\beta|} \right) + \frac{1}{2|\beta|} \left\{ \frac{4a_{m\ell}}{3\gamma^2} \eta^i \eta^k - \frac{8}{9\gamma^2} a_{m\ell} \frac{\partial a_{m\ell}}{\partial z^j} \eta^i \eta^k + b_m \frac{\partial b_i}{\partial z^j} \eta^i + \beta \frac{\partial b_m}{\partial z^j} \eta^i - \frac{\beta}{2|\beta|} b_m \frac{\partial b_i}{\partial z^j} \eta^i \right\}.
\]
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Substituting (3.18) and (4.2) in (4.1), we have

\[
C_F^i N^i_j = \frac{3\gamma^2}{4F|\beta|} \left\{ a_{mj} + An^j m + Bb^j b^m + (\beta b^i \eta^m + b^m \eta^i) \right\} \\
\times \left\{ \left( \frac{2|\beta|}{3\gamma^2} \frac{\partial g_{\mu \pi}}{\partial z^j} \eta^\mu \eta^\pi \eta^k + \beta \frac{\partial g_{\mu \pi}}{\partial z^j} \eta^\mu \eta^\pi \eta^k \right) + \frac{4a_{m\pi}}{3\gamma^2} \frac{\beta b^m}{|\beta|} \right\} \\
+ F \left( \frac{4}{3\gamma^2} \frac{\partial g_{m\pi}}{\partial z^j} \eta^\mu \eta^\pi \eta^k - \frac{8}{9\gamma^2} a_{m\pi} \frac{\partial a_{m\pi}}{\partial z^j} \eta^\mu \eta^\pi \eta^k + \frac{b^m b^j b^k}{2|\beta|} \frac{\partial b_m}{\partial z^j} \eta^k \right) \\
+ \frac{\beta}{|\beta|} \frac{\partial b_m}{\partial z^j} - \frac{\beta}{2|\beta|} \frac{\partial b_m}{\partial z^j} \eta^k \right\}
\]

Next, we consider the following complex Cartan tensors [2]

\[
C_{\pi k} = \frac{\partial g_{j\pi}}{\partial \eta^k} = \frac{\partial g_{j\pi}}{\partial \gamma} \frac{\partial \gamma}{\partial \eta^k} + \frac{\partial g_{j\pi}}{\partial |\beta|} \frac{\partial |\beta|}{\partial \eta^k}
\]

Differentiating (3.10) with respect to \( \eta^k \) and using (3.1) in (4.4), we have

\[
C_{j\pi k} = \frac{1}{3\gamma^2} \left\{ \frac{\beta}{|\beta|} b_k - \frac{2(F + |\beta|)}{3\gamma^2} a_k \right\} a_j \pi \\
+ \frac{4}{27\gamma^4} F \left( 5\gamma^2 + 10|\beta|^2 - 16|\gamma| \right) a_k a_j a_{\pi k} \\
+ \frac{1}{4F|\beta|} (\beta|\beta| - F^2) b_k b_j b_{\pi k} \\
- \frac{1}{9F|\beta|\gamma^4} \left\{ \frac{2\beta}{F} (F + |\beta|) b_k a_j a_{\pi k} + \frac{\beta}{\gamma}(2F + \gamma)a_k a_j b_{\pi k} + 2\gamma a_k b_{\pi k} \right\} \\
+ \frac{1}{3F\gamma} \left\{ \frac{1}{2|\beta|} a_k b_j b_{\pi k} + \frac{|\beta|^2 - F}{\gamma |\beta|} b_k b_j a_{\pi k} + \frac{1}{2\gamma} b_k a_j b_{\pi k} \right\}.
\]

Also, the vertical coefficients of Chern- Finsler connections are defined as

\[
C_{jik} = g^{m j} \frac{\partial g_{mj}}{\partial \eta^k} = g^{m j} C_{j m k}.
\]
The coefficients of Chern-Finsler connection, complex Cartan tensors and the vertical coefficients of Chern-Finsler connections of a complex Finsler space $(M, F)$ with $(\gamma, |\beta|)$-metric are given by (4.3), (4.5) and (4.7) respectively.

Thus, we have

\[ R^C_{jk} = -\frac{3\gamma^2}{4F|\beta|} \left\{ \frac{2F}{\gamma^2} a^g + A\eta^i + Bb^i\overline{b}^j + C(\beta b^i\overline{\eta}^j + \overline{\beta} b^j\overline{\eta}^i) \right\} \delta_{\eta} \]

\[ \times \left\{ \frac{\beta}{\gamma^2} \frac{\partial a^g}{\partial z^k} \eta^i \overline{\eta}^j + \beta \frac{\partial b^i}{\partial z^k} \eta^j + \overline{\beta} \frac{\partial b^j}{\partial z^k} \eta^i \right\} \left( \frac{4a^g}{3\gamma^2} + \frac{\beta b^i}{|\beta|} \right) \]

\[ + F \left( \frac{b^g}{2|\beta|} \frac{\partial b^i}{\partial z^k} \eta^j + \frac{\beta}{|\beta|} \frac{\partial b^j}{\partial z^k} \eta^i \right) \eta^h. \]

Thus, we have

Theorem 4.2. The holomorphic curvature of a complex Finsler space $(M, F)$ with $(\gamma, |\beta|)$-metric is given by (2.4) together with (4.8).

5 Euler-Lagrange equations

The Lagrangian $L$ given by (3.3), $L = F^2 = (\gamma + |\beta|)^2$ depends on the parameter $t \in R$ by means of $z^g(t)$ and $\eta^h(t)$ and their conjugates. Differentiating (3.3) with
respect to $t$, we have

$$
\frac{dL}{dt} = \frac{\partial L}{\partial z^i} \eta^i + \frac{\partial L}{\partial \eta^i} \frac{d\eta^i}{dt} + \frac{\partial L}{\partial \eta} \frac{d\eta}{dt}.
$$

Since, $L$ is homogeneous of degree one in $\eta^i$, $\frac{\partial L}{\partial \eta^i} \eta^i = L$, on differentiating with respect to $t$, we get

$$
\frac{dL}{dt} = \frac{d}{dt} \left( \frac{\partial L}{\partial \eta^i} \right) \eta^i + \frac{\partial L}{\partial \eta} \frac{d\eta}{dt}.
$$

But $E_i(L) = 0$, implies $\frac{d}{dt} \left( \frac{\partial L}{\partial \eta^i} \right) = \frac{\partial L}{\partial \eta} \frac{d\eta}{dt}$ along the extremal curve $c(t)$ on $T'M$. Therefore (5.2) gives

$$
\frac{dL}{dt} = \frac{\partial L}{\partial \eta^i} \eta^i + \frac{\partial L}{\partial \eta} \frac{d\eta}{dt}.
$$

By conjugation (as $t$ and $L$ are real valued functions), we have

$$
\frac{dL}{dt} = \frac{\partial L}{\partial \eta^i} \bar{\eta}^i + \frac{\partial L}{\partial \eta} \frac{d\eta}{dt}.
$$

Adding (5.3) and (5.4) and using in (5.1), we conclude $\frac{dL}{dt} = 0$, which further implies $\frac{dF}{dt}$. This leads to

**Theorem 5.1.** For the complex Finsler space $(M, F)$ with $(\gamma, |\beta|)$-metric, $\frac{dL}{dt} = 0 = \frac{dF}{dt}$ along an extremal curve $c(t)$ on $T'M$.

Now, we derive the Euler-Lagrange equations for $L = F^2 = (\gamma + |\beta|)^2$.

Differentiating $L$ with respect to $z^i$ and $\eta^i$ respectively, we have

$$
\frac{\partial L}{\partial z^i} = L_i \frac{\partial \gamma}{\partial z^i} + L_{i|\beta|} \frac{\partial |\beta|}{\partial z^i} = 2F \left( \frac{\partial \gamma}{\partial z^i} + \frac{\partial |\beta|}{\partial z^i} \right)
$$

$$
\frac{\partial L}{\partial \eta^i} = L_i \frac{\partial \gamma}{\partial \eta^i} + L_{i|\beta|} \frac{\partial |\beta|}{\partial \eta^i} = 2F \left( \frac{\partial \gamma}{\partial \eta^i} + \frac{\partial |\beta|}{\partial \eta^i} \right).
$$

Further differentiation of (5.6) with respect to $t$ implies

$$
\frac{d}{dt} \left( \frac{\partial L}{\partial \eta^i} \right) = 2 \frac{d}{dt} \left( \frac{\partial \gamma}{\partial \eta^i} + \frac{\partial |\beta|}{\partial \eta^i} \right)
$$

$$
+ 2F \left\{ \frac{d}{dt} \left( \frac{\partial \gamma}{\partial \eta^i} \right) + \frac{d}{dt} \left( \frac{\partial |\beta|}{\partial \eta^i} \right) \right\}.
$$

In view of theorem 5.1, (5.7) gives

$$
\frac{d}{dt} \left( \frac{\partial L}{\partial \eta^i} \right) = 2F \left\{ \frac{d}{dt} \left( \frac{\partial \gamma}{\partial \eta^i} \right) + \frac{d}{dt} \left( \frac{\partial |\beta|}{\partial \eta^i} \right) \right\}.
$$
Complex Finsler spaces with \((\gamma, |\beta|)\)-metric

Now,

\[
E_i(\gamma) = \frac{1}{3\gamma^2} E_i(\gamma^3) + \frac{1}{\gamma^2} \frac{d\gamma^2}{dt} \frac{\partial \gamma}{\partial \eta^i},
\]

\[
E_i(|\beta|) = \frac{1}{2|\beta|} E_i(|\beta|^2) + \frac{1}{|\beta|} \frac{d|\beta|}{dt} \frac{\partial |\beta|}{\partial \eta^i}
\]

respectively. For the Lagrangian \(L = F^2 = (\gamma + |\beta|)^2\), (2.6) gives

\[
E_i (L) \equiv 2F \{E_i(\gamma) + E_i(|\beta|)\} = 0.
\]

Substituting values of \(E_i(\gamma)\) and \(E_i(|\beta|)\) from (5.9) and (5.10) in (5.11), we obtain

\[
2|\beta| E_i(\gamma^3) + 3\gamma^2 E_i(|\beta|^2) + 6|\beta| \frac{d\gamma^2}{dt} \frac{\partial \gamma}{\partial \eta^i} + 6\gamma^2 \frac{d|\beta|}{dt} \frac{\partial |\beta|}{\partial \eta^i} = 0.
\]

Thus, we have

Theorem 5.2. The Euler-Lagrange equations of the complex Finsler space \((M, F)\) with \((\gamma, |\beta|)\)-metric are given by (5.12).

References


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