Warped product pseudo-slant submanifolds in a locally product Riemannian manifold

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Abstract. Non-existence of warped product semi-slant submanifolds of locally product Riemannian manifolds is proved in [3, 21]. In this paper, we study warped products of slant and anti-invariant submanifolds of a locally product Riemannian manifold and we prove the existence of such kind of warped products by a characterization. Also, we construct examples and derive an inequality for the squared norm of the second fundamental form of such immersions in terms of the warping function.

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1 Introduction

Recently, Sahin studied slant submanifolds of locally product Riemannian manifolds [19] and this idea was extended by Li and Li for semi-slant submanifolds of locally product Riemannian manifolds [16], which are generalizations of the semi-invariant submanifolds studied in [2, 5, 18]. They obtained some characterization results on the underlying submanifolds of locally product Riemannian manifolds.

On the other hand, the warped product submanifolds in a locally product Riemannian manifold were studied in [1, 4, 20]. For a survey on warped product submanifolds we also refer to [13, 14]. Recently, Sahin [21] and Atceken [3] studied warped product semi-slant submanifolds of a locally product Riemannian manifold. They proved that the warped products of the form $M_T \times_f M_\theta$ and $M_\perp \times_f M_\theta$ do not exist in a locally product Riemannian manifold $\tilde{M}$, where M_T, $M_\perp$ and M_θ are invariant, anti-invariant and proper slant submanifolds of $\widetilde{M}$, respectively. They provided some examples on the existence of warped product submanifolds of the form $M_\theta \times_f M_T$ and $M_\theta \times_f M_\perp$.

In this paper, we study the warped product submanifolds of the form $M = M_\theta \times_f M_\perp$ (we call such warped products *pseudo-slant warped products*) of a locally product Riemannian manifold $\tilde{M}$, where M_θ and $M_\perp$ are proper slant and anti-invariant submanifolds of $\tilde{M}$, respectively. We give first some preparatory results for later use, provide two examples of such warped product immersions and obtain a

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characterization for warped products. Also, we establish a relationship between the squared norm of the second fundamental form and the warping function, and consider, furthermore, the equality case of the inequality.

2 Preliminaries

Let $\tilde{M}$ be an n-dimensional Riemannian manifold with a tensor field of type $(1,1)$, such that $F^2 = I$ $(F \neq \pm I)$, where I denotes the identity transformation. Then we say that $\tilde{M}$ is an almost product manifold with almost product structure F. If an almost product manifold $\tilde{M}$ admits a Riemannian metric g such that

$$g(FU, FV) = g(U, V), \quad g(FU, V) = g(U, FV), \tag{2.1}$$

for any vector fields U and V on $\tilde{M}$, then $\tilde{M}$ is called an almost product Riemannian manifold. Let $\tilde{\nabla}$ denote the Levi Cevita connection on $\tilde{M}$ with respect to g. If $(\tilde{\nabla}_U F)V = 0$, for all $U, V \in \Gamma(T\tilde{M})$, where $\Gamma(T\tilde{M})$ denotes the set of all vector fields of $\tilde{M}$, then $(\tilde{M}, g)$ is called a locally product Riemannian manifold with Riemannian metric g [5].

Let M be a submanifold of a locally product Riemannian manifold $\tilde{M}$ with the induced Riemannian metric g. If ∇ and $\nabla^\perp$ are the induced Riemannian connections on the tangent bundle TM and on the normal bundle $T^\perp M$ of M respectively, then the Gauss and Weingarten formulas are:

$$\tilde{\nabla}_U V = \nabla_U V + h(U, V), \;\; \tilde{\nabla}_U N = -A_N U + \nabla^\perp_U N, \tag{2.2}$$

for each $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively, for the immersion of M into $\tilde{M}$. They are related by

$$g(h(U, V), N) = g(A_N U, V). \tag{2.3}$$

Now, for any $X \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, we write

$$(i) \;\; FU = PU + \omega U, \quad (ii) \;\; FN = tN + fN, \tag{2.4}$$

where $PU(tN)$ and $\omega U(fN)$ are the tangential and the normal components of $FU(FN)$, respectively. The covariant derivatives of the endomorphisms F, T and ω are respectively defined as

$$\begin{cases} (\tilde{\nabla}_U F)V = \tilde{\nabla}_U FV - F\tilde{\nabla}_U V, \;\; \forall\, U, V \in \Gamma(T\tilde{M}) \\ (\tilde{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \;\; \forall\, U, V \in \Gamma(TM) \\ (\tilde{\nabla}_U \omega)V = \nabla^\perp_U \omega V - \omega\nabla_U V, \;\; \forall\, U, V \in \Gamma(TM). \end{cases} \tag{2.5}$$

A submanifold M of a locally product Riemannian manifold $\tilde{M}$ is said to be *totally umbilical* and *totally geodesic* respectively, if

$$h(U, V) = g(U, V)H, \;\; \text{and} \;\; h(U, V) = 0,$$

for any $U, V \in \Gamma(TM)$, where H is the mean curvature vector of M defined by $H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i)$, with $\dim M = m$ and $\{e_1, \cdots, e_m\}$ is a local orthonormal frame of vector fields on M. Furthermore, if $H = 0$, then M is *minimal* in $\tilde{M}$.

Let M be a submanifold of a locally product Riemannian manifold $\tilde{M}$. Then, for each nonzero vector U tangent to M at a point $p \in M$, we define an angle $\theta(U)$ between FU and the tangent space T_pM known as *the Wirtinger angle* of U in M. If the angle $\theta(U)$ is constant, i.e., it is independent of the choice of $U \in T_pM$ and $p \in M$, then M is said be a *slant submanifold* of $\tilde{M}$. In particular, the invariant and the anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. M is *proper slant* if it is neither invariant nor anti-invariant. The normal bundle $T^{\perp}M$ of a slant submanifold M is decomposed as

$$T^{\perp}M = \omega(TM) \oplus \mu,$$

where μ is the invariant normal subbundle with respect to F orthogonal to $\omega(TM)$.

We recall the following result for a slant submanifold of a locally product Riemannian manifold.

Theorem 2.1. *[19] Let M be a submanifold of a locally product Riemannian manifold $\tilde{M}$. Then M is slant if and only if there exists a constant $\delta \in [0, 1]$ such that $P^2 = \delta I$. In this case, θ is slant angle of M and satisfies $\delta = \cos^2\theta$.*

The following relations are consequences of the above theorem:

$$g(PU, PV) = \cos^2\theta g(U, V),$$
$$g(\omega U, \omega V) = \sin^2\theta g(U, V),$$

for any $U, V \in \Gamma(TM)$. Also, for a slant submanifold, (2.4) (i) and Theorem 2.1 yield

$$t\omega U = \sin^2\theta U, \quad \omega PU = -f\omega U. \tag{2.6}$$

Also, if $\{e_1, \cdots e_m\}$ is an orthonormal basis of the tangent space TM of M, and e_r belong to the orthonormal basis $\{e_{m+1}, \cdots, e_n\}$ of the normal bundle $T^{\perp}M$, we define

$$h_{ij}^r = g(h(e_i, e_j), e_r) \quad \textit{and} \quad ||h||^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.7}$$

As a consequence, for a differentiable function φ on M, we have

$$||\vec{\nabla}\varphi||^2 = \sum_{i=1}^{m}(e_i(\varphi))^2,$$

where $\vec{\nabla}\varphi$ is the gradient of φ, defined by $g(\vec{\nabla}\varphi, U) = U\varphi$, for any $U \in \Gamma(TM)$.

3 Pseudo-slant submanifolds

Recently, semi-slant subamnifolds of locally product Riemannian manifolds were studied by Li and Li [16]. They defined these submanifolds as follows:

Definition 3.1. A submanifold M of a locally product Riemannian manifold $\tilde{M}$ is said to be a *semi-slant submanifold*, if there exist two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}^\theta$,

(ii) the distribution $\mathcal{D}$ is invariant, i.e. $F(\mathcal{D}) = \mathcal{D}$,

(iii) the distribution $\mathcal{D}^\theta$ is slant with slant angle $\theta \neq 0$.

On a similar line, we define pseudo-slant submanifolds as follows:

Definition 3.2. Let M be a submanifold of a locally product Riemannian manifold $\tilde{M}$ with a pair of orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$. Then M is said to be a *pseudo-slant submanifold* of $\tilde{M}$ if

(i) $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta$,

(ii) *the distribution $\mathcal{D}^\perp$ is anti-invariant under F, i.e., $F(\mathcal{D}^\perp) \subseteq T^\perp M$;*

(iii) *$\mathcal{D}^\theta$ is a slant distribution with slant angle $\theta \neq 0$.*

Let us denote by m_1 and m_2, the dimensions of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$. Then M is *anti-invariant* if $m_2 = 0$ and *proper slant* if $m_1 = 0$. It is *proper pseudo-slant*, if the slant angle is different from 0 and $\pi/2$ and $m_1 \neq 0$.

Moreover, if μ is an *invariant* normal subbundle under F of the normal bundle $T^\perp M$, then in case of pseudo-slant submanifolds, the normal bundle $T^\perp M$ can be decomposed as $T^\perp M = F\mathcal{D}^\perp \oplus \omega\mathcal{D}^\theta \oplus \mu$.

A pseudo-slant submanifold of a locally product Riemannian manifold is said to be *mixed totally geodesic* if $h(X, Z) = 0$, for any $X \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

First, we give the following example of a proper pseudo-slant submanifold.

Example 3.3. Consider a submanifold M of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates (x_1, x_2, y_1, y_2), and the product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, \quad F\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial y_i}, \quad i = 1, 2.$$

For any $\theta \in (0, \frac{\pi}{4})$, consider a submanifold M into $\mathbb{R}^4$, which is given by the immersion

$$f(u, v) = (u\cos\theta, v, u\sin\theta, v), \qquad u, v \neq 0.$$

Then the tangent space TM of M is spanned by the following vector fields

$$e_1 = \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial y_1}, \quad e_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}.$$

With respect to the product Riemannian structure F, we find

$$Fe_1 = \cos\theta\frac{\partial}{\partial x_1} - \sin\theta\frac{\partial}{\partial y_1}, \quad Fe_2 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_2}$$

It is easy to see that Fe_2 is orthogonal to TM, thus the anti-invariant distribution is $\mathcal{D}^\perp = \text{span}\{e_2\}$ and $\mathcal{D}^{\theta_1} = \text{span}\{e_1\}$ is the slant distribution with slant angle $\theta_1 = \arccos\left(\frac{g(Fe_1, e_1)}{\|Fe_1\|\|e_1\|}\right) = 2\theta$, and hence M is a proper pseudo-slant submanifold with slant angle $\theta_1 = 2\theta$.

The detailed study of pseudo-slant submanifolds of a locally product Riemannian manifold is given in [22] under the name of *hemi-slant submanifolds*. Now, we have the following result for later use.

Proposition 3.1. *On a proper pseudo-slant submanifold M of a locally product Riemannian manifold $\tilde{M}$, we have*

$$g(\nabla_X Y, Z) = \sec^2\theta\left(g(A_{FZ}PY, X) + g(A_{\omega PY}Z, X)\right),$$

for any $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. For any $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z) \quad = g(F\tilde{\nabla}_X Y, FZ) = g(\tilde{\nabla}_X FY, FZ) - g((\tilde{\nabla}_X F)Y, FZ).$$

Using (2.4) and the locally product Riemannian structure, we get

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\tilde{\nabla}_X PY, FZ) + g(\tilde{\nabla}_X \omega Y, FZ) \\ &= g(h(X, PY), FZ) + g(F\tilde{\nabla}_X \omega Y, Z) \\ &= g(A_{FZ}PY, X) + g(\tilde{\nabla}_X F\omega Y, Z) - g((\tilde{\nabla}_X F)\omega Y, Z). \end{aligned}$$

Similarly, by using (2.4) and the locally product Riemannian structure, we derive

$$g(\nabla_X Y, Z) = g(A_{FZ}PY, X) + g(\tilde{\nabla}_X t\omega Y, Z) + g(\tilde{\nabla}_X f\omega Y, Z).$$

Then from (2.6), we obtain

$$g(\nabla_X Y, Z) = g(A_{FZ}PY, X) + \sin^2\theta g(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_X \omega PY, Z).$$

Using (2.2), we infer

$$\cos^2\theta \cdot g(\nabla_X Y, Z) = g(A_{FZ}PY, X) + g(A_{\omega PY}X, Z),$$

and thus the assertion follows from the last relation. □

4 Warped product submanifolds

In this section, we study warped products of slant and anti-invariant submanifolds of a locally product Riemannian manifold. We first we give a brief introduction of warped product manifolds, which were first introduced by R. L. Bishop and B. O'Neill in [6]. They defined these manifolds as follows. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, and $f : M_1 \to (0, \infty)$ a positive differentiable function on M_1. Consider the product manifold $M_1 \times M_2$ with the canonical projections $\pi_1 : M_1 \times M_2 \to M_1$, $\pi_2 : M_1 \times M_2 \to M_2$, with the projection maps given by $\pi_1(p, q) = p$ and $\pi_2(p, q) = q$, for every $t = (p, q) \in M_1 \times M_2$. Then the warped product $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ equipped with the Riemannian structure, such that

$$||U||^2 = ||\pi_1 * (U)||^2 + f^2(\pi_1(p))||\pi_2 * (U)||^2,$$

for any tangent vector $U \in \Gamma(TM)$, where $*$ is the the tangent map symbol. Thus, we have $g = g_1 + f^2 g_2$. The function f is called *the warping function* on M. The following result from [6] is useful in our further study.

Lemma 4.1. *[6] Let $M = M_1 \times_f M_2$ be a warped product manifold. For any $X, Y \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$, we have*

(i) $\nabla_X Y \in \Gamma(TM_1)$,

(ii) $\nabla_Z X = \nabla_X Z = (X \ln f)Z$,

(iii) $\nabla_Z W = \nabla'_Z W - g(Z, W)\nabla \ln f$,

where ∇ and ∇' denote the Levi-Civita connections on M and M_2, respectively.

Now, we will consider the warped product pseudo-slant submanifolds of the form $M = M_\theta \times_f M_\perp$, where M_θ and $M_\perp$ are proper slant and anti-invariant submanifolds of a locally product Riemannian manifold $\tilde{M}$, respectively. In [3], Atceken has given an example for the existence of such warped products. In the following, $M = M_\theta \times_f M_\perp$ is called *proper warped product pseudo-slant* if $M_\perp$ and M_θ are anti-invariant and proper slant submanifolds of $\tilde{M}$, respectively.

For a proper warped product pseudo-slant submanifold $M = M_\theta \times_f M_\perp$, we have the following useful lemmas.

Lemma 4.2. *On a warped product pseudo-slant submanifold $M = M_\theta \times_f M_\perp$ of a locally product Riemannian manifold $\tilde{M}$, we have*

(i) $g(h(X, Y), FZ) = -g(h(X, Z), \omega Y)$

(ii) $g(h(PX, Y), FZ) = -g(h(PX, Z), \omega Y)$,

for any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. For any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, we have

$$g(h(X, Y), FZ) = g(\tilde{\nabla}_X Y, FZ).$$

Then from (2.1) and (2.4)(i), we obtain

$$g(h(X, Y), FZ) = g(\tilde{\nabla}_X PY, Z) + g(\tilde{\nabla}_X \omega X, Z).$$

Since M_θ is totally geodesic in M, using this fact in the above relation, from (2.2) we get

$$g(h(X, Y), FZ) = -g(A_{\omega Y} X, Z) = -g(h(X, Z), \omega Y),$$

which is exactly the assertion (i). If we interchange X by PX in (i), we can get (ii). Thus, the lemma is proved. □

Lemma 4.3. *Let $M = M_\theta \times_f M_\perp$ be a proper warped product pseudo-slant submanifold of a locally product Riemannian manifold $\tilde{M}$. Then*

(i) $g(h(Z, W), \omega X) = -g(h(X, Z), FW) + (PX \ln f)g(Z, W)$

(ii) $g(h(Z, W), \omega PX) = -g(h(PX, Z), FW) + \cos^2\theta(X \ln f)g(Z, W)$,

for any $X, Y \in \Gamma(TM_\theta)$ and $Z, W \in \Gamma(TM_\perp)$.

Proof. For any $X \in \Gamma(TM_\theta)$ and $Z, W \in \Gamma(TM_\perp)$, we have

$$g(h(Z,W),\omega X) = g(\tilde{\nabla}_Z W, \omega X).$$

From (2.4)(i), we get

$$g(h(Z,W),\omega X) = g(\tilde{\nabla}_Z W, FX) - g(\tilde{\nabla}_Z W, PX) = g(F\tilde{\nabla}_Z W, X) + g(\tilde{\nabla}_Z PX, W).$$

By Lemma 4.1 (ii), we derive

$$g(h(Z,W),\omega X) = g(\tilde{\nabla}_Z FW, X) + (PX \ln f) g(Z,W).$$

Using (2.2), we get

$$\begin{aligned} g(h(Z,W),\omega X) &= -g(A_{FW}Z, X) + (PX \ln f) g(Z,W) \\ &= -g(h(X,Z), FW) + (PX \ln f) g(Z,W), \end{aligned}$$

which proves (i). If we interchange X by PX in the above relation and using Theorem 2.1, we can easily get the second part, which completely proves the lemma. □

Now, we construct the following example of a proper warped product pseudo-slant submanifold in a locally product Riemannian manifold.

Example 4.1. Consider a submanifold M of $\mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^4$ with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, y_4)$ and the product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, \quad F\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial y_j}, \quad i = 1, 2, 3 \ \text{ and } \ j = 1, 2, 3, 4.$$

Let us consider the immersion f of M into $\mathbb{R}^7$ as follows

$$f(u,\varphi) = (u\cos\varphi, u\sin\varphi, 2u, \sqrt{2}u, -u, u\sin\varphi, u\cos\varphi), \quad \varphi \neq 0 \quad u \neq 0.$$

Then the tangent space TM of M is spanned by the following vector fields

$$Z_1 = \cos\varphi\frac{\partial}{\partial x_1} + \sin\varphi\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3} + \sqrt{2}\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} + \sin\varphi\frac{\partial}{\partial y_3} + \cos\varphi\frac{\partial}{\partial y_4}$$

and

$$Z_2 = -u\sin\varphi\frac{\partial}{\partial x_1} + u\cos\varphi\frac{\partial}{\partial x_2} + u\cos\varphi\frac{\partial}{\partial y_3} - u\sin\varphi\frac{\partial}{\partial y_4}.$$

Then with respect to the product Riemannian structure F, we get

$$FZ_1 = \cos\varphi\frac{\partial}{\partial x_1} + \sin\varphi\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3} - \sqrt{2}\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} - \sin\varphi\frac{\partial}{\partial y_3} - \cos\varphi\frac{\partial}{\partial y_4}$$

and

$$FZ_2 = -u\sin\varphi\frac{\partial}{\partial x_1} + u\cos\varphi\frac{\partial}{\partial x_2} - u\cos\varphi\frac{\partial}{\partial y_3} + u\sin\varphi\frac{\partial}{\partial y_4}.$$

It is easy to see now that FZ_2 is orthogonal to TM, and thus the anti-invariant distribution satisfies $\mathcal{D}^\perp = \text{span}\{Z_2\}$ and the slant distribution verifies $\mathcal{D}^\theta = \text{span}\{Z_1\}$, with slant angle $\theta = \arccos\left(\frac{g(FZ_1,Z_1)}{\|FZ_1\|\|Z_1\|}\right) = \arccos\left(\frac{1}{9}\right)$. It is easy to see that both the distributions are integrable. If the integral manifolds of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ are denoted by $M_\perp$ and M_θ respectively, then the induced metric tensor g_M on M is given by

$$g_M = 9du^2 + 2u^2 d\varphi^2 = g_{M_\theta} + \left(\sqrt{2}u\right)^2 g_{M_\perp}.$$

Thus M is a warped product submanifold of the form $M = M_\theta \times_{f_1} M_\perp$, with warping function $f_1 = \sqrt{2}u$.

Theorem 4.4. *Let M be a pseudo-slant submanifold of a locally product Riemannian manifold $\tilde{M}$. Then M is locally a mixed totally geodesic warped product submanifold if and only if*

$$(i)\ A_{FZ}X = 0, \quad (ii)\ A_{\omega PX}Z = \cos^2\theta(X\lambda)Z, \tag{4.1}$$

for each $X \in \Gamma(TM_\theta)$ and λ a C^∞-function on M, with $Z\lambda = 0$, for each $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. If M is a mixed totally geodesic warped product submanifold of a locally product Riemannian manifold $\tilde{M}$, then for any $X \in \Gamma(TM_\theta)$ and $Z, W \in \Gamma(TM_\perp)$, we have $g(A_{FZ}X, W) = g(h(X,W), FZ) = 0$, i.e., $A_{FZ}X$ has no component in $TM_\perp$. Also, from Lemma 4.2 we have $g(A_{FZ}X, Y) = 0$, i.e., $A_{FZ}X$ has no component in TM_θ. Therefore, it follows that $A_{FZ}X = 0$, which is first part of (4.1). Similarly, $g(A_{\omega PX}Z, Y) = g(h(Y,Z), \omega PX) = 0$, i.e., $A_{\omega PX}Z$ has no component in TM_θ for any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$. Therefore, the second part of (4.1) follows from Lemma 4.3 (ii).

Conversely, let M be a proper pseudo-slant submanifold of a locally product Riemannian manifold $\tilde{M}$ such that (4.1) holds. Then by Proposition 3.1 and the relation (4.1) we find $g(\nabla_X Y, Z) = 0$, which means that the leaves of $\mathcal{D}^\theta$ are totally geodesic in M. On the other hand, for any $X \in \Gamma(\mathcal{D}^\theta)$ and $Z, W \in \Gamma(D^\perp)$ we have

$$g([Z,W], PX) = g(\tilde{\nabla}_Z W, PX) - g(\tilde{\nabla}_W Z, PX).$$

From (2.4) (i), we get

$$g([Z,W], PX) = g(\tilde{\nabla}_Z W, FX) - g(\tilde{\nabla}_Z W, \omega X) - g(\tilde{\nabla}_W Z, FX) + g(\tilde{\nabla}_Z W, \omega X).$$

Using (2.1), we obtain

$$g([Z,W], PX) = g(\tilde{\nabla}_Z FW, X) - g(\tilde{\nabla}_Z \omega X, W) - g(\tilde{\nabla}_W FZ, X) + g(\tilde{\nabla}_W \omega X, Z).$$

Thus by (2.2), we derive

$$g([Z,W], PX) = -g(A_{FW}X, Z) + g(A_{\omega X}Z, W) + g(A_{FZ}X, W) - g(A_{\omega X}W, Z).$$

The first and the third terms of right hand side are identically zero by using (4.1) (i), and the second and fourth terms can be evaluated from (4.1) (ii) by interchanging X by PX, as follows

$$g([Z,W], PX) = (PX\lambda)g(Z,W) - (PX\lambda)g(W,Z) = 0,$$

which means that the distribution $\mathcal{D}^\perp$ is integrable. Thus, if we consider $M_\perp$ a leaf of $\mathcal{D}^\perp$ in M, and $h^\perp$ the second fundamental form of $M_\perp$ in M then, for any $Z, W \in \Gamma(D^\perp)$, we have

$$g(A_{FW}PX, Z) = g(h(Z, PX), FW) = g(\tilde{\nabla}_Z PX, FW).$$

Then by (2.1), we derive

$$g(A_{FW}PX, Z) = g(F\tilde{\nabla}_Z PX, W).$$

From (2.5) and the structure equation of the locally product Riemannian manifold, we obtain

$$g(A_{FW}PX, Z) = g(\tilde{\nabla}_Z FPX, W).$$

Using (2.4) (i), we get

$$g(A_{FW}PX, Z) = g(\tilde{\nabla}_Z P^2X, W) + g(\tilde{\nabla}_Z \omega PX, W).$$

Thus by Theorem 2.1 and (2.2), we derive

$$\begin{aligned} g(A_{FW}PX, Z) &= \cos^2\theta g(\tilde{\nabla}_Z X, W) - g(A_{\omega PX}Z, W) \\ &= -\cos^2\theta g(\tilde{\nabla}_Z W, X) - g(A_{\omega PX}Z, W). \end{aligned}$$

Since $\mathcal{D}^\perp$ is integrable, by using (2.2), we get

$$g(A_{FW}PX, Z) = -\cos^2\theta g(h^\perp(Z, W), X) - g(A_{\omega PX}Z, W).$$

From (4.1) (i) and (4.1) (iii), we obtain

$$\cos^2\theta g(h^\perp(Z, W), X) + \cos^2\theta (X\lambda) g(Z, W) = 0.$$

Since M is proper slant, we find

$$g(h^\perp(Z, W), X) = -X(\lambda) g(Z, W) = -g(Z, W) g(\nabla\lambda, X),$$

which means $h^\perp(Z, W) = -g(Z, W)\nabla\lambda$, that is, $M_\perp$ is totally umbilical in M with the mean curvature vector $H^\perp = -\nabla\lambda$. We can easily see that $H^\perp$ is a parallel mean curvature vector corresponding to the normal connection of $M_\perp$ into M. Hence, by a result of Hiepko [15], M is a warped product pseudo-slant submanifold, which completely proves the theorem. □

5 An inequality for warped products

In this section, we obtain a sharp estimation between the squared norm of the second fundamental form of the warped product immersion and the warping function. In order to obtain the relation for the squared norm of the second fundamental form, we first construct the following orthonormal frame fields for a proper warped product pseudo-slant submanifold $M = M_\theta \times_f M_\perp$.

Let $M = M_\theta \times_f M_\perp$ be a m-dimensional warped product pseudo-slant submanifold of an n-dimensional locally product Riemannian manifold $\tilde{M}$, such that $p = \dim M_\theta$

and $q = \dim M_\perp$, where M_θ and $M_\perp$ are proper slant and anti-invariant submanifolds of $\tilde{M}$, respectively. Denote the tangent bundles of M_θ and $M_\perp$ by $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$, respectively. Consider the orthonormal frame fields $\{e_1, \cdots, e_q\}$ and $\{e_{q+1} = e_1^* = \sec\theta P e_1^*, \cdots, e_m = e_p^* = \sec\theta P e_p^*\}$ of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively. Then the orthonormal frame fields of the normal subbundles $F\mathcal{D}^\perp$, $\omega\mathcal{D}^\theta$ and μ respectively are $\{e_{m+1} = Fe_1, \cdots, e_{m+q} = Fe_q\}$, $\{e_{m+q+1} = \tilde{e}_1 = \csc\theta\omega e_1^*, \cdots, e_{m+p+q} = \tilde{e}_p = \csc\theta\omega e_p^*\}$ and $\{e_{2m+1}, \cdots, e_n\}$.

Theorem 5.1. *Let $M = M_\theta \times_f M_\perp$ be a mixed totally geodesic warped product pseudo-slant submanifold of a locally product Riemannian manifold $\tilde{M}$ such that $M_\perp$ is an anti-invariant submanifold and let M_θ be a proper slant submanifold of $\tilde{M}$. Then, we have:*

(i) The squared norm of the second fundamental form h of M satisfies

$$||h||^2 \geq q\cot^2\theta ||\nabla^\theta \ln f||^2$$

where $q = \dim M_\perp$ and $\nabla^\theta \ln f$ is the gradient of $\ln f$ along M_θ.

(ii) If the equality sign of (i) holds identically, then M_θ is totally geodesic and $M_\perp$ is totally umbilical in $\tilde{M}$.

Proof. From the definition, we have

$$||h||^2 = ||h(\mathcal{D}^\theta, \mathcal{D}^\theta)||^2 + ||h(\mathcal{D}^\perp, \mathcal{D}^\perp)||^2 + 2||h(\mathcal{D}^\theta, \mathcal{D}^\perp)||^2.$$

Since M is mixed totally geodesic, then the second term in the above relation is identically zero, and thus we find

$$||h||^2 = ||h(\mathcal{D}^\perp, \mathcal{D}^\perp)||^2 + ||h(\mathcal{D}^\theta, \mathcal{D}^\theta)||^2.$$

Then from (2.7), we obtain

$$||h||^2 = \sum_{r=m+1}^{n} \sum_{i,j=1}^{q} g(h(e_i, e_j), e_r)^2 + \sum_{r=m+1}^{n} \sum_{i,j=1}^{p} g(h(e_i^*, e_j^*), e_r)^2.$$

The above relation can be expressed in terms of the components of $F\mathcal{D}^\perp$, $\omega\mathcal{D}^\theta$ and μ, as follows

$$\begin{aligned} ||h||^2 = \; & \textstyle\sum_{r=1}^{q}\sum_{i,j=1}^{q} g(h(e_i,e_j), Fe_r)^2 + \sum_{r=1}^{p}\sum_{i,j=1}^{q} g(h(e_i,e_j), \csc\theta\,\omega e_r^*)^2 \\ & + \textstyle\sum_{r=2m+1}^{n}\sum_{i,j=1}^{q} g(h(e_i,e_j), e_r)^2 + \sum_{r=1}^{q}\sum_{i,j=1}^{p} g(h(e_i^*,e_j^*), Fe_r)^2 \\ & + \textstyle\sum_{r=1}^{p}\sum_{i,j=1}^{p} g(h(e_i^*,e_j^*), \tilde{e}_r)^2 + \sum_{r=2m+1}^{n}\sum_{i,j=1}^{p} g(h(e_i^*,e_j^*), e_r)^2. \end{aligned} \tag{5.1}$$

we shall leave terms $g(h(e_i, e_j), Fe_r)$, for any $i, j, r = 1, \cdots q$ and $g(h(e_i^*, e_j^*), \tilde{e}_r)$, for any $i, j, r = i, \cdots, p$ in the above relation (5.1) unchanged. Also, the third and sixth terms have μ-components, and therefore we also leave these two terms, and by using Lemma 4.2 and Lemma 4.3, we derive

$$||h||^2 \geq \csc^2\theta \sum_{i,j=1}^{q} \sum_{r=1}^{p} (Pe_r^* \ln f)^2 \, g(e_i, e_j)^2.$$

From the orthonormal frame fields of $\mathcal{D}^\theta$, we have $Pe_r^* = \cos\theta\, e_r^*$, for $r = 1, \cdots, p$. Using this fact, we find

$$||h||^2 \geq \cot^2\theta \sum_{i,j=1}^{q} \sum_{r=1}^{p} (e_r^* \ln f)^2\, g(e_i, e_j)^2 = q\cot^2\theta \|\nabla^\theta \ln f\|^2$$

which is exactly the inequality (i). If the equality in (i) holds, then from the remaining fifth and sixth terms of (5.1), we obtain the following conditions,

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp \omega\mathcal{D}^\theta, \quad h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp \mu,$$

which means that

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) \subset F\mathcal{D}^\perp. \tag{5.2}$$

Also, from Lemma 4.2, we get

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp F\mathcal{D}^\perp. \tag{5.3}$$

The,n from (5.2) and (5.3), we get

$$h(\mathcal{D}^\theta, \mathcal{D}^\theta) = 0. \tag{5.4}$$

Then M_θ is totally geodesic in $\tilde{M}$, by using the fact that M_θ is totally geodesic in M [6,9] and (5.4). Also, from the remaining first and third terms in (5.1), we infer

$$h(\mathcal{D}^\perp, \mathcal{D}^\perp) \perp F\mathcal{D}^\perp, \quad h(\mathcal{D}^\perp, \mathcal{D}^\perp) \perp \mu \Rightarrow h(\mathcal{D}^\perp, \mathcal{D}^\perp) \subset \omega\mathcal{D}^\theta. \tag{5.5}$$

From Lemma 4.3, for a mixed totally geodesic warped product submanifold, we have

$$g(h(Z, W), \omega PX) = \cos^2\theta (X \ln f) g(Z, W). \tag{5.6}$$

Hence, by using the fact that $M_\perp$ is totally umbilical in M [6, 9], with (5.5) and (5.6), it follows that $M_\perp$ is totally totally umbilical in $\tilde{M}$. Thus the proof is complete. □

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