On curvature-type invariants in hyperbolic Kähler spaces (II)

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Abstract. In our previous paper [10], we considered (F, g)-holomorphically-structurally semi-symmetric and $F-$holomorphically-structurally semi-symmetric connections on hyperbolic Kählerian spaces. Under certain assumptions, each of these two types of connections had curvature-type invariants. The case of $g-$holomorphically-structurally semi-symmetric connection, like on anti-Kählerian and product spaces, exhibits a prominent complexity. In this work, we provide the construction of a curvature-type invariant for such a connection.

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1 Introduction

Numerous works were dedicated to the study of hyperbolic Kählerian spaces and of their related structures ([2]-[10]). The present paper investigates such spaces, following the idea from [1]. A hyperbolic Kählerian space is an even-dimensional space endowed with an involutive and a skew-symmetric structure. On such space, like on the standard (or elliptic) Kählerian space, it is impossible to introduce a conformal transformation in a natural way.

The metric tensor (g_{ij}) and the structure tensor (F_j^i) on a hyperbolic Kählerian space are satisfying the following conditions:

$$F_j^t F_i^s g_{ts} = -g_{ji}, \quad F_j^t F_t^i = \delta_j^i, \quad \overset{\circ}{\nabla}_k F_j^i = 0,$$

where the symbol $\overset{\circ}{\nabla}$ denotes the covariant derivative with respect to Levi-Civita connection.

The scalar square of any tangent vector of such a space is opposite to the scalar square of its image by the structure. So, such a space can be considered as a product of a space with positive definite and a space with negative definite metrics. Moreover, as the structure has n (even dimension of the space) eigenvectors, they have to be null-vectors (isotropic).

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Since there is no way to define a conformal transformation of such a space just by exponential change of its metrics, we introduce the following transformation:

$$\overline{g}_{ji} = e^{2p} g_{ji}; \overline{F}_j^i = F_j^i; \overline{F}_{ji} = e^{2p} F_{ji}.$$

Then it is easy to obtain a metric connection (for $\overline{g}$) with torsion tensor $\overline{S}_{ji}^a = 2F_{ji}q^a$, where (*) $q^a = p_b F^{ab}$ and p_k is a gradient vector field, $\frac{\partial p}{\partial x^k}$. The coefficients of such a connection are

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + p_k \delta^i_j + p_j \delta^i_k - g_{jk} p^i + F^i_j q_k + F^i_k q_j + F_{jk} q^i. \tag{1.1}$$

Taking into account (*), such a connection is also an $F-$connection. We call such a connection a conformal connection on a hyperbolic Kählerian space. The author has investigated properties of this and similar connection and its curvature tensor in [4, 5, 6]. Also, some similar problems had been solved in [1, 2]. In [3, 7, 8, 9] we investigated properties of metric and $F-$connections, which were holomorphically semi-symmetric on anti-Kähler and product space. Some of these connections (namely, F and (F, g)) have invariants of curvature-type which are equal to conformal invariants of such spaces. $g-$holomorphically semi-symmetric connection had different invariants in both cases. But, both anti-Kähler and product spaces have symmetric structure tensors. As the structure of hyperbolic Kählerian space is skew-symmetric, its geometry is rather different. In our previous paper [10], we have found curvature-type invariant tensors of F and the $(F, g)-$holomorphically-structurally semi-symmetric connections. They are different between themselves and different from the curvature-type invariant of the connection (1.1) in this paper.

Here we are going to construct a curvature-type invariant of $g-$holomorphically-structurally semi-symmetric connection. This task is the most difficult one.

2 The curvature of $g-$holomorphically-structurally semi-symmetric connections

In our previous paper [10], we have constructed connections whose coefficients were consisting of the following addends: Christoffel symbols, $p_j \delta^i_k$ and $p^i g_{jk}$, where p_j is a gradient ("generator") and $q_j F^i_k$, $q^i F_{jk}$, where $q_j = F^a_j p_a$. We have found conditions for such connections to be metric and to be F-connection.

The connection which is metric, but not F-connection, according to these results, will have coefficients

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + p_j \delta^i_k - p^i g_{jk} - q_j F^i_k - q^i F_{jk}. \tag{2.1}$$

Its torsion will be

$$T^i_{jk} = p_j \delta^i_k - p_k \delta^i_j - q_j F^i_k + q_k F^i_j - 2q^i F_{jk}.$$

and we shall call such a connection a g-holomorphically-structurally semi-symmetric connection. The reasons for titles "holomorphically" and "semi-symmetric" are obvious. The adverb "structurally" is used because the last addend in (2.1), which is skew-symmetric, depends on the structure tensor and cannot be eliminated.

We can calculate components of the curvature tensor of the connection given by the coefficients (2.1). If we denote

$$S_{lj} = -p_l p_j + q_l q_j + \tfrac{1}{2} p_s p^s g_{lj},$$
$$S_{la} F^a_j = -p_l q_j + q_l p_j - \tfrac{1}{2} p_s p^s F_{lj},$$

and

$$2S_{ja} F^a_i = -2p_j q_i + 2q_j p_i - p_s p^s F_{ji};$$
$$2(p_i q_j - p_j q_i) = 2S_{ja} F^a_i + p_s p^s F_{ji}.$$

We obtain, after lowering the upper index

(2.2)

$$R_{ijkl} = K_{ijkl} + g_{ki} \overset{\circ}{\nabla}_l p_j - g_{li} \overset{\circ}{\nabla}_k p_j + g_{lj} \overset{\circ}{\nabla}_k p_i - g_{kj} \overset{\circ}{\nabla}_l p_i -$$
$$-F_{ki} \overset{\circ}{\nabla}_l q_j + F_{kj} \overset{\circ}{\nabla}_l q_i - F_{lj} \overset{\circ}{\nabla}_k q_i + F_{li} \overset{\circ}{\nabla}_k q_j + g_{ki} S_{lj} - g_{li} S_{kj} - g_{kj} S_{li} + g_{lj} S_{ki}$$
$$+F_{ki} S_{la} F^a_j - F_{kj} S_{la} F^a_i + F_{lj} S_{ka} F^a_i - F_{li} S_{ka} F^a_j + 2F_{kl} S_{ja} F^a_i + p_s p^s F_{ji} F_{kl}.$$

Now the tensor (2.2) should be skew-symmetric in first two indices. If we change their places, we shall obtain that there holds

(2.3)

$$R_{jikl} = K_{jikl} + g_{kj} \overset{\circ}{\nabla}_l p_i - g_{lj} \overset{\circ}{\nabla}_k p_i + g_{li} \overset{\circ}{\nabla}_k p_j - g_{ki} \overset{\circ}{\nabla}_l p_j$$
$$-F_{kj} \overset{\circ}{\nabla}_l q_i + F_{ki} \overset{\circ}{\nabla}_l q_j - F_{li} \overset{\circ}{\nabla}_k q_j + F_{lj} \overset{\circ}{\nabla}_k q_i + g_{kj} S_{li} - g_{lj} S_{ki} + g_{li} S_{kj} - g_{ki} S_{lj}$$
$$+F_{kj} S_{la} F^a_i - F_{ki} S_{la} F^a_j + F_{li} S_{ka} F^a_j - F_{lj} S_{ka} F^a_i + 2F_{kl} S_{ia} F^a_j + p_s p^s F_{ij} F_{kl}.$$

Every term from (2.2) has its pair in the expression (2.3) with opposite sign, if we take into account skew-symmetry of the tensor $S_{ia} F^a_j$.

We have proved that there holds the following

Lemma 2.1. *The curvature tensor of g−holomorphically-structurally semi-symmetric connection with gradient generator on hyperbolic Kählerian space is skew-symmetric in the first two indices.*

Now we want the components of the curvature tensor of this connection to be invariant under changing places of first and second pair of indices. If we change places of these pairs of indices, we shall obtain

$$R_{klij} = K_{klji} + g_{ki} \overset{\circ}{\nabla}_j p_l - g_{jk} \overset{\circ}{\nabla}_i p_l + g_{jl} \overset{\circ}{\nabla}_i p_k - g_{il} \overset{\circ}{\nabla}_k p_j$$
$$-F_{ik} \overset{\circ}{\nabla}_j q_l + F_{il} \overset{\circ}{\nabla}_j q_k - F_{jl} \overset{\circ}{\nabla}_i q_k + F_{jk} \overset{\circ}{\nabla}_i q_l$$
$$+g_{ik} S_{jl} - g_{jk} S_{il} + g_{jl} S_{ik} - g_{il} S_{jk} + F_{ik} S_{ja} F^a_l - F_{il} S_{ja} F^a_k$$
$$+F_{jl} S_{ia} F^a_k - F_{jk} S_{ia} F^a_l + 2F_{ij} S_{la} F^a_k + p_s p^s F_{lk} F_{ij}.$$

If we compare the upper relation with (2.2), we can notice that all the terms containing of just metric tensor and tensors $\overset{\circ}{\nabla} p$ and S are same. So, we can substract them and also the last members on both sides. The rest of members will give us

$$\begin{aligned}&-F_{ik}(\overset{\circ}{\nabla}_j q_l + \overset{\circ}{\nabla}_l q_j) + F_{il}(\overset{\circ}{\nabla}_j q_k + \overset{\circ}{\nabla}_k q_j) - F_{jl}(\overset{\circ}{\nabla}_i q_k + \overset{\circ}{\nabla}_k q_i)\\&+F_{jk}(\overset{\circ}{\nabla}_i q_l + \overset{\circ}{\nabla}_l q_i) + F_{ik}(S_{ja}F^a_l + S_{la}F^a_j) - F_{il}(S_{ja}F^a_k + S_{ka}F^a_j)\\&+F_{jl}(S_{ia}F^a_k + S_{ka}F^a_i) - F_{jk}(S_{ia}F^a_l + S_{la}F^a_i)\\&= 2F_{kl}S_{ja}F^a_i - 2F_{ij}S_{la}F^a_k.\end{aligned}$$

On the left side, every of last four members vanishes because the tensor $S_{ka}F^a_j$ is skew-symmetric. Then, there holds

$$\begin{aligned}&-F_{ik}(\overset{\circ}{\nabla}_j q_l + \overset{\circ}{\nabla}_l q_j) + F_{il}(\overset{\circ}{\nabla}_j q_k + \overset{\circ}{\nabla}_k q_j) - F_{jl}(\overset{\circ}{\nabla}_i q_k + \overset{\circ}{\nabla}_k q_i)\\&+F_{jk}(\overset{\circ}{\nabla}_i q_l + \overset{\circ}{\nabla}_l q_i) = 2F_{kl}S_{ja}F^a_i - 2F_{ij}S_{la}F^a_k.\end{aligned}$$

We shall transvect the upper equation by F^{ki} and obtain

$$\begin{aligned}&-n(\overset{\circ}{\nabla}_j q_l + \overset{\circ}{\nabla}_l q_j) + (\overset{\circ}{\nabla}_j q_l + \overset{\circ}{\nabla}_l q_j) - F_{jl}(\overset{\circ}{\nabla}_i p_a F^a_k F^{ki} + \overset{\circ}{\nabla}_k p_a F^a_i F^{ki})\\&+(\overset{\circ}{\nabla}_j q_l + \overset{\circ}{\nabla}_l q_j) = 2F_{kl}F^{ki}S_{ja}F^a_i - 2F_{ij}S_{la}F^a_k F^{ki}.\end{aligned}$$

From the upper equation, we obtain

$$\begin{aligned}&(2-n)(\overset{\circ}{\nabla}_j q_l + \overset{\circ}{\nabla}_l q_j) - F_{jl}(\overset{\circ}{\nabla}_i p^i - \overset{\circ}{\nabla}_k p^k)\\&= -2S_{ja}F^a_l - 2S_{la}F^a_j.\end{aligned}$$

or

$$\overset{\circ}{\nabla}_j q_l + \overset{\circ}{\nabla}_l q_j = \frac{2}{n-2}(S_{ja}F^a_l + S_{la}F^a_j) = 0,$$

since the tensor $S_{la}F^a_j$ is skew-symmetric. So, we have proved

Lemma 2.2. *If the curvature tensor of holomorphically-structurally semi-symmetric connection on a hyperbolic Kählerian space is invariant under changing places of the first and second pair of indices, then there holds*

$$\overset{\circ}{\nabla}_j q_l = -\overset{\circ}{\nabla}_l q_j. \tag{2.4}$$

In the future calculations, we shall suppose that the relation (2.4) is satisfied.

3 Two ways to relate $\overset{\circ}{\nabla}_s p^s$ and $p_s p^s$

We shall use the formula (2.2). After contraction of this formula by g^{il}, we obtain

$$\begin{aligned}&R_{jk} = K_{jk} - (n-3)\overset{\circ}{\nabla}_k p_j - g_{kj}\overset{\circ}{\nabla}_s p^s - F^a_k F^b_j \overset{\circ}{\nabla}_a p_b -\\&-(n-2)S_{kj} - g_{kj}S^s_s - p_s p^s g_{kj}.\end{aligned}$$

We shall express $\overset{\circ}{\nabla}_k\, p_j$ from the last formula and obtain

$$\begin{aligned}\overset{\circ}{\nabla}_k\, p_j &= \tfrac{1}{n-3}[K_{jk} - R_{jk} - g_{kj}\overset{\circ}{\nabla}_s\, p^s - F^a_k F^b_j \overset{\circ}{\nabla}_a\, p_b \\ &-(n-2)S_{kj} - g_{kj}S^s_s - p_s p^s g_{kj}].\end{aligned} \tag{3.1}$$

This formula is recurrent. There will hold

$$\begin{aligned}\overset{\circ}{\nabla}_a\, p_b &= \tfrac{1}{n-3}[K_{ba} - R_{ba} - g_{ab}\overset{\circ}{\nabla}_s\, p^s - F^r_a F^s_b \overset{\circ}{\nabla}_r\, p_s \\ &-(n-2)S_{ab} - g_{ab}S^s_s - p_s p^s g_{ab}]\end{aligned}$$

and, consequently

$$\begin{aligned}-F^a_k F^b_j \overset{\circ}{\nabla}_a\, p_b &= \tfrac{1}{n-3}(R_{ba} - K_{ba})F^a_k F^b_j + \tfrac{1}{n-3}g_{ab}F^a_k F^b_j \overset{\circ}{\nabla}_s\, p^s \\ &+\tfrac{1}{n-3}F^r_a F^s_b F^a_k F^b_j \overset{\circ}{\nabla}_r\, p_s + \tfrac{n-2}{n-3}S_{ab}F^a_k F^b_j \\ &+F^a_k F^b_j g_{ab}\tfrac{S^s_s}{n-3} + \tfrac{p_s p^s}{n-3}g_{ab}F^a_k F^b_j \\ &= \tfrac{1}{n-3}(R_{ba} - K_{ba})F^a_k F^b_j - \tfrac{1}{n-3}g_{kj}\overset{\circ}{\nabla}_s\, p^s + \tfrac{1}{n-3}\overset{\circ}{\nabla}_k\, p_j \\ &-\tfrac{n-2}{n-3}S_{kj} - \tfrac{1}{n-3}g_{kj}S^s_s - \tfrac{p_s p^s}{n-3}g_{kj}.\end{aligned}$$

We have used the equality $S_{ab}F^a_k F^b_j = -S_{kj}$, which yields from the fact of skew-symmetry of the tensor $S_{ja}F^a_i$. If we substitute the upper equality into (3.1), we obtain

$$\begin{aligned}\overset{\circ}{\nabla}_k\, p_j &= \tfrac{1}{n-3}(K_{jk} - R_{jk}) - \tfrac{1}{n-3}g_{kj}\overset{\circ}{\nabla}_s\, p^s + \tfrac{1}{(n-3)^2}(R_{ba} - K_{ba})F^a_k F^b_j \\ &-\tfrac{1}{(n-3)^2}g_{kj}\overset{\circ}{\nabla}_s\, p^s + \tfrac{1}{(n-3)^2}\overset{\circ}{\nabla}_k\, p_j - \tfrac{n-2}{(n-3)^2}S_{kj} - \tfrac{1}{(n-3)^2}g_{kj}S^s_s \\ &-\tfrac{p_s p^s}{(n-3)^2}g_{kj} - \tfrac{n-2}{n-3}S_{kj} - \tfrac{1}{n-3}g_{kj}S^s_s - \tfrac{p_s p^s}{n-3}g_{kj}.\end{aligned}$$

Here we shall also use the relation which we can obtain from upper equality, and get

$$\begin{aligned}\overset{\circ}{\nabla}_k\, p_j = \;& \tfrac{n-3}{(n-2)(n-4)}(K_{jk} - R_{jk}) \\ &-\tfrac{1}{(n-2)(n-4)}(K_{ba} - R_{ba})F^a_k F^b_j \\ &-\tfrac{1}{n-4}g_{kj}\overset{\circ}{\nabla}_s\, p^s - \tfrac{n-2}{n-4}S_{kj} - \tfrac{1}{n-4}g_{kj}S^s_s - \tfrac{1}{n-4}p_s p^s g_{kj}.\end{aligned} \tag{3.2}$$

If we transvect (3.2) by g^{jk}, we obtain

$$\begin{aligned}\overset{\circ}{\nabla}_s\, p^s = \;& [\tfrac{n-3}{(n-2)(n-4)} + \tfrac{1}{(n-2)(n-4)}](K - R) - \tfrac{n}{n-4}\overset{\circ}{\nabla}_s\, p^s \\ &-\tfrac{n-2}{n-4}S^s_s - \tfrac{n}{n-4}S^s_s - \tfrac{n}{n-4}p_s p^s.\end{aligned}$$

Since $S^s_s = \frac{n-4}{2}p_s p^s$, we obtain that there holds

$$\overset{\circ}{\nabla}_s\, p^s = \frac{1}{2(n-2)}(K - R) - \frac{n-2}{2}p_s p^s. \tag{3.3}$$

There is also another way to find the relationship between $\overset{\circ}{\nabla}_s p^s$ and $p_s p^s$ and, using (3.1), express them through some scalar functions which are not depending on the generator. As we just have one relationship between these scalar functions, we really need another one. For this purpose, we shall introduce two new tensors, namely

$$\overline{R}_{jk} = R_{ijkl} F^{il}; \quad \overline{K}_{jk} = K_{ijkl} F^{il}.$$

Then, we shall transvect the formula (2.2) by F^{il} and obtain

$$\overline{R}_{jk} = \overline{K}_{jk} + (n-4) \overset{\circ}{\nabla}_k q_j - F_{kj} \overset{\circ}{\nabla}_s p^s - (n-2) S_{ka} F^a_j + F_{kj} S^s_s + p_s p^s F_{kj},$$

as the tensor $S_{ka} F^a_j$ is skew-symmetric and as there holds the upper relation between S^s_s and $p_s p^s$; then there will hold

$$\overset{\circ}{\nabla}_k q_j = \frac{\overline{R}_{jk} - \overline{K}_{jk}}{n-4} + \frac{\overset{\circ}{\nabla}_s p^s}{n-4} F_{kj} + \frac{n-2}{n-4} S_{ka} F^a_j - \frac{n-2}{2(n-4)} p_s p^s F_{kj}.$$

From the upper equality, we get

$$\overset{\circ}{\nabla}_k p_j = \frac{\overline{R}_{tk} - \overline{K}_{tk}}{n-4} F^t_j - \frac{\overset{\circ}{\nabla}_s p^s}{n-4} g_{kj} + \frac{n-2}{n-4} S_{kj} + \frac{n-2}{2(n-4)} p_s p^s g_{kj}. \tag{3.4}$$

If we transvect this equality by g^{kj}, we obtain, after introducing the following notation

$$\overset{\approx}{R} = \overline{R}_{jk} F^{kj}; \quad \overset{\approx}{K} = \overline{K}_{jk} F^{kj}$$

that there holds

$$\overset{\circ}{\nabla}_s p^s = \frac{\overset{\approx}{R} - \overset{\approx}{K}}{n-4} - \frac{n}{n-4} \overset{\circ}{\nabla}_s p^s + \frac{n-2}{n-4} S^s_s + \frac{n(n-2)}{2(n-4)} p_s p^s.$$

We shall get on the left side

$$\frac{2(n-2)}{n-4} \overset{\circ}{\nabla}_s p^s$$

and on the right side

$$\frac{\overset{\approx}{R} - \overset{\approx}{K}}{n-4} + \frac{(n-2)}{2} p_s p^s + \frac{n(n-2)}{2(n-4)} p_s p^s = \frac{\overset{\approx}{R} - \overset{\approx}{K}}{n-4} + \frac{(n-2)^2}{n-4} p_s p^s.$$

So, we obtain that there holds

$$\overset{\circ}{\nabla}_s p^s = \frac{\overset{\approx}{R} - \overset{\approx}{K}}{2(n-2)} + \frac{(n-2)}{2} p_s p^s. \tag{3.5}$$

Comparing this with (3.3), we shall obtain that there holds

$$\frac{1}{2(n-2)} (K - R + \overset{\approx}{K} - \overset{\approx}{R}) = (n-2) p_s p^s$$

and consequently,

$$p_s p^s = \frac{1}{2(n-2)^2} (K - R + \overset{\approx}{K} - \overset{\approx}{R}). \tag{3.6}$$

Substituting this result into (3.5), we infer

$$\overset{\circ}{\nabla}_s p^s = \frac{1}{4(n-2)} (K - R - \overset{\approx}{K} + \overset{\approx}{R}). \tag{3.7}$$

4 The invariant of $g-$holomorphically-structurally semi-symmetric connections

From (3.2), it is easy to get that there holds

$$\overset{\circ}{\nabla}_k p_j = \frac{n-3}{(n-2)(n-4)}(K_{jk} - R_{jk}) - \frac{1}{(n-2)(n-4)}(K_{ba} - R_{ba})F^a_k F^b_j$$
$$-\frac{1}{n-4}g_{kj}\overset{\circ}{\nabla}_s p^s - \frac{n-2}{n-4}S_{kj} - \frac{n-2}{2(n-4)}p_s p^s g_{kj}.$$

If we use the expression (3.6), we obtain

$$\begin{aligned}\overset{\circ}{\nabla}_k p_j &= \frac{n-3}{(n-2)(n-4)}(K_{jk} - R_{jk}) \\ &-\frac{1}{(n-2)(n-4)}(K_{ba} - R_{ba})F^a_k F^b_j - \frac{1}{n-4}g_{kj}\overset{\circ}{\nabla}_s p^s \\ &-\frac{n-2}{n-4}S_{kj} - \frac{1}{4(n-2)(n-4)}(K - R + \tilde{\tilde{K}} - \tilde{\tilde{R}})g_{kj},\end{aligned} \tag{4.1}$$

where $\overset{\circ}{\nabla}_k p_j$ has been given by another expression, (3.4). We can give another form to (3.4), taking into account (3.6):

$$\begin{aligned}\overset{\circ}{\nabla}_k p_j = &\frac{\overline{R}_{tk} - \overline{K}_{tk}}{n-4}F^t_j - \frac{\overset{\circ}{\nabla}_s p^s}{n-4}g_{kj} \\ &+\frac{1}{4(n-2)(n-4)}(K - R + \tilde{\tilde{K}} - \tilde{\tilde{R}})g_{kj} + \frac{n-2}{n-4}S_{kj}.\end{aligned} \tag{4.2}$$

Now, we are getting

$$\begin{aligned}S_{kj} &= \frac{n-3}{2(n-2)^2}(K_{jk} - R_{jk}) + \frac{1}{2(n-2)}(\overline{K}_{tk} - \overline{R}_{tk})F^t_j \\ &-\frac{1}{2(n-2)^2}(K_{ba} - R_{ba})F^a_k F^b_j - \frac{1}{4(n-2)^2}(K - R + \tilde{\tilde{K}} - \tilde{\tilde{R}})g_{kj}.\end{aligned} \tag{4.3}$$

Now we can calculate $\overset{\circ}{\nabla}_k p_j$. By using the expressions (4.2), (4.3), (4.1), (3.6) and (3.7), we obtain that there holds:

$$\overset{\circ}{\nabla}_k p_j = \frac{n-3}{2(n-2)(n-4)}(K_{jk} - R_{jk}) - \frac{1}{2(n-4)}(\overline{K}_{tk} - \overline{R}_{tk})F^t_j$$
$$-\frac{1}{2(n-2)(n-4)}(K_{ba} - R_{ba})F^a_k F^b_j - \frac{1}{4(n-2)(n-4)}(K - R - \tilde{\tilde{K}} + \tilde{\tilde{R}})g_{kj}.$$

From the upper equality, we easily obtain

$$\overset{\circ}{\nabla}_k q_j = F^s_j \overset{\circ}{\nabla}_k p_s = \frac{n-3}{2(n-2)(n-4)}(K_{sk} - R_{sk})F^s_j - \frac{1}{2(n-4)}(\overline{K}_{jk} - \overline{R}_{jk})$$
$$-\frac{1}{2(n-2)(n-4)}(K_{ja} - R_{ja})F^a_k + \frac{1}{4(n-2)(n-4)}(K - R - \tilde{\tilde{K}} + \tilde{\tilde{R}})F_{kj}.$$

Also, we shall need the expression for $S_{ka}F^a_j$:

$$\begin{aligned}S_{ka}F^a_j &= \frac{n-3}{2(n-2)^2}(K_{sk} - R_{sk})F^s_j + \frac{1}{2(n-2)}(\overline{K}_{jk} - \overline{R}_{jk}) \\ &-\frac{1}{2(n-2)^2}(K_{ja} - R_{ja})F^a_k + \frac{1}{4(n-2)^2}(K - R + \tilde{\tilde{K}} - \tilde{\tilde{R}})F_{kj}.\end{aligned} \tag{4.4}$$

From (2.2), it is obvious that we can express the curvature tensor of such connection in the form

$$
\begin{aligned}
R_{ijkl} = \;& K_{ijkl} + g_{ki}p_{lj} - g_{li}p_{kj} + g_{lj}p_{ki} - g_{kj}p_{li} \\
& -F_{ki}q_{lj} + F_{kj}q_{li} - F_{lj}q_{ki} + F_{li}q_{kj} + 2F_{kl}S_{ja}F_i^a + p_s p^s F_{ji}F_{kl},
\end{aligned} \tag{4.5}
$$

where the new abbreviations denote

$$
p_{lj} = \overset{\circ}{\nabla}_l\, p_j + S_{lj} \;\; \text{(symmetric)}
$$

$$
q_{lj} = \overset{\circ}{\nabla}_l\, q_j - S_{la}F_j^a \;\; \text{(skew-symmetric)}.
$$

Then, we can easily obtain

$$
\begin{aligned}
p_{kj} = \;& \tfrac{1}{(n-2)^2(n-4)}[(n-3)^2(K_{jk} - R_{jk}) \\
& -(n-2)(\overline{K}_{tk} - \overline{R}_{tk})F_j^t - (n-3)(K_{ba} - R_{ba})F_k^a F_j^b \\
& -(\tfrac{n-3}{2}(K - R) - \tfrac{1}{2}(\widetilde{\widetilde{K}} - \widetilde{\widetilde{R}}))g_{jk}]
\end{aligned} \tag{4.6}
$$

and, consequently,

$$
\begin{aligned}
q_{kj} = \;& \tfrac{1}{(n-2)^2(n-4)}[(n-3)(K_{sk} - R_{sk})F_j^s - (n-2)(n-3)(\overline{K}_{jk} \\
& -\overline{R}_{jk}) - (K_{ja} - R_{ja})F_k^a + (\tfrac{K-R}{2} - \tfrac{n-3}{2}(\widetilde{\widetilde{K}} - \widetilde{\widetilde{R}}))F_{kj}].
\end{aligned} \tag{4.7}
$$

Moreover, we can easily calculate the sum of the last two terms on the right-hand side of (4.5). From (4.4) and (3.6), we obtain

$$
\begin{aligned}
& (2S_{ja}F_i^a + p_s p^s F_{ji})F_{kl} \\
& = \tfrac{1}{(n-2)^2}[(n-3)(K_{sj} - R_{sj})F_i^s F_{kl} + (n-2)(\overline{K}_{ij} - \overline{R}_{ij})F_{kl} \\
& -(K_{ia} - R_{ia})F_j^a F_{kl} + (K - R + \widetilde{\widetilde{K}} - \widetilde{\widetilde{R}})F_{ji}F_{kl}].
\end{aligned} \tag{4.8}
$$

If we substitute (4.6), (4.7) and (4.8) into (4.5), we shall obtain that there holds

$$
\begin{aligned}
R_{ijkl} = \;& K_{ijkl} + \tfrac{g_{ki}}{(n-2)^2(n-4)}[(n-3)^2(K_{jl} - R_{jl}) - (n-2)(\overline{K}_{tl} - \overline{R}_{tl})F_j^t \\
& -(n-3)(K_{ba} - R_{ba})F_l^a F_j^b - (\tfrac{n-3}{2}(K - R) - \tfrac{1}{2}(\widetilde{\widetilde{K}} - \widetilde{\widetilde{R}}))g_{lj}] \\
& -\tfrac{g_{li}}{(n-2)^2(n-4)}[(n-3)^2(K_{jk} - R_{jk}) - (n-2)(\overline{K}_{tk} - \overline{R}_{tk})F_j^t \\
& -(n-3)(K_{ba} - R_{ba})F_k^a F_j^b - (\tfrac{n-3}{2}(K - R) - \tfrac{1}{2}(\widetilde{\widetilde{K}} - \widetilde{\widetilde{R}}))g_{kj}] \\
& +\tfrac{g_{lj}}{(n-2)^2(n-4)}[(n-3)^2(K_{ik} - R_{ik}) - (n-2)(\overline{K}_{tk} - \overline{R}_{tk})F_i^t \\
& -(n-3)(K_{ba} - R_{ba})F_k^a F_i^b - (\tfrac{n-3}{2}(K - R) - \tfrac{1}{2}(\widetilde{\widetilde{K}} - \widetilde{\widetilde{R}}))g_{ki}] \\
& -\tfrac{g_{kj}}{(n-2)^2(n-4)}[(n-3)^2(K_{il} - R_{il}) - (n-2)(\overline{K}_{tl} - \overline{R}_{tl})F_i^t \\
& -(n-3)(K_{ba} - R_{ba})F_l^a F_i^b - (\tfrac{n-3}{2}(K - R) - \tfrac{1}{2}(\widetilde{\widetilde{K}} - \widetilde{\widetilde{R}}))g_{li}] \\
& -\tfrac{F_{ki}}{(n-2)^2(n-4)}[(n-3)(K_{sl} - R_{sl})F_j^s - (n-2)(n-3)(\overline{K}_{jl} - \overline{R}_{jl})
\end{aligned}
$$

$$-(K_{ja}-R_{ja})F^a_l+(\tfrac{K-R}{2}-\tfrac{n-3}{2}(\widetilde{\widetilde{K}}-\widetilde{\widetilde{R}}))F_{lj}]$$
$$+\tfrac{F_{kj}}{(n-2)^2(n-4)}[(n-3)(K_{sl}-R_{sl})F^s_i-(n-2)(n-3)(\overline{K}_{il}-\overline{R}_{il})$$
$$-(K_{ia}-R_{ia})F^a_l+(\tfrac{K-R}{2}-\tfrac{n-3}{2}(\widetilde{\widetilde{K}}-\widetilde{\widetilde{R}}))F_{li}]$$
$$-\tfrac{F_{lj}}{(n-2)^2(n-4)}[(n-3)(K_{sk}-R_{sk})F^s_i-(n-2)(n-3)(\overline{K}_{ik}-\overline{R}_{ik})$$
$$-(K_{ia}-R_{ia})F^a_k+(\tfrac{K-R}{2}-\tfrac{n-3}{2}(\widetilde{\widetilde{K}}-\widetilde{\widetilde{R}}))F_{ki}]$$
$$+\tfrac{F_{li}}{(n-2)^2(n-4)}[(n-3)(K_{sk}-R_{sk})F^s_j-(n-2)(n-3)(\overline{K}_{jk}-\overline{R}_{jk})$$
$$-(K_{ja}-R_{ja})F^a_k+(\tfrac{K-R}{2}-\tfrac{n-3}{2}(\widetilde{\widetilde{K}}-\widetilde{\widetilde{R}}))F_{kj}]$$
$$+\tfrac{1}{(n-2)^2}[(n-3)(K_{sj}-R_{sj})F^s_i+(n-2)(\overline{K}_{ij}-\overline{R}_{ij})$$
$$-(K_{ia}-R_{ia})F^a_j+(K-R+\widetilde{\widetilde{K}}-\widetilde{\widetilde{R}})F_{ji}]F_{kl}.$$

If we divide the members of such formula (Levi-Civita and g connection) on different sides, we get

$$\begin{aligned}
&R_{ijkl}+\tfrac{1}{(n-2)^2(n-4)}[(n-3)^2(g_{ki}R_{lj}-g_{li}R_{kj}+g_{lj}R_{ik}-g_{kj}R_{il})\\
&-(n-2)(g_{ki}\overline{R}_{tl}F^t_j-g_{li}\overline{R}_{tk}F^t_j+g_{lj}\overline{R}_{tk}F^t_i-g_{kj}\overline{R}_{tl}F^t_i)\\
&-(n-3)R_{ba}(F^a_lF^b_jg_{ki}-F^a_kF^b_jg_{li}+F^a_kF^b_lg_{lj}-F^a_lF^b_ig_{kj})\\
&-((n-3)R-\widetilde{\widetilde{R}})(g_{ki}g_{lj}-g_{li}g_{kj})\\
&-(n-3)(F_{ki}R_{sl}F^s_j-F_{kj}R_{sl}F^s_i+F_{lj}R_{sk}F^s_i-F_{li}R_{sk}F^s_j)\\
&+(n-2)(n-3)(F_{ki}\overline{R}_{jl}-F_{kj}\overline{R}_{il}+F_{lj}\overline{R}_{ik}-F_{li}\overline{R}_{jk})\\
&+(F_{ki}R_{ja}F^a_l-F_{kj}R_{ia}F^a_l+F_{lj}R_{ia}F^a_k-F_{li}R_{ja}F^a_i)\\
&+(R-(n-3)\widetilde{\widetilde{R}})(F_{li}F_{kj}-F_{ki}F_{lj})]\\
&+\tfrac{1}{(n-2)^2}[(n-3)R_{sj}F^s_i+(n-2)\overline{R}_{ij}-R_{ia}F^a_j+(R+\widetilde{\widetilde{R}})F_{ji}]\\
&=K_{ijkl}+\tfrac{1}{(n-2)^2(n-4)}[(n-3)^2(g_{ki}K_{lj}-g_{li}K_{kj}+g_{lj}K_{ik}-g_{kj}K_{il})\\
&-(n-2)(g_{ki}\overline{K}_{tl}F^t_j-g_{li}\overline{K}_{tk}F^t_j+g_{lj}\overline{K}_{tk}F^t_i-g_{kj}\overline{K}_{tl}F^t_i)\\
&-(n-3)K_{ba}(F^a_lF^b_jg_{ki}-F^a_kF^b_jg_{li}+F^a_kF^b_lg_{lj}-F^a_lF^b_ig_{kj})\\
&-((n-3)K-\widetilde{\widetilde{K}})(g_{ki}g_{lj}-g_{li}g_{kj})\\
&-(n-3)(F_{ki}K_{sl}F^s_j-F_{kj}K_{sl}F^s_i+F_{lj}K_{sk}F^s_i-F_{li}K_{sk}F^s_j)\\
&+(n-2)(n-3)(F_{ki}\overline{K}_{jl}-F_{kj}\overline{K}_{il}+F_{lj}\overline{K}_{ik}-F_{li}\overline{K}_{jk})\\
&+(F_{ki}K_{ja}F^a_l-F_{kj}K_{ia}F^a_l+F_{lj}K_{ia}F^a_k-F_{li}K_{ja}F^a_i)\\
&+(K-(n-3)\widetilde{\widetilde{K}})(F_{li}F_{kj}-F_{ki}F_{lj})]\\
&+\tfrac{1}{(n-2)^2}[(n-3)K_{sj}F^s_i+(n-2)\overline{K}_{ij}-K_{ia}F^a_j+(K+\widetilde{\widetilde{K}})F_{ji}].
\end{aligned}\tag{4.9}$$

Hence, we have proved that there holds the following:

Theorem 4.1. *If the generator of a $g-$holomorhically-structurally semi-symmetric connection is a gradient and if its curvature tensor is invariant under changing places of the first and the second pair of indices, then the tensor on the left-hand side of* (4.9) *does not depend on the choice of the generator.*

5 On the first Bianchi identity

We suppose that the curvature tensor of a $g-$holomorphically-structurally semi-symmetric connection on such a kind of space satisfies the first Bianchi identity. We shall use it and will obtain, after dividing by 2,

$$\begin{aligned}(5.1)\qquad 0 = \; & -F_{ki} \overset{\circ}{\nabla}_l q_j + F_{kj} \overset{\circ}{\nabla}_l q_i - F_{lj} \overset{\circ}{\nabla}_k q_i + F_{li} \overset{\circ}{\nabla}_k q_j \\ & -F_{kl} \overset{\circ}{\nabla}_j q_i + F_{ji} \overset{\circ}{\nabla}_l q_k + F_{ki} S_{la} F^a_j - 2F_{kj} S_{la} F^a_i \\ & +2F_{lj} S_{ka} F^a_i - F_{li} S_{ka} F^a_j + 2F_{kl} S_{ja} F^a_i - F_{ji} S_{la} F^a_k \\ & +\tfrac{1}{2} p_s p^s (F_{ji} F_{kl} + F_{ki} F_{lj} + F_{li} F_{jk}).\end{aligned}$$

If we transvect (5.1) by F^{ik}, taking into account skew-symmetry of the tensor $S_{ja} F^a_k$, we obtain

$$\begin{aligned}(5.2)\qquad 0 = \; & (4-n) \overset{\circ}{\nabla}_l q_j + F_{lj} \overset{\circ}{\nabla}_s p^s + (n-6) S_{la} F^a_j \\ & -2F_{lj} S^s_s + \tfrac{n-2}{2} p_s p^s F_{lj}.\end{aligned}$$

If we now transvect the equality (5.2) with F^{jl}, we obtain

$$0 = (n-4) \overset{\circ}{\nabla}_j p^j + n \overset{\circ}{\nabla}_s p^s - 2n S^s_s - (n-6) S^s_s + n \frac{n-2}{2} p_s p^s.$$

Then, taking into account that $S^s_s = \frac{n-4}{2} p_s p^s$, we obtain that there holds

$$2(n-2) \overset{\circ}{\nabla}_s p^s + \frac{-3(n-2)(n-4)}{2} p_s p^s + \frac{n(n-2)}{2} p_s p^s = 0$$

or, since $n > 2$,

$$2 \overset{\circ}{\nabla}_s p^s - \frac{3n}{2} p_s p^s + 6 p_s p^s + \frac{n}{2} p_s p^s = 0$$

and hence $\overset{\circ}{\nabla}_s p^s = \frac{n-6}{2} p_s p^s$. Then, if we use (3.3), we obtain

$$(5.3)\qquad p_s p^s = \frac{1}{2(n-2)(n-4)} (K - R)$$

and, consequently

$$(5.4)\qquad \overset{\circ}{\nabla}_s p^s = \frac{n-6}{4(n-2)(n-4)} (K - R).$$

After expanding the relation (5.2), we obtain

$$\overset{\circ}{\nabla}_l q_j = \frac{1}{n-4}[F_{lj} \overset{\circ}{\nabla}_s p^s - 2F_{lj}S^s_s + (n-6)S_{la}F^a_j + \frac{n-2}{2}p_s p^s F_{lj}].$$

By using the relations (5.3) and (5.4) and taking into account that $S^s_s = \frac{n-4}{2}p_s p^s$, we get

$$\begin{aligned}\overset{\circ}{\nabla}_l q_j = \; & \tfrac{1}{n-4}[\tfrac{n-6}{4(n-2)(n-4)}(K-R)F_{lj} - \tfrac{K-R}{2(n-2)}F_{lj} \\ & +(n-6)S_{la}F^a_j + \tfrac{K-R}{4(n-4)}F_{lj}].\end{aligned}$$

The upper relation yields $\overset{\circ}{\nabla}_l q_j = \frac{n-6}{n-4}S_{la}F^a_j$, which leads to $\overset{\circ}{\nabla}_l p_j = \frac{n-6}{n-4}S_{lj}$. Then, there will also hold

$$\begin{aligned} p_{lj} = & \overset{\circ}{\nabla}_l p_j + S_{lj} = \tfrac{2(n-5)}{n-4}S_{lj}; \\ q_{lj} = & \overset{\circ}{\nabla}_l q_j - S_{la}F^a_j = -\tfrac{2}{n-4}S_{la}F^a_j. \end{aligned}$$

After substituting these equations into the expression (4.5), there holds

$$\begin{aligned} R_{ijkl} = \; & K_{ijkl} + \tfrac{2(n-5)}{(n-4)}(g_{ki}S_{lj} - g_{li}S_{kj} + g_{lj}S_{ki} - g_{kj}S_{li}) \\ & + \tfrac{2}{(n-4)}(F_{ki}S_{la}F^a_j - F_{li}S_{ka}F^a_j + F_{lj}S_{ka}F^a_i - F_{kj}S_{la}F^a_i) \\ & + 2F_{kl}S_{ja}F^a_i + \tfrac{K-R}{2(n-2)(n-4)}F_{ji}F_{kl}. \end{aligned} \tag{5.5}$$

Now we shall transvect the upper equality by g^{il} and get

$$\begin{aligned} R_{jk} = \; & K_{jk} + \tfrac{2(n-5)}{n-4}(S_{kj} - nS_{kj} + S_{kj} - g_{kj}S^s_s] \\ & + \tfrac{2}{(n-4)}[S_{la}F^l_k F^a_j - F_{kj}S_{la}F^{la} + F_{lj}S_{ka}F^{la} - F_{li}S_{ka}F^a_j g^{li}] \\ & + 2F^i_k F^a_i S_{ja} - \tfrac{1}{2(n-2)(n-4)}(K-R)g_{kj}. \end{aligned}$$

If we take into account skew-symmetry of the structure tensor, symmetry of the tensor S_{kj}, the value of S^s_s and the equality (5.3), we shall obtain

$$\begin{aligned} R_{jk} = \; & K_{jk} + [-\tfrac{2(n-2)(n-5)}{n-4} - \tfrac{4}{n-4} + 2]S_{kj} \\ & - \tfrac{n-5}{2(n-2)(n-4)}(K-R)g_{kj} - \tfrac{1}{2(n-2)(n-4)}(K-R)g_{kj}. \end{aligned}$$

The expression in the first parentheses will be

$$\Big(-\frac{2(n-2)(n-5)}{n-4} - \frac{4}{n-4} + 2\Big)S_{kj} = \frac{-2(n^2-8n+16)}{n-4}S_{kj} = -2(n-4)S_{kj}.$$

So, from the upper equation there holds $R_{jk} = K_{jk} - \frac{K-R}{2(n-2)}g_{kj} - 2(n-4)S_{kj}$ and, consequently

$$\begin{aligned} S_{kj} = \; & \tfrac{K_{jk}-R_{jk}}{2(n-4)} - \tfrac{K-R}{4(n-2)(n-4)}g_{kj}; \\ S_{ka}F^a_j = \; & \tfrac{K_{ak}-R_{ak}}{2(n-4)}F^a_j + \tfrac{K-R}{4(n-2)(n-4)}F_{kj}. \end{aligned}$$

Now, after substituting the last two relations into (5.5), there will hold:

$$
\begin{aligned}
&R_{ijkl} + \tfrac{n-5}{(n-4)^2}[g_{ki}R_{lj} - g_{li}R_{kj} + g_{lj}R_{ki} - g_{kj}R_{li} - \tfrac{n-4}{n-2}R(g_{ki}g_{lj} - g_{li}g_{kj})] \\
&+\tfrac{1}{(n-4)^2}[R_{al}F^a_jF_{ki} - R_{ak}F^a_jF_{li} + R_{ak}F^a_iF_{lj} - R_{al}F^a_iF_{lj} \\
&+\tfrac{R}{n-2}(F_{ki}F_{lj} - F_{kj}F_{li})] + \tfrac{1}{n-4}(R_{aj}F^a_iF_{kl} + \tfrac{1}{(n-2)}RF_{ji}F_{kl}) \\
&\quad = K_{ijkl} + \tfrac{n-5}{(n-4)^2}[g_{ki}K_{lj} - g_{li}K_{kj} + g_{lj}K_{ki} - g_{kj}K_{li} \\
&-\tfrac{n-4}{n-2}K(g_{ki}g_{lj} - g_{li}g_{kj})] + \tfrac{1}{(n-4)^2}[K_{al}F^a_jF_{ki} - K_{ak}F^a_jF_{li} + K_{ak}F^a_iF_{lj} \\
&-K_{al}F^a_iF_{lj} + \tfrac{K}{n-2}(F_{ki}F_{lj} - F_{kj}F_{li})] + \tfrac{1}{n-4}(K_{aj}F^a_iF_{kl} + \tfrac{1}{(n-2)}KF_{ji}F_{kl}).
\end{aligned} \tag{5.6}
$$

Theorem 5.1. *If the generator of a $g-$holomorphically-structurally semi-symmetric connection on a hyperbolic Kählerian space is a gradient and if its curvature tensor, besides of invariance under changing places of the first and the second pair of indices, satisfies the first Bianchi identity, then the tensor on the left-hand side of* (5.6) *does not depend on the choice of the generator.*

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