On the classification of some totally umbilical submanifolds of C-manifolds

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Abstract. In this paper, we give some classifications of submanifolds of C -manifolds, particularly totally umbilical slant submanifolds and totally umbilical CR-submanifolds of globally framed type. Firstly, we show that a totally umbilical slant submanifold M of a C-manifold $\overline{M}$ is either an anti-invariant submanifold or an s-dimensional submanifold. Then we prove that every totally umbilical proper slant submanifold of a C -manifold is totally geodesic. Last, we give the characterization of a totally umbilical globally framed CR-submanifold M of a C-manifold $\overline{M}$.

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1 Introduction

The submanifold theory is an important research topic in Differential Geometry since the famous Nash's embedding theorem. In time, several authors introduced some different classes of submanifolds. In this work, we particularly focus on the slant submanifolds and CR-submanifolds.

The notion of slant submanifolds in the complex spaces was initiated by Chen ([7], [8]). A slant submanifold is a natural generalization of both holomorphic and totally real submanifolds. Since then, many researchers focused on this area and proved the existence of these submanifolds in different known spaces ([5], [14], [19], [17]). Recently, the first author defined the slant submanifolds of globally framed metric f-manifolds in his Ph.D. thesis [1].

On the other hand, in 1978, Bejancu introduced the notion of CR -submanifolds of a Kähler manifold as a natural generalization of both the holomorphic and totally real submanifolds of a Kähler manifold. [2]. Then, a lot of researchers studied on the CR-submanifolds using different structures ([3], [9], [10], [18]).

In the present paper, firstly, we consider slant submanifolds of C -submanifolds and we give some classifications of these type submanifolds under some certain conditions. Secondly, we prove that a globally CR-submanifold of a C-submanifold is totally geodesic or the anti-invariant distribution $D^{\perp}$ is s-dimensional or a mean curvature

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vector $H \in \Gamma(\mu)$, where μ denotes the orthogonal complementary distribution of $\varphi D^{\perp}$ in $T^{\perp}M$. Finally, we give some non-trivial examples.

2 Preliminaries

Let $\overline{M}$ be a $(2n+s)$-dimensional manifold and φ is a non-null $(1,\ 1)$ tensor field on $\overline{M}$. If φ satisfies

$$\varphi^3 + \varphi = 0, \tag{2.1}$$

then φ is called an f-structure and M is called a f-manifold [20]. If $rank\varphi = 2n$, namely $s = 0$, φ is called an almost complex structure, and if $rank\varphi = 2n+1$, namely $s = 1$, then φ reduces an almost contact structure [12], where $rank\varphi$ is always a constant [16].

On an f-manifold $\overline{M}$, the operators P_1 and P_2 are defined by

$$P_1 = -\varphi^2, \quad P_2 = \varphi^2 + I, \tag{2.2}$$

which satisfy

$$\begin{array}{ccc} P_1 + P_2 = I, & P_1^2 = P_1, & P_2^2 = P_2, \\ \varphi P_1 = P_1\varphi = \varphi, & P_2\varphi = \varphi P_2 = 0. & \end{array} \tag{2.3}$$

These properties show that P_1 and P_2 are the complement projection operators. So there are D and $D^{\perp}$ distributions with respect to P_1 and P_2 operators, respectively [21]. Also, we have $\dim(D) = 2n$ and $\dim\left(D^{\perp}\right) = s$.

Suppose that $\overline{M}$ is a $(2n+s)$-dimensional f-manifold and φ is a $(1,\ 1)$ tensor field, ξ_k is a vector field and η^i is 1-form for each $1 \leq k \leq s$ on $\overline{M}$, respectively. If $\left(\varphi,\ \xi_k,\ \eta^k\right)$ satisfies

$$\eta^j(\xi_k) = \delta_k^j, \tag{2.4}$$

and

$$\varphi^2 = -I + \sum_{k=1}^{s} \eta^k \otimes \xi_k, \tag{2.5}$$

then $\left(\varphi,\ \xi_k,\ \eta^k\right)$ is called a globally framed f-structure or simply framed f-structure and M is called a globally framed f-manifold or simply framed f-manifold [15]. For a framed f-manifold $\overline{M}$, the following properties are satisfied [15].

$$\varphi\xi_k = 0, \tag{2.6}$$

$$\eta^k \circ \varphi = 0. \tag{2.7}$$

On a framed f-manifold M, if there exists a Riemannian metric satisfying

$$\eta^k(X) = g(X,\ \xi_k), \tag{2.8}$$

and

$$g(\varphi X,\ \varphi Y) = g(X,\ Y) - \sum_{k=1}^{s} \eta^k(X)\,\eta^k(Y)\,, \tag{2.9}$$

for all vector fields $X,\ Y$ on $\overline{M}$, then $\overline{M}$ is called a framed metric f-manifold [11]. On a framed metric f -manifold, the fundamental 2-form Φ can be defined by

$$\Phi(X,\ Y) = g(X,\ \varphi Y)\,, \tag{2.10}$$

for all vector fields $X,\ Y \in \chi\left(\overline{M}\right)$ [11]. For a framed metric f-manifold, if

$$N_\varphi + 2\sum_{k=1}^{s} d\eta^k \otimes \xi_k, \tag{2.11}$$

holds, then M is called a normal framed metric f-manifold, where N_φ denotes the Nijenhuis torsion tensor of φ [13].

If a globally framed metric f-manifold satisfies $d\Phi = 0$ and $d\eta^k = 0$ for each $1 \leq k \leq s$, then it is called a C-manifold. On a C-manifold M,

$$\left(\overline{\nabla}_X \varphi\right) Y = 0, \tag{2.12}$$

holds for each vector fields $X,\ Y \in \chi(M)$ [4].

Now, let M be a submanifold immersed in $\overline{M}$. We denote the induced metric on M by g. Let TM be the Lie algebra of vector fields in M and $T^{\perp}M$ the set of all vector fields normal to M. Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and $\overline{M}$, respectively. Then, the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,\ Y) \tag{2.13}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.14}$$

respectively, for any $X,\ Y \in TM$ and any $V \in T^{\perp}M$. Here, $\nabla^{\perp}$ is the normal connection in the normal bundle, h is the second fundamental form of M and A_V is the Weingarten endomorphism associated with V [6]. On the other hand, there is a relation between A_V and h in the following [6]

$$g(A_V X,\ Y) = g(h(X,\ Y)\,,\ V)\,. \tag{2.15}$$

The mean curvature vector H is defined by $H = \dfrac{1}{m} trace h$, where m is the dimension of M. M is said to be minimal, totally geodesic or totally umbilical if H vanishes identically, $h = 0$ and

$$h(X,\ Y) = g(X,\ Y)\,H, \tag{2.16}$$

respectively [6]. Furthermore, the second fundamental form h satisfies [6]

$$\left(\overline{\nabla}_X h\right)(Y,\ Z) = \nabla^{\perp}_X h(Y,\ Z) - h(\nabla_X Y,\ Z) - h(Y,\ \nabla_X Z)\,. \tag{2.17}$$

3 Submanifolds of globally framed metric f-manifolds

In this section, we recall some basic properties of submanifolds of a globally framed metric f-manifold. We give the following definition, corollary and propositions from [1].

Definition 3.1. Let $\overline{M}$ be a globally framed metric f-manifold and M a submanifold of $\overline{M}$. For all $X \in \Gamma(TM)$, we can write

$$\varphi X = TX + NX, \tag{3.1}$$

where TX and NX are called the tangent and normal components of φX, respectively. Similarly, for each $V \in \Gamma\left(T^{\perp}M\right)$, we have

$$\varphi V = tV + nV, \tag{3.2}$$

where tV is the tangent component and nV is the normal component of φV.

Corollary 3.1. *Let $\overline{M}$ be a globally framed metric f-manifold and M a submanifold of $\overline{M}$. Then the following identities hold*

$$T^2 = -I + \sum_{k=1}^{s} \eta^k \otimes \xi_k - tN, \quad NT + nN = 0, \tag{3.3}$$

$$Tt + tn = 0, \quad Nt + n^2 = -I, \tag{3.4}$$

where I denotes the identity transformation.

Proposition 3.2. *Let $\overline{M}$ be a globally framed metric f-manifold and M a submanifold of $\overline{M}$. Then, T and n are the skew-symmetric tensor fields.*

Proposition 3.3. *Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. Then, for $X \in \Gamma(TM)$ and $V \in \Gamma\left(T^{\perp}M\right)$, we have*

$$g(NX,\ V) = -g(X,\ tV), \tag{3.5}$$

which gives the relation between N and t.

Proposition 3.4. *Let $\overline{M}$ be a globally framed metric f-manifold and M is a submanifold of $\overline{M}$. Then, for $X,\ Y \in \Gamma(TM)$ and $V \in \Gamma\left(T^{\perp}M\right)$, the following identities hold:*

$$\left(\overline{\nabla}_X \varphi\right) Y = \overline{\nabla}_X \varphi Y - \varphi \overline{\nabla}_X Y \tag{3.6}$$

$$\left(\overline{\nabla}_X T\right) Y = \overline{\nabla}_X TY - T\overline{\nabla}_X Y, \tag{3.7}$$

$$\left(\overline{\nabla}_X N\right) Y = \overline{\nabla}^{\perp}_X NY - N\overline{\nabla}_X Y, \tag{3.8}$$

$$\left(\overline{\nabla}_X t\right) V = \overline{\nabla}_X tV - t\overline{\nabla}^{\perp}_X V, \tag{3.9}$$

$$\left(\overline{\nabla}_X n\right) V = \overline{\nabla}^{\perp}_X nV - n\overline{\nabla}^{\perp}_X V, \tag{3.10}$$

$$\left(\overline{\nabla}_X T\right) Y + \left(\overline{\nabla}_Y T\right) X = A_{NX}Y + A_{NY}X + 2th\left(X,\ Y\right), \tag{3.11}$$

$$\left(\overline{\nabla}_X N\right) Y + \left(\overline{\nabla}_Y N\right) X = 2nh\left(X,\ Y\right) - h\left(X,\ TY\right) - h\left(Y,\ TX\right), \tag{3.12}$$

$$\left(\overline{\nabla}_X t\right) V = A_{nV}X - TA_V X, \tag{3.13}$$

$$\left(\overline{\nabla}_X n\right) V = -h\left(tV,\ X\right) - NA_V X, \tag{3.14}$$

where h is the second fundamental form, ∇ is the Levi-Civita connection and A_V is the shape operator corresponding to the normal vector field V.

4 Geometry of slant submanifolds of C-manifolds

Firstly, we recall the following two definitions, theorem and corollary from [1].

Definition 4.1. Let $\overline{M}$ be a globally framed metric f-manifold and M a submanifold of $\overline{M}$. Then, the tangent bundle TM of M can be decomposed as

$$TM = \sum_{k=1}^{s} D_\theta \oplus \xi_k, \tag{4.1}$$

where for each $1 \leq k \leq s$ the ξ_k is the distribution spanned by the structure vector fields ξ_k and D_θ is the complementary of distribution ξ_k in TM, which isknown as the slant distribution on M.

Theorem 4.1. *Let $\overline{M}$ be a globally framed metric f-manifold and M a submanifold of $\overline{M}$. Then, M is a slant submanifold if and only if there exists a constant $\mu \in [0,\ 1]$ such that*

$$T^2 = -\mu\left(I - \sum_{k=1}^{s} \eta^k \otimes \xi_k\right). \tag{4.2}$$

Moreover, if θ is the slant angle of M, then $\mu = \cos^2\theta$.

Corollary 4.2. *Let M be a slant submanifold of a globally framed metric f-manifold $\overline{M}$ with a slant angle θ. Then for any vector fields $X,\ Y \in \Gamma\left(TM\right)$, we find*

$$g\left(TX,\ TY\right) = \cos^2\theta\left\{g\left(X,\ Y\right) - \sum_{k=1}^{s}\eta^k\left(X\right)\eta^k\left(Y\right)\right\} \tag{4.3}$$

and

$$g\left(NX,\ NY\right) = \sin^2\theta\left\{g\left(X,\ Y\right) - \sum_{k=1}^{s}\eta^k\left(X\right)\eta^k\left(Y\right)\right\}. \tag{4.4}$$

Definition 4.2. Suppose that M is a submanifold of a globally framed metric f-manifold $\overline{M}$ and tangent to the structure vector fields ξ_k for each $1 \leq k \leq s$. For each nonzero vector X tangent to M at p, we denote by $\theta(X)$ with $0 \leq \theta(X) \leq \frac{\pi}{2}$, the angle between φX and T_pM, known as the Wirtinger angle of X. If the $\theta(X)$ is constant, that is, independent of the choice of $p \in M$ and $X \in T_pM - \{\xi_k\}$, for each $1 \leq k \leq s$, then M is said to be a slant submanifold and the constant angle θ is called a slant angle of the slant submanifold.

Here, if $\theta = 0$, M is an invariant submanifold and if $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold. A slant submanifold is proper slant if it is neither an invariant nor an anti-invariant submanifold.

Now, we give the main results for this section. Firstly, we prove the following.

Theorem 4.3. *Let M be a totally umbilical slant submanifold of a C-manifold $\overline{M}$. Then at least one of the following statements holds.*

i) M is an anti-invariant submanifold.

iii) M is an s-dimensional submanifold.

iii) If M is a proper slant submanifold, then $H \in \Gamma(\mu)$, where H is the mean curvature vector of the submanifold of M and μ denotes the orthogonal complementary distribution of $\varphi D^{\perp}$ in $T^{\perp}M$.

Proof. Let M be a totally umbilical slant submanifold of a C-manifold $\overline{M}$. Then we have

$$h(TX, TY) = g(TX,\ TY)\,H, \tag{4.5}$$

for any $X,\ Y \in \Gamma(TM)$. From (2.13) and (4.3), we obtain that

$$\overline{\nabla}_{TX}TX - \nabla_{TX}TX = \cos^2\theta\left\{g(X,\ X) + \sum_{k=1}^{s}\eta^k(X)\,\eta^k(X)\right\}H. \tag{4.6}$$

Using (3.1) and since $\overline{M}$ is a C-manifold, we deduce that

$$\varphi\overline{\nabla}_{TX}X - \overline{\nabla}_{TX}NX - \nabla_{TX}TX = \cos^2\theta\left\{\|X\|^2 + \sum_{k=1}^{s}\left[\eta^k(X)\right]^2\right\}H. \tag{4.7}$$

On the other hand, using (2.13) and (2.14) then we derive

$$\begin{aligned}(4.8)\qquad & \varphi\nabla_{TX}X + \varphi h(X,\ TX) + A_{NX}TX - \nabla^{\perp}_{TX}NX - \nabla_{TX}TX \\ &= \cos^2\theta\left\{\|X\|^2 + \sum_{k=1}^{s}\left[\eta^k(X)\right]^2\right\}H.\end{aligned}$$

Thus by (3.1) and (2.16), we get

$$\begin{aligned}(4.9)\qquad & T\nabla_{TX}X + N\nabla_{TX}X + g(TX,\ X)\,\varphi H + A_{NX}TX - \nabla^{\perp}_{TX}NX - \nabla_{TX}TX \\ &= \cos^2\theta\left\{\|X\|^2 + \sum_{k=1}^{s}\left[\eta^k(X)\right]^2\right\}H.\end{aligned}$$

Considering the normal components of the last equation, we have

$$N\nabla_{TX}X - \nabla^{\perp}_{TX}NX = \cos^2\theta\left\{\|X\|^2 + \sum_{k=1}^{s}\left[\eta^k(X)\right]^2\right\}H. \tag{4.10}$$

Moreover, using (4.4), we deduce

$$g(NX,\ NX) = \sin^2\theta\left\{g(X,\ X) - \sum_{k=1}^{s}\eta^k(X)\,\eta^k(X)\right\}, \tag{4.11}$$

for any vector field $X \in \Gamma(TM)$. Taking the covariant derivative of the above equation with respect to TX, we conclude

$$\begin{aligned} 2g\left(\overline{\nabla}_{TX}NX,\ NX\right) &= 2\sin^2\theta g\left(\overline{\nabla}_{TX}X,\ X\right) \\ &\quad -2\sin^2\theta\sum_{k=1}^{s}\left\{\eta^k(X)\,g\left(\overline{\nabla}_{TX}X,\ \xi_k\right) + \eta^k(X)\,g\left(X,\ \overline{\nabla}_{TX}\xi_k\right)\right\}. \end{aligned} \tag{4.12}$$

In view of the property of metric connection $\overline{\nabla}$, since the last two terms of the right-hand side are cancelling each other, then we get

$$g\left(\overline{\nabla}_{TX}NX,\ NX\right) = 2\sin^2\theta g\left(\overline{\nabla}_{TX}X,\ X\right). \tag{4.13}$$

Now, by using (2.13) and (2.14), we obtain

$$g\left(\nabla^{\perp}_{TX}NX,\ NX\right) = 2\sin^2\theta g\left(\nabla_{TX}X,\ X\right). \tag{4.14}$$

Taking the inner product of (4.10) with NX, then we derive

$$g(N\nabla_{TX}X,\ NX) - g\left(\nabla^{\perp}_{TX}NX,\ NX\right) = \cos^2\theta\left\{\|X\|^2 + \sum_{k=1}^{s}\left[\eta^k(X)\right]^2\right\}g(H,\ NX) \tag{4.15}$$

for any vector field $X \in \Gamma(TM)$. Then from (4.4) and (4.14), we get

$$-\sin^2\theta\sum_{k=1}^{s}\eta^k(X)\,g(\nabla_{TX}X,\ \xi_k) = \cos^2\theta\left\{\|X\|^2 + \sum_{k=1}^{s}\left[\eta^k(X)\right]^2\right\}g(H,\ NX \tag{4.16}$$

From (2.13), we get

$$-\sin^2\theta\sum_{k=1}^{s}\eta^k(X)\,g\left(\overline{\nabla}_{TX}X,\ \xi_k\right) = \cos^2\theta\left\{\|X\|^2 + \sum_{k=1}^{s}\left[\eta^k(X)\right]^2\right\}g(H,\ NX). \tag{4.17}$$

The fact that $\overline{\nabla}$ is the metric connection, then the last equation can be written as

$$-\sin^2\theta\sum_{k=1}^{s}\eta^k(X)\,g\left(X,\ \overline{\nabla}_{TX}\xi_k\right) = \cos^2\theta\left\{\|X\|^2 + \sum_{k=1}^{s}\left[\eta^k(X)\right]^2\right\}g(H,\ NX). \tag{4.18}$$

For each $1 \leq k \leq s$, since $\overline{M}$ is a C-manifold. thus we have $\overline{\nabla}_{TX}\xi_k = 0$ and using this fact, the left hand side of the above equation vanishes identically, then we get

$$\cos^2\theta \left\{ \|X\|^2 + \sum_{k=1}^{s} \left[\eta^k(X)\right]^2 \right\} g(H,\ NX) = 0. \tag{4.19}$$

So, from (4.19), we deduce that either $\theta = \frac{\pi}{2}$ or $X = \xi_k$ or $H \in \Gamma(\mu)$, where μ is the invariant normal subbundle orthogonal to NTM. Thus, the proof is completed. □

Theorem 4.4. *Every totally umbilical proper slant submanifold M of a C-manifold $\overline{M}$ is totally geodesic, if $\nabla^{\perp}_X H \in \Gamma(\mu)$, for $X \in \Gamma(TM)$.*

Proof. By the fact that $\overline{M}$ is a C-manifold, then we get

$$\overline{\nabla}_X \varphi Y = \varphi \overline{\nabla}_X Y, \tag{4.20}$$

for any $X,\ Y \in \Gamma\left(\overline{TM}\right)$. Using (4.20) and from (2.13) and (3.1), we obtain

$$\overline{\nabla}_X TY + \overline{\nabla}_X NY = T\nabla_X Y + N\nabla_X Y + \varphi h(X,\ Y), \tag{4.21}$$

for any $X,\ Y \in \Gamma(TM)$. Then taking in account of (2.13), (2.14) and (2.16), we have

$$\nabla_X TY + h(X,\ TY) - A_{NY}X + \nabla^{\perp}_X NY = T\nabla_X Y + N\nabla_X Y + g(X,\ Y)\varphi H. \tag{4.22}$$

Taking the inner product of (4.22) with φH and using $H \in \Gamma(\mu)$ and in view of Theorem 2, we conclude

$$g(h(X,\ TY),\ \varphi H) + g\left(\nabla^{\perp}_X NY,\ \varphi H\right) = g(X,\ Y)\, g(\varphi H,\ \varphi H). \tag{4.23}$$

Then by using (2.9) and (2.16), we get

$$g(X,\ TY)\, g(H,\ \varphi H) + g\left(\nabla^{\perp}_X NY,\ \varphi H\right) = g(X,\ Y)\|H\|^2, \tag{4.24}$$

which means

$$g\left(\nabla^{\perp}_X NY,\ \varphi H\right) = g(X,\ Y)\|H\|^2. \tag{4.25}$$

Now, considering

$$\overline{\nabla}_X \varphi H = \varphi \overline{\nabla}_X H, \tag{4.26}$$

for any $X \in \Gamma(TM)$. From (2.14), we have

$$-A_{\varphi H}X + \nabla^{\perp}_X \varphi H = \varphi\left(-A_H X + \nabla^{\perp}_X H\right). \tag{4.27}$$

Thus, by using (3.1) and (3.2), we obtain

$$-A_{\varphi H}X + \nabla^{\perp}_X \varphi H = -TA_H X - NA_H X + t\nabla^{\perp}_X H + n\nabla^{\perp}_X H. \tag{4.28}$$

Taking the inner product with NY, for any $Y \in \Gamma(TM)$, then we derive

$$g\left(\nabla^{\perp}_X \varphi H,\ NY\right) = -g(NA_H X,\ NY) + g\left(n\nabla^{\perp}_X H,\ NY\right). \tag{4.29}$$

As $n\nabla^{\perp}_{X}H \in \Gamma(\mu)$, then from (4.4) the above equation takes the form

$$g\left(\nabla^{\perp}_{X}\varphi H,\ NY\right) = -\sin^{2}\theta\left\{g\left(A_{H}X,\ Y\right) - \sum_{k=1}^{s}\eta^{k}\left(A_{H}X\right)\eta^{k}\left(Y\right)\right\}. \tag{4.30}$$

Using (2.14), (2.15) and (2.16), we deduce

$$g\left(\overline{\nabla}^{\perp}_{X}\varphi H,\ NY\right) = -\sin^{2}\theta\left\{g\left(X,\ Y\right) - \sum_{k=1}^{s}\eta^{k}\left(X\right)\eta^{k}\left(Y\right)\right\}\left\|H\right\|^{2}. \tag{4.31}$$

The last equation can be written as

$$g\left(\overline{\nabla}^{\perp}_{X}NY,\ \varphi H\right) = -\sin^{2}\theta\left\{g\left(X,\ Y\right) - \sum_{k=1}^{s}\eta^{k}\left(X\right)\eta^{k}\left(Y\right)\right\}\left\|H\right\|^{2}. \tag{4.32}$$

Now using $H \in \Gamma(\mu)$, then by (2.14), we get

$$g\left(\nabla^{\perp}_{X}NY,\ \varphi H\right) = -\sin^{2}\theta\left\{g\left(X,\ Y\right) - \sum_{k=1}^{s}\eta^{k}\left(X\right)\eta^{k}\left(Y\right)\right\}\left\|H\right\|^{2}. \tag{4.33}$$

Finally, by using (4.25) and (4.33), we conclude

$$\left\{\cos^{2}\theta g\left(X,\ Y\right) + \sin^{2}\theta\sum_{k=1}^{s}\eta^{k}\left(X\right)\eta^{k}\left(Y\right)\right\}\left\|H\right\|^{2} = 0, \tag{4.34}$$

which implies that either $H = 0$ or $\theta = \tan^{-1}\left(\sqrt{g\left(X,\ Y\right)/\sum_{k=1}^{s}\eta^{k}\left(X\right)\eta^{k}\left(Y\right)}\right)$. This is not possible, because the slant angle $\theta \in (0,\ \pi/2)$. Hence, M is totally geodesic in $\overline{M}$. □

5 Globally framed CR-submanifolds of C-manifolds

In this section, we focus on the globally framed CR-submanifolds. Firstly, we recall the definition and then we give some basic properties.

Definition 5.1. Let $\overline{M}$ be a globally framed metric f-manifold and M be an isometric immersed submanifold of $\overline{M}$. If there is a differentiable distribution $D : p \to D_p \subseteq T_pM$ on M satisfying the following conditions, then M is called a globally framed CR-submanifold of $\overline{M}$[1].

1) For each $1 \leq k \leq s, \xi_k \in D$.

2) D is invariant under φ tensor field, i.e., for any point $p \in M \varphi(D_p) \subset T_pM$.

3) The orthogonal distribution $D^{\perp} : p \to D_p^{\perp} \subseteq T_pM$ satisfies $\varphi\left(D_p^{\perp}\right) \subseteq T_p^{\perp}M$ for each point $p \in M$.

If neither $D = \{0\}$ nor $D^{\perp} = \{0\}$, then M is called a proper globally framed CR-submanifold.

Lemma 5.1. *Let $\overline{M}$ be a globally framed metric f-manifold and M be a globally framed CR-submanifold of $\overline{M}$. Then we have*

$$\varphi\xi_k = T\xi_k + N\xi_k = 0 \tag{5.1}$$

and

$$T\xi_k = 0, \quad N\xi_k = 0 \tag{5.2}$$

for each $1 \leq k \leq s$ and $\xi_k \in \Gamma(D) \subseteq \Gamma(TM)$ [1].

Definition 5.2. Let $\overline{M}$ be a globally framed metric f-manifold and M be a globally framed CR-submanifold of $\overline{M}$. Also, let μ be orthogonal distribution of $\varphi\left(D^{\perp}\right)$ in $T^{\perp}M$ and then we have

$$T^{\perp}M = \varphi\left(D^{\perp}\right) \otimes \mu, \quad \mu \perp \varphi\left(D^{\perp}\right). \tag{5.3}$$

The dimension of $\left(\varphi\left(D^{\perp}\right)\right)^{\perp}$ is even since it is invariant subbundle [1].

Now, let $\overline{M}$ be a globally framed metric f-manifold and M a proper globally framed CR-submanifold of $\overline{M}$. Then, for any $X \in \Gamma(TM)$, we have

$$X = P_1X + P_2X + \sum_{k=1}^{s} \eta^k(X)\,\xi_k, \tag{5.4}$$

where P_1 and P_2 are the orthogonal projections from TM to D and $D^{\perp}$, respectively. For a globally framed CR-submanifold, from (3.1) and (5.4), we get

$$TX = \varphi P_1X, \qquad NX = \varphi P_2X.$$

Let M be a globally framed CR-submanifold of a C-manifold $\overline{M}$, then for any $X,\ Y \in \Gamma\left(D^{\perp}\right)$ and $Z \in \Gamma(TM)$, we deduce

$$g(A_{\varphi Y}X,\ Z) = g(h(X,\ Z),\ \varphi Y).$$

By using (2.13), we get

$$g(A_{\varphi Y}X,\ Z) = g\left(\overline{\nabla}_Z X,\ \varphi Y\right) = -g\left(\varphi\overline{\nabla}_Z X,\ Y\right).$$

From (2.12), we have

$$g(A_{\varphi Y}X,\ Z) = -g\left(\overline{\nabla}_Z \varphi X,\ Y\right).$$

Hence, using (2.14), we obtain

$$g(A_{\varphi Y}X,\ Z) = g(A_{\varphi X}Z,\ Y) = g(h(Y,\ Z),\ \varphi X).$$

Now, from (2.13), we derive

$$g(A_{\varphi Y}X,\ Z) = g\left(\overline{\nabla}_Y Z,\ \varphi X\right) = -g\left(Z,\ \overline{\nabla}_Y \varphi X\right).$$

Then, using (2.14), we have

$$g\left(A_{\varphi Y}X,\ Z\right)=g\left(A_{\varphi X}Y,\ Z\right).$$

Thus, for a globally framed CR-submanifold of a C-manifold we infer

$$A_{\varphi Y}X=A_{\varphi X}Y, \tag{5.5}$$

for all vector fields $X,\ Y\in\Gamma\left(D^{\perp}\right)$. On the other hand, we obtain

$$\begin{aligned} g\left(\left[X,\ Y\right],\ \varphi Z\right) &= g\left(\overline{\nabla}_X Y-\overline{\nabla}_Y X,\ \varphi Z\right) \\ &= g\left(\varphi\overline{\nabla}_Y X-\varphi\overline{\nabla}_X Y,\ Z\right), \end{aligned}$$

for any vector field $Z\in\Gamma\left(D\oplus\langle\xi_k\rangle\right)$ $1\leq k\leq s$ and for all vector fields $X,\ Y\in\Gamma\left(D^{\perp}\right)$. Hence, from (2.12) and (3.6), we deduce

$$\begin{aligned} g\left(\left[X,\ Y\right],\ \varphi Z\right) &= g\left(\overline{\nabla}_Y\varphi X-\overline{\nabla}_X\varphi Y,\ Z\right) \\ &= g\left(A_{\varphi Y}X-A_{\varphi X}Y,\ Z\right). \end{aligned}$$

Thus, using (5.5), we obtain $g\left(\left[X,\ Y\right],\ \varphi Z\right)=0$ which means $\left[X,\ Y\right]\in\Gamma\left(D^{\perp}\right)$ for any $X,\ Y\in\Gamma\left(D^{\perp}\right)$. This implies that $D^{\perp}$ is integrable.

Hence, we have the following theorem.

Theorem 5.2. *Let M be a globally framed CR-submanifold of a C-manifold $\overline{M}$. Then, the distribution $D^{\perp}$ is completely integrable.*

Now, for all vector fields $Z,\ W\in\Gamma\left(D\oplus\langle\xi_k\rangle\right)$ $1\leq k\leq s$, we get

$$h\left(Z,\ TW\right)+\nabla_Z TW=\overline{\nabla}_Z TW=\overline{\nabla}_Z\varphi W.$$

Since $\overline{M}$ is a C-manifold, then using (2.12) and (3.6), we conclude

$$h\left(Z,\ TW\right)+\nabla_Z TW=\varphi\overline{\nabla}_Z W.$$

Now, in view of (2.13), (3.1) and (3.2), we have

$$h\left(Z,\ TW\right)+\nabla_Z TW=T\nabla_Z W+N\nabla_Z W+th\left(Z,\ W\right)+nh\left(Z,\ W\right).$$

Considering the normal components of the last equation, we derive

$$N\nabla_Z W=h\left(Z,\ TW\right)-nh\left(Z,\ W\right). \tag{5.6}$$

In a similar way, we get

$$N\nabla_W Z=h\left(W,\ TZ\right)-nh\left(Z,\ W\right) \tag{5.7}$$

Hence from (5.6) and (5.7), we deduce

$$N\left[Z,\ W\right]=h\left(Z,\ TW\right)-h\left(W,\ TZ\right). \tag{5.8}$$

Thus, we give the following theorem.

Theorem 5.3. *Let M be a globally framed CR-submanifold of a C-manifold $\overline{M}$. Then for each $1 \leq k \leq s$, the distribution $D \oplus \langle \xi_k \rangle$ is integrable if and only if $h(Z,\ TW) = h(W,\ TZ)$ for any vector fields $Z,\ W \in \Gamma(D \oplus \langle \xi_k \rangle)$.*

Now, we give main theorem of this section in the following:

Theorem 5.4. *Let M be a totally umbilical globally framed CR-submanifold of a C-manifold $\overline{M}$. Then at least one of the following statements is true*
i) M is totally geodesic,
ii) $\dim(D^{\perp}) = s$,
iii) the mean curvature vector $H \in \Gamma(\mu)$.

Proof. As $\overline{M}$ is a C-manifold, we get

$$\overline{\nabla}_X \varphi Y = \varphi \overline{\nabla}_X Y,$$

for all vector fields $X,\ Y \in \Gamma(D^{\perp})$. Since M is totally umbilical, from (2.13) and (2.14), we obtain

$$-g(H,\ \varphi Y)X + \nabla^{\perp}_X \varphi Y = \varphi \nabla_X Y + g(X,\ Y)\varphi H. \tag{5.9}$$

Taking the inner product of (5.9) with $X \in \Gamma(D^{\perp})$, we have

$$g(H,\ \varphi Y)\|X\|^2 = g(X,\ Y)\, g(H,\ \varphi X). \tag{5.10}$$

Interchanging X and Y in (5.10), we derive

$$g(H,\ \varphi X)\|Y\|^2 = g(X,\ Y)\, g(H,\ \varphi Y). \tag{5.11}$$

Then, from (5.10) and (5.11), we conclude

$$g(H,\ \varphi X) = \frac{g(X,\ Y)^2}{\|X\|^2 \|Y\|^2} g(H,\ \varphi X),$$

which implies

$$g(H,\ \varphi X)\left\{1 - \frac{g(X,\ Y)^2}{\|X\|^2 \|Y\|^2}\right\} = 0. \tag{5.12}$$

If at least one of the following

$$i)\ H = 0, \quad ii)\ \ X \| Y, \quad iii)\ H \perp \varphi D^{\perp}.$$

holds, then the last equation has a solution. This completes the proof. □

6 Examples

Example 6.1. Let $(C^4,\ J,\ h)$ be a 4-dimensional Kähler manifold with complex structure J and the Euclidean Hermitian metric h. Then it is a flat manifold of real dimension 8. Hence $\overline{M} = C^4 \times R^s$ is a C-manifold with structure vector fields $\xi_k =$

$\frac{\partial}{\partial t_k}$, the dual 1-forms $\eta^k = dt_k$ for each $1 \le k \le s$ and the metric $g = h + \sum_{k=1}^{s} dt_k^2$. Now, consider $M = R^4 \times S^2$, where S^2 is 2-dimensional unit sphere which is a totally real submanifold of C^4. Then M is a globally framed CR-submanifold of $\overline{M}$ with the invariant distribution $D = R^3$, anti-invariant distribution $D^\perp\ \Gamma\left(S^2\right)$ and the 2-dimensional distribution $\langle \xi_1,\ \xi_2 \rangle = R^2$.

Example 6.2. Let $\overline{M}$ be a Kähler manifold. Then $\overline{M} \times R^s$ is a C-manifold with the usual product structure. Let M be a slant submanifold of a Kähler manifold $\overline{M}$. Then $M \times R^s$ is a slant submanifold of globally framed type of C-manifold $\overline{M} \times R^s$.

References

[1] Y. S. Balkan, *Geometry of nearly C-manifolds*, Ph.D. thesis, Düzce University 2016.

[2] A. Bejancu, *CR-submanifolds of a Kaehler manifold*, Proc. Amer. Math. Soc., 69 (1978), 135-142.

[3] A. Bejancu, *Umbilical CR-submanifolds of a Kaehler manifold*, Rend Mat. J., 13 (1980), 431-466.

[4] D. E. Blair,*Geometry of manifolds with structural group* $U(n) \times O(s)$, Journal of Differential Geometry, 4 (2) (1970), 155-167.

[5] J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández, *Slant submanifolds in Sasakian manifolds*, Glasgow Math. Journal, 42 (1) (2000), 125–138.

[6] B. Y. Chen, *Geometry of submanifolds*, Marcel Dekker. INC, 1973.

[7] B. Y. Chen, *Slant immersions*, Bulletin of the Australian Math. Soc., 41(1) (1990), 135–147.

[8] B. Y. Chen, *Geometry of Slant Submanifolds*, Katholieke Universiteit Leuven, Leuven, Belgium, 1990.

[9] B. Y. Chen, *Totally umbilical submanifolds of Kaehler manifolds*, Arch. Math. J., 36 (1981), 83-91.

[10] S. Deshmukh and S. I. Husain, *Totally umbilical CR-submanifolds of a Kaehler manifold*, Kodai Math. J., 9 (1986), 425-429.

[11] S. I. Goldberg and K. Yano, *On normal globally framed f -manifolds*, Tohoku Mathematical Journal, 22 (3) (1970), 362-370.

[12] S. I. Goldberg and K. Yano, *On normal globally framed f -manifolds*, Illinois J. Math., 15 (3) (1971), 456-474.

[13] S. Ishihara, *Normal structure f satisfying* $f^3 + f = 0$, Kodai Math. Sem. Rep., 18 (1) (1966), 36-47.

[14] A. Lotta, *Slant submanifolds in contact geometry*, Bulletin Math. Soc. Roumanie, 39 (1996), 183–198.

[15] H. Nakagawa, *On framed f-manifolds*, Kodai Math. Sem. Rep., 18 (4) (1966), 293-306.

[16] R. E. Stong, *The rank of f-srtucture*, Kodai Math. Sem. Rep., 29 (1-2) (1977), 207-209.

[17] B. Şahin, *Every totally umbilical proper slant submanifold of a Kahler manifold is totally geodesic*, Results in Math., 54 (1-2) (2009), 167–172.

[18] S. Uddin and C. Özel, *A classification theorem on totally umbilical submanifolds in a cosymplectic manifold*, Hacettepe J. Math. Statistic, 43 (4) (2014), 635-640.

[19] S. Uddin, C. Özel and V. A. Khan, *A Classification of a Totally Umbilical Slant Submanifold of Cosymplectic Manifolds*, Abstract and Appl. Analysis, 2012 Article ID 716967 8 pages doi:10.1155/2012/716967.

[20] K. Yano, *On a structure* f *satisfying* $f^3 + f = 0$, Technical Report No. 12, University of Washington, Washington-USA, 1961.

[21] K. Yano, *On a structure defined by a tensor field of* f *of type* $(1, 1)$ *satisfying* $f^3 + f = 0$, Tensor, 14 (1963), 99-109.

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