On Einstein Kropina change of $m$-th root Finsler metrics

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Abstract. In the present paper, we consider Kropina change of $m$-th root metric and prove that if it is an Einstein metric (or weak Einstein metric), then it is Ricci-flat.

M.S.C. 2010: 53B40, 53C60.

Key words: Finsler space; Kropina metrics; $m$-th root metrics; Einstein metrics.

1 Introduction

The theory of $m$-th root Finsler metrics has been developed by Shimada [11] in 1979, applied to ecology by Antonelli [1] and studied by several authors ([12],[14],[11]). It is regarded as a generalization of Riemannian metric in the sense that for $m = 2, 3$ and 4, it is called Riemannian metric, cubic metric and quartic metric respectively [7]. In the four-dimension, a special fourth root metric of the form $F = \sqrt[4]{y^1y^2y^3y^4}$ is called the Berwald-Moór metric [4], which is considered by physicists as an important subject for a possible model of space time. Recent studies show that $m$-th root Finsler metrics plays a very important role in physics, space-time and general relativity as well as in unified field theory ([3],[2]). Z. Shen and B. Li have studied the geometric properties of locally projectively flat fourth root metrics in the form $F = \sqrt[4]{a_{ijkl}(x)y^iy^jy^ky^l}$ and generalized fourth root metrics in the form $F = \sqrt[4]{a_{ijkl}(x)y^iy^jy^ky^l + b_{ij}(x)y^iy^j}$ [7].

Recently, B. Tiwari and M. Kumar [13] have studied Randers change of a Finsler space with $m$-th root metric. Also, A. Tayebi, T. Tabatabaeifar and E. Peyghan introduced the Kropina change of $m$-th root metric and established conditions on Kropina change of $m$-th root metric, to be locally dually flat and locally projectively flat.

Let $(M, F) = F^n$ be an n-dimensional Finsler manifold. For a 1-form $\beta(x, y) = b_i(x)y^i$ on $M$, define a Finsler change as follows

$$ F(x, y) \rightarrow \tilde{F}(x, y) = f(F, \beta), $$

Differential Geometry - Dynamical Systems, Vol.18, 2016, pp. 139-146.
where \( f(F, \beta) \) is a positively homogeneous function of \( F \) and \( \beta \). A Finsler change is called Kropina change if \( f(F, \beta) = \frac{F^2}{\beta} \). The purpose of the present paper is to investigate Kropina change of \( m \)-th root metrics, defined by

\[
F = \frac{F^2}{\beta},
\]

where \( F = \sqrt[2]{A} \) is an \( m \)-th root metric on the manifold \( M \), for which we shall restrict our consideration to the domain where \( \beta = \beta(x) > 0 \).

The Einstein metrics are solutions to Einstein field equation in General Relativity, which closely connect Riemannian geometry with gravitation in General Relativity. C. Robles studied a special class of Einstein Finsler metrics, that is, Einstein Randers metrics, and proved that for a Randers metric on a 3-dimensional manifold, it is Einstein if and only if it has constant flag curvature. E. Guo, X. Mo and X. Zhang have explicitly constructed an Einstein Finsler metrics of non-constant flag curvature in terms of navigation representation [6]. Recently, Z. Shen and C. Yu, using certain transformation, have constructed a large class of Einstein metrics [9]. In this paper, we establish following theorems

**Theorem 1.1.** Let \( F = \frac{F^2}{\beta} \) be a non-Riemannian Kropina change of \( m \)-th root Finsler metric \( F \) on a manifold of dimension \( n \geq 2 \), with \( m \geq 3 \). If \( F \) is Einstein metric, then it is Ricci-flat.

**Theorem 1.2.** Let \( F = \frac{F^2}{\beta} \) be a non-Riemannian Kropina change of \( m \)-th root Finsler metric \( F \) on a manifold of dimension \( n \geq 2 \), with \( m \geq 3 \). If \( F \) is a weak Einstein metric, then it is Ricci-flat.

**Theorem 1.3.** Let \( F = \frac{F^2}{\beta} \) be a non-Riemannian Kropina change of \( m \)-th root Finsler metric \( F \) on a manifold of dimension \( n \geq 2 \), with \( m \geq 3 \). If \( F \) is of scalar flag curvature \( K(x,y) \) and isotropic \( S \)-curvature, then \( K = 0 \).

Throughout the paper we call the Finsler metric \( F \) as Kropina change of \( m \)-th root metric and \( F^n = (M, F) \) as Kropina transformed Finsler space. We restrict ourselves for \( m \geq 3 \) throughout the paper and also the quantities corresponding to the Kropina transformed Finsler space \( F^n \) will be denoted by putting bar on the top of that quantity.

## 2 Preliminaries

Let \( M \) be an \( n \)-dimensional \( C^\infty \)-manifold. Denote by \( T_xM \) the tangent space at \( x \in M \) and by \( TM := \bigcup_{x \in M} T_xM \) the tangent bundle of \( M \). Each element of \( TM \) has the form \((x, y)\), where \( x \in M \) and \( y \in T_xM \). Let \( TM_0 = TM \setminus \{0\} \).

**Definition.** A Finsler metric on \( M \) is a function \( F : TM \to [0, \infty) \) with the following properties:

(i) \( F \) is \( C^\infty \) on \( TM_0 \),

(ii) \( F \) is positively 1-homogeneous on the fibers of tangent bundle \( TM \), and

(iii) the Hessian of \( \frac{F^2}{2} \) with components \( g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y_i \partial y_j} \) is positive definite on \( TM_0 \).
Therefore, by using the equations (3.1) and (3.2), we obtain the metric tensor $g_{ij}$. Clearly, $A_i$ with $a_i$ is called the fundamental tensor of the Finsler space $F^n$. The normalized supporting element $l_i$ and the angular metric tensor $h_{ij}$ of $F$ are defined, respectively as $l_i = \frac{\partial F}{\partial y^i}$, and $h_{ij} = F \frac{\partial^2 F}{\partial y^i \partial y^j}$. The $S$-curvature $S = S(x, y)$ in Finsler geometry has been introduced by Shen [8] as a non-Riemannian quantity, defined as

$$S(x, y) = \left. \frac{d}{dt} [\tau(\sigma(t), \dot{\sigma}(t))] \right|_{t=0},$$

where $\tau = \tau(x, y)$ is a scalar function on $T_x M \setminus \{0\}$, called distortion of $F$ and $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. A Finsler metric $F$ is called of isotropic if $S = (n + 1)cF$, for some scalar function $c = c(x)$, on $M$. Let $F$ be a Finsler metric defined by $F = \sqrt{A}$, where $A$ is given by $A = a_{ij} y^i y^j$, with $a_{ij}$ symmetric in all its indices [11]. Then $F$ is called an $m$-th root Finsler metric. Clearly, $A$ is homogeneous of degree $m$ in $y$. Let

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i.$$

Then the following relations hold

$$g_{ij} = A_i A_j + (2 - m) A_i A_j,$$
$$y^i A_i = m A_i y^i, \quad g_{ij} = (m - 1) A_i , \quad y_i = A_i,$$
$$A^{ij} A_k = \delta^i_k, \quad A^{ij} A_i = \frac{1}{m} y^j, \quad A_i A_j A^{ij} = \frac{m}{m-1} A.$$

3 Fundamental metric tensors and geodesic sprays of Kropina changed $m$-th root Finsler metrics

The differentiation of equation (1.1) with respect to $y^i$ yields the normalized supporting element $\bar{l}_i$ given by

$$\bar{l}_i = F \left( \frac{2 A_i}{m A} - \frac{b_i}{\beta} \right)$$

and the angular metric tensor $\bar{h}_{ij}$ given by

$$\bar{h}_{ij} = F^2 \left[ \frac{2}{m A} A_i A_j + \frac{2(2 - m)}{m^2 A^2} A_i A_j - \frac{2}{m A \beta} (A_i b_j + A_j b_i) + \frac{2}{\beta^2} b_i b_j \right].$$

Also the fundamental metric tensor $\bar{g}_{ij}$ of Finsler space $F^n$ is given by $\bar{g}_{ij} = \bar{h}_{ij} + \bar{l}_i \bar{l}_j$. Therefore, by using the equations (3.1) and (3.2), we obtain the metric tensor $\bar{g}_{ij}$ as

$$\bar{g}_{ij} = F^2 \left[ \frac{2}{m A} A_i A_j + \frac{2(4 - m)}{m^2 A^2} A_i A_j - \frac{4}{m A \beta} (A_i b_j + A_j b_i) + \frac{3}{\beta^2} b_i b_j \right].$$

This equation can be rewritten as

$$\bar{g}_{ij} = F^2 \left[ \frac{2}{m A} A_i A_j + \frac{2(4 - 3m)}{m^2 A^2} A_i A_j + \left( \frac{4}{\sqrt{3m A}} A_i - \sqrt{\frac{3}{\beta}} b_i \right) \left( \frac{4}{\sqrt{3m A}} A_j - \sqrt{\frac{3}{\beta}} b_j \right) \right] .$$
In order to calculate the components of inverse metric tensor $g^{ij}$, we use the following Proposition twice.

**Proposition.** [8] Let $G = (g_{ij})$ and $H = (h_{ij})$ be symmetric $n \times n$ matrices and $C = (c_i)$ be an $n$-vector. Assume that $H$ is invertible with $H^{-1} = (h^{ij})$ and $g_{ij} = h_{ij} + \delta c_i c_j$. Then $det(g_{ij}) = (1 + \delta c^2)det(h_{ij})$, where $c = \sqrt{h^{ij}c_ic_j}$. If $1 + \delta c^2 \neq 0$, then $G$ is invertible. The inverse matrix $G^{-1} = (g^{ij})$ is given by

$$g^{ij} = h^{ij} - \frac{\delta c^2 c^j}{1 + \delta c^2}, \text{ where } c' = h^{ij}c_j.$$

Let

$$H_{ij} = \frac{2}{m^2} A_{ij} + \frac{2(4 - 3m)}{3m^2 A^2} A_i A_j. \tag{3.4}$$

By using the Proposition from above, we infer $H^{ij} = \frac{m^2 A^{ij}}{2} - \frac{4 - 3m}{2(m - 1)} y^i y^j$. Thus, in view of equations (3.3) and (3.4), $\overline{g}_{ij}$ can be written as $\overline{g}_{ij} = \frac{F^2}{2} [H_{ij} + K_i K_j]$, where $K_i = \left( \frac{4}{\sqrt{3mA}} A_i - \frac{2 A}{\sqrt{3m}} b_i \right)$. By direct computation, we have

$$\overline{g}^{ij} = \frac{1}{F^2} \left[ a_0 A^{ij} + a_1 y^i y^j + a_2 B^i B^j + a_3 (y^i B^j + y^j B^i) \right],$$

where,

$$a_0 = \frac{m A}{2}, a_1 = \left\{ \frac{2(m - 4)\beta^2 + m(3m - 4) A B^2}{2((m - 2)\beta^2 + m(m - 1) A B^2)} \right\}, a_2 = \frac{m^2 (1 - m) A^2}{2((m - 2)\beta^2 + m(m - 1) A B^2)},$$

$$a_3 = \frac{m^2 \beta A}{2((m - 2)\beta^2 + m(m - 1) A B^2)}, B^i = A^{ij} b_j, B^2 = B^i b_i. \tag{3.5}$$

Thus we have

**Proposition 3.1.** The covariant metric tensor $\overline{g}_{ij}$ and the contravariant metric tensor $\overline{g}^{ij}$ of Kropina transformed Finsler space $\tilde{F}^n$ are given as

$$\overline{g}_{ij} = \frac{F^2}{2} \left[ \frac{2}{m^2 A} A_{ij} + \frac{2(4 - m)}{m^2 A} A_i A_j - \frac{4}{m A^2} (A_i b_j + A_j b_i) + \frac{3}{\beta^2} b_i b_j \right]$$

and

$$\overline{g}^{ij} = \frac{1}{F^2} \left[ a_0 A^{ij} + a_1 y^i y^j + a_2 B^i B^j + a_3 (y^i B^j + y^j B^i) \right],$$

where $a_0, a_1, a_2, a_3, B^i$ and $B^2$ are given by equation (3.5).

In local coordinates, the geodesics of a given Finsler metric $F = F(x, y)$ are characterized by the equations

$$\frac{d^2 x^i}{dt^2} + 2G^i \left( \frac{dx^i}{dt} \right) = 0,$$

where

$$G^i = \frac{1}{4} g^{ij} \left( [F^2]_{x^i y^j} y^k - [F^2]_{x^i x^j} \right) \tag{3.6}$$
are called spray coefficients. To calculate the spray coefficients $G^i$, we use the relations

$$ (F^2)_{x^k} = F^2 \left( \frac{4A_{x^k}}{mA} - \frac{2\beta_{x^k}}{\beta} \right) $$

and

$$ (F^2)_{x^k y^k} = F^2 \left( \frac{4(A_0)_{y^k}}{mA} + \frac{(16 - 4m)_{A_0 A_0}}{m^2 A^2} - \frac{2(b_0)_0}{\beta^2} + \frac{6b_0 \beta_0 + A_0 b_1}{m \beta} \right). $$

In view of equations (3.6), (3.7), (3.8) and Proposition 3.1, we have

$$ G^i = \left[ \frac{1}{4} \left[ a_0 A^i + a_1 y^i y^j + a_2 B^i B^j + a_3 (y^i B^j + y^j B^i) \right] \times \right] $$

$$ \left[ \frac{4(A_0)_0 + (16 - 4m)_{A_0 A_0}}{m A} + \frac{2(b_0)_0}{\beta^2} + \frac{6b_0 \beta_0 + A_0 b_1}{m A \beta} \right]. $$

**Proposition 3.2** Let $F = F^2$ be a non-Riemannian Kropina change of an $m$-th root Finsler metric $F$ on a manifold of dimension $n \geq 2$, with $m \geq 3$. Then the spray coefficients $G^i$ of $F^n$ are given by equation (3.9).

**Remark 3.1** It is remarkable to note that the metric tensors $\bar{g}_{ij}$ and $\bar{g}_{ij}$ of $F^n$ are not necessarily rational functions of $y$, but the spray coefficients $G^i$ of $F^n$ are rational functions of $y$.

## 4 Einstein metrics

For a Finsler metric $F = F(x, y)$, its Riemann curvature $R^i_k = \frac{\partial G^i}{\partial x^k} \otimes dx^k$ is defined by

$$ R^i_k = 2 \frac{\partial G^i}{\partial x^k} - y^i \frac{\partial^2 G^i}{\partial x^j \partial x^k} + 2G^i \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^i}{\partial y^k}. $$

The Finsler metric $F = F(x, y)$ is said to be of scalar flag curvature if there is a scalar function $K = K(x, y)$ such that

$$ R^i_k = K(x, y) F^2 \left( \delta^i_k - \frac{F_k y^i}{F} \right). $$

The Ricci curvature is the trace of the Riemann curvature, $Ric = R^i_k$. In view of the definition of Riemann curvature, Ricci curvature and Remark 3.1, we have

**Lemma 4.1.** Let $F = F^2$ be a non-Riemannian Kropina change of an $m$-th root Finsler metric $F$ on a manifolds of dimension $n \geq 2$, with $m \geq 3$. Then $R^i_k$ and $Ric = R^i_k$ are rational functions of $y$.

A Finsler metric $F = F(x, y)$ on an $n$-dimensional manifold $M$ is called an Einstein metric if there is a scalar function $\lambda = \lambda(x)$ on $M$ such that $Ric = (n - 1)\lambda F^2$. $F$ is said to be Ricci constant (resp. flat) if $\lambda = \text{constant}$ (resp. zero).

By definition, every 2-dimensional Riemann metric is an Einstein metric, but generally not of Ricci constant. In dimension $n \geq 3$, the second Schur Lemma ensures that...
every Riemannian Einstein metric must be Ricci constant. In particular, in dimension $n = 3$, a Riemann metric is Einstein if and only if it is of constant sectional curvature.

**Proof of Theorem 1.1.** By Lemma 4.1, $\text{Ric}$ is a rational function of $y$. Suppose $F$ is an Einstein metric, that is $\text{Ric} = (n - 1)\lambda F^2$ and $F^2$ is not a rational function. Therefore $\lambda = 0$. □

In Finsler geometry, the flag curvature is an analogue of the sectional curvature from Riemannian geometry. A natural problem is to study and characterize Finsler metrics of constant flag curvature. There are only three local Riemannian metrics of constant sectional curvature, up to a scaling. However there are lots of non-Riemannian Finsler metrics of constant flag curvature. For example, the Funk metric is positively complete and non-reversible with $K = \frac{1}{4}$, and the Hilbert-Klein metric is complete and reversible with $K = 1$. Clearly, if a Finsler metric is of constant flag curvature, then it is an Einstein metric. We obtain

**Corollary 4.2.** Let $F = \frac{F^2}{a^2}$ be a non-Riemannian Kropina change of $m$-th root Finsler metric on a manifold of dimension $n \geq 2$, where $m \geq 3$. If $F$ is of constant flag curvature $K$, then $K = 0$.

**Example 1.** Let $F = \sqrt{\sum_{i=1}^{n} (y^i)^4}$, for any fixed $j$, $1 \leq j \leq n$. By direct computation, we get $G^i = 0$ and $R^i_k = 0$. Thus the flag curvature of $F$ is zero.

**Example 2.** Let $F = \frac{\sqrt{\sum_{i=1}^{n} (x^i)^2 (y^i)^4}}{x^j y^j}$, for any fixed $j$, $1 \leq j \leq n$. By direct computation, we get $\tilde{G}^i = \frac{(y^j)^2}{4x^j}$ and $\tilde{R}^i_k = 0$. Thus the flag curvature of $F$ is zero. It is known that every Berwald metric with $K = 0$ is locally Minkowskian. So $F$ is locally Minkowskian.

## 5 Weak Einstein metrics

A weakly Einstein metric is the generalization of the Einstein metric. A Finsler metric $F$ is called a weakly Einstein metric if its Ricci curvature $\text{Ric}$ is of the form $\text{Ric} = (n - 1)(\frac{\theta}{2} + \lambda)F^2$, where $\theta$ is a 1-form and $\lambda = \lambda(x)$ is a scalar function. In general, a weak Einstein metric is not necessarily an Einstein metric and vice versa.

**Proof of Theorem 1.2.** Suppose $F$ is a weak Einstein metric, then

$$\text{Ric} = (n - 1)(\theta F + \lambda F^2),$$

where $\theta$ is an 1-form and $\lambda = \lambda(x)$ is a scalar function. By Lemma 4.1 Ric is rational function of $y$. If $\lambda \neq 0$, we get

$$F = \frac{-3(n-1)\theta \pm \sqrt{9(n-1)^2 \theta^2 + 4(n-1)\lambda \text{Ric}}}{2(n-1)\lambda}.$$

On the other hand, $F = \frac{(a_{11,2\ldots n}(x)y^{i_1}y^{i_2}\ldots y^{i_n})^\frac{1}{\beta}}{\beta}$, so we get...
\[
\left( a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m} \right)^{\frac{1}{m}} = \left( \frac{-3(n-1)\theta \pm \sqrt{9(n-1)^2 \theta^2 + 4(n-1)\lambda \Ric}}{2(n-1)\lambda} \right) \beta.
\]

Here the left hand side is purely irrational for \( m \geq 3 \). Then the right hand side will be irrational if and only if \( \theta = 0 \). Thus we have that \( \mathcal{F} \) is an Einstein metric. Using Theorem 1.1, we obtain \( \Ric = 0 \).

\[\square\]

6 Scalar flag curvature

For a tangent plane \( P = \text{span}(y, u) \), \( y \) and \( u \) are linearly independent vectors of tangent space \( T_x M \) of \( M \) at point \( x \in M \), the flag curvature \( K = K(P, u) \) depends on plane \( P \) as well as vector \( u \in P \).

(a) A Finsler metric \( F \) is of scalar flag curvature if for any non-zero vector \( y \in T_x M \), \( K = K(x, y) \) is independent of \( P \) containing \( y \in T_x M \).

(b) \( F \) is called of almost isotropic flag curvature if \( K = \frac{3c(x)y^m}{p} + \lambda \), where \( c = c(x) \) and \( \lambda = \lambda(x) \) are some scalar functions on \( M \).

(c) \( F \) is of weakly isotropic flag curvature if \( K = \frac{3c(x)y^m}{p} + \lambda \), where \( \theta \) is an 1-form and \( \lambda = \lambda(x) \) is a scalar function.

Clearly, if a Finsler metric is of weakly isotropic flag curvature, then it is a weak Einstein metric.

**Lemma 6.1.** Let \( \mathcal{F} = F^2 \) be a non-Riemannian Kropina change of \( m \)-th root Finsler metric on a manifold of dimension \( n \geq 2 \), where \( m \geq 3 \). If \( \mathcal{F} \) is of almost isotropic flag curvature \( K \), then \( K = 0 \).

The S-curvature \( S = S(x, y) \) in Finsler geometry was introduced by Shen [8] as a non-Riemannian quantity, defined as
\[
S(x, y) = \frac{d}{dt} |r(\sigma(t), \dot{\sigma}(t))|_{t=0}
\]

where \( r = r(x, y) \) is a scalar function on \( T_x M \setminus \{0\} \), called distortion of \( F \) and \( \sigma = \sigma(t) \) is the geodesic with \( \sigma(0) = x \) and \( \dot{\sigma}(0) = y \). A Finsler metric \( F \) is called of isotropic S-curvature if \( S = (n+1)cF \), for some scalar function \( c = c(x) \), on \( M \).

**Theorem 6.2.** [5] Let \( (M, F) \) be an \( n \)-dimensional Finsler manifold of scalar flag curvature \( K(x, y) \). Suppose that the S-curvature is isotropic, \( S = (n+1)c(x)F \), then there is a scalar function \( \lambda(x) \) on \( M \) such that \( K = \frac{3c(x)y^m}{p} + \lambda \). In particular, \( c(x) = c \) is a constant if and only if \( K = K(x) \) is a scalar function on \( M \).

In dimension \( n \geq 3 \), a Finsler metric \( F \) is of isotropic flag curvature if and only if \( F \) is of constant flag curvature by Schur’s Lemma. In general, a Finsler metric of weakly isotropic flag curvature and that of isotropic flag curvature are not equivalent.

**Proof of Theorem 1.3.** Lemma 6.1 and Theorem 6.2 yield the claimed result. \( \square \)
References


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