Minimal timelike surfaces in a certain homogeneous Lorentzian 3-manifold (II)

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Abstract. The 2-parameter family of certain homogeneous Lorentzian 3-manifolds which includes Minkowski 3-space and anti-de Sitter 3-space, is considered. Each homogeneous Lorentzian 3-manifold in the 2-parameter family has a solvable Lie group structure with left invariant metric. A generalized integral representation formula for minimal timelike surfaces in the homogeneous Lorentzian 3-manifolds is obtained. The normal Gauss map of minimal timelike surfaces and its harmonicity are discussed.

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Introduction

In [5], the author considered the 2-parameter family of homogeneous Lorentzian 3-manifolds $(\mathbb{R}^3(x^0, x^1, x^2), g_{(\mu_1,\mu_2)})$ with Lorentzian metric

$$g_{(\mu_1,\mu_2)} = -(dx^0)^2 + e^{-2\mu_1 x^0}(dx^1)^2 + e^{-2\mu_2 x^0}(dx^2)^2.$$

Every homogeneous Lorentzian 3-manifold in this family can be represented as a solvable matrix Lie group with left invariant metric. This family of homogeneous Lorentzian 3-manifolds includes Minkowski 3-space $\mathbb{E}^3_1$, de Sitter 3-space $\mathbb{S}^3_1(c^2)$ of constant sectional curvature c^2, and $\mathbb{S}^2_1(c^2)\times\mathbb{E}^1$, the direct product of de Sitter 2-space $\mathbb{S}^2_1(c^2)$ of constant curvature c^2 and the real line $\mathbb{E}^1$. (In the family, only Minkowski 3-space and de Sitter 3-space have constant sectional curvature.) These three spaces may be considered as Lorentzian counterparts of Euclidean 3-space $\mathbb{E}^3$, hyperbolic 3-space $\mathbb{H}^3(-c^2)$, and the direct product $\mathbb{H}^2(-c^2)\times\mathbb{E}^1$, respectively, of Thurston's eight model geometries [7]. In [5], the author obtained a generalized integral representation formula that includes Weierstraß representation formula for minimal timelike surfaces in Minkowski 3-space studied independently by Inoguchi-Toda [2] and by the author [4], and Weierstraß representation formula for minimal timelike surfaces in de Sitter 3-space.

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In this paper, we consider the 2-parameter family of homogeneous Lorentzian 3-manifolds $(\mathbb{R}^3(x^0, x^1, x^2), g_{(\mu_1,\mu_2)})$ with Lorentzian metric

$$g_{(\mu_1,\mu_2)} = -e^{-2\mu_1 x^2}(dx^0)^2 + e^{-2\mu_2 x^2}(dx^1)^2 + (dx^2)^2.$$

Every homogeneous Lorentzian manifold in this family can also be represented as a solvable matrix Lie group with left invariant metric. This family of homogeneous Lorentzian 3-manifolds includes Minkowski 3-space $\mathbb{E}^3_1$, anti-de Sitter 3-space $\mathbb{H}^3_1(-c^2)$ of constant sectional curvature $-c^2$, $\mathbb{H}^2(-c^2) \times \mathbb{E}^1_1$, the direct product of hyperbolic plane $\mathbb{H}^2(-c^2)$ of constant curvature $-c^2$ and the timeline $\mathbb{E}^1_1$, and $\mathbb{H}^2_1(-c^2) \times \mathbb{E}^1$, the direct product of anti-de Sitter 2-space $\mathbb{H}^2_1(-c^2)$ of constant curvature $-c^2$ and the real line $\mathbb{E}^1$. (In the family, only Minkowski 3-space and anti-de Sitter 3-space have constant sectional curvature.) These four spaces may be considered as Lorentzian counterparts of Euclidean 3-space $\mathbb{E}^3$, 3-sphere $\mathbb{S}^3$, the direct product $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$, and $\mathbb{S}^2 \times \mathbb{E}^1$, the direct product of 2-sphere $\mathbb{S}^2$ and the real line $\mathbb{E}^1$, respectively, of Thurston's eight model geometries [7]. We obtain a generalized integral representation formula that includes, in particular, representation formulas for minimal spacelike surfaces in Minkowski 3-space ([2], [4]) and in anti-de Sitter 3-space. The normal Gauss map of minimal timelike surfaces in $G(\mu_1, \mu_2)$ is discussed. It is shown that Minkowski 3-space $G(0, 0)$, anti-de Sitter 3-space $G(c, c)$, and $G(c, -c)$ are the only homogeneous Lorentzian 3-manifolds among the 2-parameter family members $G(\mu_1, \mu_2)$ in which the (projected) normal Gauss map of minimal timelike surfaces is harmonic. The harmonic map equations for those cases are also obtained.

1 Solvable Lie groups

In this section, we study the two-parameter family of certain homogeneous Lorentzian 3-manifolds.

Let us consider the two-parameter family of homogeneous Lorentzian 3-manifolds

$$\left\{(\mathbb{R}^3(x^0, x^1, x^2), g_{(\mu_1,\mu_2)}) \mid (\mu_1, \mu_2) \in \mathbb{R}^2\right\}, \tag{1.1}$$

where the metric $g_{(\mu_1,\mu_2)}$ is defined by

$$g_{(\mu_1,\mu_2)} := -e^{-2\mu_1 x^2}(dx^0)^2 + e^{-2\mu_2 x^2}(dx^1)^2 + (dx^2)^2. \tag{1.2}$$

Proposition 1.1. *Each homogeneous space $(\mathbb{R}^3, g_{(\mu_1,\mu_2)})$ is isometric to the following solvable matrix Lie group:*

$$G(\mu_1, \mu_2) = \left\{ \begin{pmatrix} e^{\mu_1 x^2} & 0 & 0 & x^0 \\ 0 & e^{\mu_2 x^2} & 0 & x^1 \\ 0 & 0 & 1 & x^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; x^0, x^1, x^2 \in \mathbb{R} \right\}$$

with left invariant metric. The group operation on $G(\mu_1, \mu_2)$ is the ordinary matrix multiplication and the corresponding group operation on $(\mathbb{R}^3, g_{(\mu_1,\mu_2)})$ is given by

$$(x^0, x^1, x^2) \cdot (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2) = (x^0 + e^{\mu_1 x^2}\tilde{x}^0, x^1 + e^{\mu_2 x^2}\tilde{x}^1, x^2 + \tilde{x}^2).$$

Proof. For $\tilde{a} = (a^0, a^1, a^2) \in G(\mu_1, \mu_2)$, denote by $L_{\tilde{a}}$ the left translation by $\tilde{a}$. Then

$$\begin{aligned} L_{\tilde{a}}(x^0, x^1, x^2) &= (a^0, a^1, a^2) \cdot (x^0, x^1, x^2) \\ &= (a^0 + e^{\mu_1 a^2} x^0, a^1 + e^{\mu_2 a^2} x^1, a^2 + x^2) \end{aligned}$$

and

$$\begin{aligned} L_{\tilde{a}}^* g_{(\mu_1,\mu_2)} &= -e^{-2\mu_1(a^2+x^2)}\{d(a^0 + e^{\mu_1 a^2} x^0)\}^2 + \\ & e^{-2\mu_2(a^2+x^2)}\{d(a^1 + e^{\mu_2 a^2} x^1)\}^2 + \{d(a^2 + x^2)\}^2 \\ &= -e^{-2\mu_1 x^2}(dx^0)^2 + e^{-2\mu_2 x^2}(dx^1)^2 + (dx^2)^2. \end{aligned}$$

Q.E.D.

The Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$ is given explicitly by

$$\mathfrak{g}(\mu_1, \mu_2) = \left\{ \left(\begin{array}{cccc} \mu_1 y^2 & 0 & 0 & y^0 \\ 0 & \mu_2 y^2 & 0 & y^1 \\ 0 & 0 & 0 & y^2 \\ 0 & 0 & 0 & 0 \end{array} \right) \middle| \; y^0, y^1, y^2 \in \mathbb{R} \right\}. \tag{1.3}$$

Then we can take the following orthonormal basis $\{E_0, E_1, E_2\}$ of $\mathfrak{g}(\mu_1, \mu_2)$:

$$E_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.4}$$

Then the commutation relation of $\mathfrak{g}(\mu_1, \mu_2)$ is given by

$$\begin{aligned} &[E_0, E_1] = 0, \; [E_1, E_2] = -\mu_2 E_1, \\ &[E_2, E_0] = \mu_1 E_0. \end{aligned}$$

$[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0$, so $\mathfrak{g}(\mu_1, \mu_2)$ is a solvable Lie algebra i.e. $G(\mu_1, \mu_2)$ is a solvable Lie group. For $X \in \mathfrak{g}(\mu_1, \mu_2)$, denote by $\mathrm{ad}(X)^*$ the *adjoint* operator of $\mathrm{ad}(X)$. Then it satisfies the equation

$$\langle [X, Y], Z \rangle = \langle Y, \mathrm{ad}(X)^*(Z) \rangle$$

for any $Y, Z \in \mathfrak{g}(\mu_1, \mu_2)$. Let U be the symmetric bilinear operator on $\mathfrak{g}(\mu_1, \mu_2)$ defined by

$$U(X, Y) := \frac{1}{2}\{\mathrm{ad}(X)^*(Y) + \mathrm{ad}(Y)^*(X)\}.$$

Lemma 1.2. *Let $\{E_0, E_1, E_2\}$ be the orthonormal basis for $\mathfrak{g}(\mu_1, \mu_2)$ defined in* (1.4). *Then*

$$\begin{aligned} &U(E_0, E_0) = \mu_1 E_2, \; U(E_1, E_1) = -\mu_2 E_2, \; U(E_2, E_2) = 0, \\ &U(E_0, E_1) = 0, \; U(E_1, E_2) = \frac{\mu_2}{2} E_1, \; U(E_2, E_0) = \frac{\mu_1}{2} E_0. \end{aligned}$$

Let $\mathfrak{D}$ be a simply connected domain and $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ an immersion. φ is said to be *timelike* if the induced metric I on $\mathfrak{D}$ is Lorentzian. The induced Lorentzian metric I determines a Lorentz conformal structure $\mathcal{C}_I$ on $\mathfrak{D}$. Let (t, x) be a Lorentz isothermal coordinate system with respect to the conformal structure $\mathcal{C}_I$. Then the first fundamental form I is written in terms of (t, x) as

$$I = e^{\omega}(-dt^2 + dx^2). \tag{1.5}$$

The conformality condition is given in terms of (t, x) by

$$\begin{aligned} &\left\langle \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x} \right\rangle = 0, \\ -&\left\langle \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial t} \right\rangle = \left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial x} \right\rangle = e^{\omega}. \end{aligned} \tag{1.6}$$

A conformal timelike surface is called a *Lorentz surface.* Let $u := t+x$ and $v := -t+x$. Then (u, v) defines a null coordinate system with respect to the conformal structure $\mathcal{C}_I$. The first fundamental form I is written in terms of (u, v) as

$$I = e^{\omega} du dv. \tag{1.7}$$

The partial derivatives $\frac{\partial \varphi}{\partial u}$ and $\frac{\partial \varphi}{\partial v}$ are computed to be

$$\frac{\partial \varphi}{\partial u} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}\right), \quad \frac{\partial \varphi}{\partial v} = \frac{1}{2}\left(-\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}\right). \tag{1.8}$$

The conformality condition (1.6) can be written in terms of null coordinates as

$$\begin{aligned} &\left\langle \frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial u} \right\rangle = \left\langle \frac{\partial \varphi}{\partial v}, \frac{\partial \varphi}{\partial v} \right\rangle = 0, \\ &\left\langle \frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial v} \right\rangle = \frac{1}{2} e^{\omega}. \end{aligned} \tag{1.9}$$

Definition 1.1. Let $\mathfrak{D}(t, x)$ be a simply connected domain. A smooth timelike immersion $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ is said to be *harmonic* if it is a critical point of the energy functional[1]

$$E(\varphi) = \int_{\mathfrak{D}} e(\varphi) dt dx, \tag{1.10}$$

where $e(\varphi)$ is the *energy density* of φ

$$e(\varphi) = \frac{1}{2}\left\{ -\left| \varphi^{-1} \frac{\partial \varphi}{\partial t} \right|^2 + \left| \varphi^{-1} \frac{\partial \varphi}{\partial x} \right|^2 \right\}. \tag{1.11}$$

$\left| \varphi^{-1} \frac{\partial \varphi}{\partial t} \right|^2 = \left\langle \varphi^{-1} \frac{\partial \varphi}{\partial t}, \varphi^{-1} \frac{\partial \varphi}{\partial t} \right\rangle < 0$ and $\left| \varphi^{-1} \frac{\partial \varphi}{\partial x} \right|^2 = \left\langle \varphi^{-1} \frac{\partial \varphi}{\partial x}, \varphi^{-1} \frac{\partial \varphi}{\partial x} \right\rangle > 0$, so $e(\varphi) > 0$ and hence $E(\varphi) \geq 0$.

[1] This is an analogue of the Dirichlet energy.

The following lemma is proved in [5]. The statement is still valid for $G(\mu_1, \mu_2)$ under consideration in this paper.

Lemma 1.3. *Let $\mathfrak{D}$ be a simply connected domain. A smooth timelike immersion $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ is harmonic if and only if it satisfies the wave equation*

$$\begin{aligned}
&-\frac{\partial}{\partial t}\left(\varphi^{-1}\frac{\partial \varphi}{\partial t}\right) + \frac{\partial}{\partial x}\left(\varphi^{-1}\frac{\partial \varphi}{\partial x}\right) \\
&- \left\{-\mathrm{ad}\left(\varphi^{-1}\frac{\partial \varphi}{\partial t}\right)^*\left(\varphi^{-1}\frac{\partial \varphi}{\partial t}\right) + \mathrm{ad}\left(\varphi^{-1}\frac{\partial \varphi}{\partial x}\right)^*\left(\varphi^{-1}\frac{\partial \varphi}{\partial x}\right)\right\} = 0.
\end{aligned} \tag{1.12}$$

In terms of null coordinates u, v, the wave equation (1.12) can be written as

$$\frac{\partial}{\partial u}\left(\varphi^{-1}\frac{\partial \varphi}{\partial v}\right) + \frac{\partial}{\partial v}\left(\varphi^{-1}\frac{\partial \varphi}{\partial u}\right) - 2U\left(\varphi^{-1}\frac{\partial \varphi}{\partial u}, \varphi^{-1}\frac{\partial \varphi}{\partial v}\right) = 0. \tag{1.13}$$

Let $\varphi^{-1}d\varphi = \alpha' du + \alpha'' dv$. Then the equation (1.13) is equivalent to

$$\alpha'_v + \alpha''_u = 2U(\alpha', \alpha''). \tag{1.14}$$

The Maurer-Cartan equation is given by

$$\alpha'_v - \alpha''_u = [\alpha', \alpha'']. \tag{1.15}$$

The equations (1.14) and (1.15) can be combined to a single equation

$$\alpha'_v = U(\alpha', \alpha'') + \frac{1}{2}[\alpha', \alpha'']. \tag{1.16}$$

The equation (1.16) is both the integrability condition for the differential equation $\varphi^{-1}d\varphi = \alpha' du + \alpha'' dv$ and the condition for φ to be a harmonic map.

Left-translating the basis $\{E_0, E_1, E_2\}$, we obtain the following orthonormal frame field:

$$e_0 = e^{\mu_1 x^2}\frac{\partial}{\partial x^0},\ e_1 = e^{\mu_2 x^2}\frac{\partial}{\partial x^1},\ e_2 = \frac{\partial}{\partial x^2}.$$

The Levi-Civita connection ∇ of $G(\mu_1, \mu_2)$ is computed to be

$$\begin{aligned}
&\nabla_{e_0} e_0 = -\mu_1 e_2,\ \nabla_{e_0} e_1 = 0,\ \nabla_{e_0} e_2 = -\mu_1 e_0, \\
&\nabla_{e_1} e_0 = 0,\ \nabla_{e_1} e_1 = \mu_2 e_2,\ \nabla_{e_1} e_2 = -\mu_2 e_1, \\
&\nabla_{e_2} e_0 = -\mu_1 e_0,\ \nabla_{e_2} e_1 = -\mu_2 e_1,\ \nabla_{e_2} e_2 = 0.
\end{aligned}$$

Let $K(e_i, e_j)$ denote the sectional curvature of $G(\mu_1, \mu_2)$ with respect to the tangent plane spanned by e_i and e_j for $i, j = 0, 1, 2$. Then

$$\begin{aligned}
K(e_0, e_1) &= g^{00} R^1_{010} = -\mu_1 \mu_2, \\
K(e_1, e_2) &= g^{11} R^2_{121} = -\mu_2^2, \\
K(e_0, e_3) &= g^{00} R^3_{030} = -\mu_1^2,
\end{aligned} \tag{1.17}$$

where $g_{ij} = g_{(\mu_1, \mu_2)}(e_i, e_j)$ denotes the metric tensor of $G(\mu_1, \mu_2)$. Hence, we see that $G(\mu_1, \mu_2)$ has a constant sectional curvature if and only if $\mu_1^2 = \mu_2^2 = \mu_1\mu_2$. If $c := \mu_1 = \mu_2$, then $G(\mu_1, \mu_2)$ is locally isometric to $\mathbb{H}^3_1(-c^2)$, the anti-de Sitter 3-space of constant sectional curvature $-c^2$. (See Example 1.3 and Remark 1.4.) If $G(\mu_1, \mu_2)$ has a constant sectional curvature and $\mu_1 = -\mu_2$, then $\mu_1 = \mu_2 = 0$, so $G(\mu_1, \mu_2) = G(0, 0) \cong \mathbb{E}^3_1$ (Example 1.2).

Example 1.2. (Minkowski 3-space) The Lie group $G(0,0)$ is isomorphic and isometric to the Minkowski 3-space

$$\mathbb{E}^3_1 = (\mathbb{R}^3(x^0, x^1, x^2), +)$$

with the metric $-(dx^0)^2 + (dx^1)^2 + (dx^2)^2$.

Example 1.3. (Anti-de Sitter 3-space) Take $\mu_1 = \mu_2 = c \neq 0$. Then $G(c,c)$ is the flat chart model of the anti-de Sitter 3-space:

$$\mathbb{H}^3_1(-c^2)_+ = (\mathbb{R}^3(x^0, x^1, x^2), e^{-2cx^2}\{-(dx^0)^2 + (dx^1)^2\} + (dx^2)^2).$$

Remark 1.4. Let $\mathbb{E}^4_2$ be the pseudo-Euclidean 4-space with the metric $\langle \cdot, \cdot \rangle$:

$$\langle \cdot, \cdot \rangle = -(du^0)^2 - (du^1)^2 + (du^2)^2 + (du^3)^2.$$

in terms of rectangular coordinate system (u^0, u^1, u^2, u^3). The *anti-de Sitter* 3*-space* $\mathbb{H}^3_1(-c^2)$ of constant sectional curvature $-c^2$ is realized as the hyperquadric in $\mathbb{E}^4_2$:

$$\mathbb{H}^3_1(-c^2) = \left\{ (u^0, u^1, u^2, u^3) \in \mathbb{E}^4_2 : \ -(u^0)^2 - (u^1)^2 + (u^2)^2 + (u^3)^2 = -\frac{1}{c^2} \right\}.$$

The anti-de Sitter 3-space $\mathbb{H}^3_1(-c^2)$ is divided into the following three regions:

$$\begin{aligned}
\mathbb{H}^3_1(-c^2)_+ &= \{(u^0, u^1, u^2, u^3) \in \mathbb{H}^3_1(-c^2) : \ c(u^1 + u^2) > 0\}; \\
\mathbb{H}^3_1(-c^2)_0 &= \{(u^0, u^1, u^2, u^3) \in \mathbb{H}^3_1(-c^2) : \ u^1 + u^2 = 0\}; \\
\mathbb{H}^3_1(-c^2)_- &= \{(u^0, u^1, u^2, u^3) \in \mathbb{H}^3_1(-c^2) : \ c(u^1 + u^2) < 0\}.
\end{aligned}$$

$\mathbb{H}^3_1(-c^2)$ is the disjoint union $\mathbb{H}^3_1(-c^2)_+ \dot{+} \mathbb{H}^3_1(-c^2)_0 \dot{+} \mathbb{H}^3_1(c^2)_-$ and $\mathbb{H}^3_1(-c^2)_\pm$ are diffeomorphic to $(\mathbb{R}^3, g_{(c,c)})$. Let us introduce a local coordinate system (x^0, x^1, x^2) on $\mathbb{H}^3_1(-c^2)_+$ by

$$\begin{aligned}
x^0 &= \frac{u^0}{c(u^1 + u^2)}, \\
x^1 &= \frac{u^3}{c(u^1 + u^2)}, \\
x^2 &= -\frac{1}{c} \ln[c(u^1 + u^2)].
\end{aligned}$$

The induced metric of $\mathbb{H}^3_1(-c^2)_+$ is expressed as:

$$g_c := e^{-2cx^2}\{-(dx^0)^2 + (dx^1)^2\} + (dx^2)^2.$$

The chart $(\mathbb{H}^3_1(-c^2)_+, g_c)$ is called the *flat chart* of $\mathbb{H}^3_1(-c^2)$. The flat chart is identified with the Lorentzian manifold $(\mathbb{R}^3, g_{(c,c)})$ of constant sectional curvature $-c^2$. This expression shows that the flat chart is a warped product $\mathbb{E}^1 \times_f \mathbb{E}^2_1$ with warping function $f(x^2) = e^{-cx^2}$. Introducing $y^0 = cx^0$, $y^1 = cx^1$, and $y^2 = e^{cx^2}$, we also obtain half-space model of anti-de Sitter 3-space $\mathbb{H}^3_1(-c^2)$ with an analogue of Poincaré metric

$$g_c := \frac{-(dy^0)^2 + (dy^1)^2 + (dy^2)^2}{c^2 (y^2)^2}.$$

Example 1.5 (Direct Product $\mathbb{H}^2(-c^2) \times \mathbb{E}^1_1$)**.** Take $(\mu_1, \mu_2) = (0, c)$ with $c \neq 0$. Then the resulting homogeneous spacetime is $\mathbb{R}^3$ with the Lorentzian metric

$$-(dx^0)^2 + e^{-2cx^2}(dx^1)^2 + (dx^2)^2.$$

$G(0, c)$ is identified with $\mathbb{H}^2(-c^2) \times \mathbb{E}^1_1$, the direct product of hyperbolic plane $\mathbb{H}^2(-c^2)$ of constant curvature $-c^2$ and the timeline $\mathbb{E}^1_1$.

Example 1.6 (Direct Product $\mathbb{H}^2_1(-c^2) \times \mathbb{E}^1$)**.** Take $(\mu_1, \mu_2) = (c, 0)$ with $c \neq 0$. Then the resulting homogeneous spacetime is $\mathbb{R}^3$ with the Lorentzian metric

$$-e^{-2cx^2}(dx^0)^2 + (dx^2)^2 + (dx^1)^2.$$

$G(c, 0)$ is identified with $\mathbb{H}^2_1(-c^2) \times \mathbb{E}^1$, the direct product of anti-de Sitter 2-space $\mathbb{H}^2_1(-c^2)$ of constant curvature $-c^2$ and the real line $\mathbb{E}^1$.

Example 1.7 (Homogeneous Spacetime $G(c, -c)$)**.** Let $\mu_1 = c$ and $\mu_2 = -c$ with $c \neq 0$. Then the resulting homogeneous spacetime $G(c, -c)$ is $\mathbb{R}^3$ with the Lorentzian metric

$$-e^{-2cx^2}(dx^0)^2 + e^{2cx^2}(dx^1)^2 + (dx^2)^2.$$

2 Integral representation formulas

Let $\mathfrak{D}(u, v)$ be a simply connected domain and $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ an immersion. Let us write $\varphi(u, v) = (x^0(u, v), x^1(u, v), x^2(u, v))$. Then

$$\begin{aligned} \alpha' &= \varphi^{-1}\frac{\partial\varphi}{\partial u} \\ &= \frac{\partial x^0}{\partial u}e^{-\mu_1 x^2}E_0 + \frac{\partial x^1}{\partial u}e^{-\mu_2 x^2}E_1 + \frac{\partial x^2}{\partial u}E_2 \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \alpha'' &= \varphi^{-1}\frac{\partial\varphi}{\partial v} \\ &= \frac{\partial x^0}{\partial v}e^{-\mu_1 x^2}E_0 + \frac{\partial x^1}{\partial v}e^{-\mu_2 x^2}E_1 + \frac{\partial x^2}{\partial v}E_2. \end{aligned} \tag{2.2}$$

It follows from (1.14) that

Lemma 2.1. *φ is harmonic if and only if it satisfies the following equations:*

$$\begin{aligned} \frac{\partial^2 x^0}{\partial u\partial v} - \mu_1\left(\frac{\partial x^0}{\partial u}\frac{\partial x^2}{\partial v} + \frac{\partial x^0}{\partial v}\frac{\partial x^2}{\partial u}\right) &= 0, \\ \frac{\partial^2 x^1}{\partial u\partial v} - \mu_2\left(\frac{\partial x^1}{\partial u}\frac{\partial x^2}{\partial v} + \frac{\partial x^1}{\partial v}\frac{\partial x^2}{\partial u}\right) &= 0, \\ \frac{\partial^2 x^2}{\partial u\partial v} - \mu_1\frac{\partial x^0}{\partial u}\frac{\partial x^0}{\partial v}e^{-2\mu_1 x^2} + \mu_2\frac{\partial x^1}{\partial u}\frac{\partial x^1}{\partial v}e^{-2\mu_2 x^2} &= 0. \end{aligned} \tag{2.3}$$

The exterior derivative d is decomposed as

$$d = \partial' + \partial'',$$

where $\partial' = \frac{\partial}{\partial u} du$ and $\partial'' = \frac{\partial}{\partial v} dv$ with respect to the conformal structure of $\mathfrak{D}$. Let

$$\begin{aligned}
(\omega^0)' &= e^{-\mu_1 x^2}\partial' x^0, \ (\omega^0)'' = e^{-\mu_1 x^2}\partial'' x^0,\\
(\omega^1)' &= e^{-\mu_2 x^2}\partial' x^1, \ (\omega^1)'' = e^{-\mu_2 x^2}\partial'' x^1,\\
(\omega^2)' &= \partial' x^2, \ (\omega^2)'' = \partial'' x^2.
\end{aligned}$$

Then by Lemma 2.1, the 1-forms $(\omega_i)'$, $(\omega_i)''$, $i = 0, 1, 2$ satisfy the differential system:

$$\partial''(\omega^i)' = \mu_{i+1}(\omega^i)'' \wedge (\omega^2)', \ i = 0, 1, \tag{2.4}$$

$$\partial''(\omega^2)' = \mu_1(\omega^0)'' \wedge (\omega^0)' - \mu_2(\omega^1)'' \wedge (\omega^1)', \tag{2.5}$$

$$\partial'(\omega^i)'' = \mu_{i+1}(\omega^i)' \wedge (\omega^2)'', \ i = 0, 1, \tag{2.6}$$

$$\partial'(\omega^2)'' = \mu_1(\omega_0)' \wedge (\omega^0)'' - \mu_2(\omega^1)' \wedge (\omega^1)''. \tag{2.7}$$

Proposition 2.2. *If* $(\omega_i)'$, $(\omega_i)''$, $i = 0, 1, 2$ *satisfy* (2.4)-(2.7) *on a simply connected domain* $\mathfrak{D}$. *Then*

$$\varphi(u, v) = \int (e^{\mu_1 x^2}(\omega^0)', e^{\mu_2 x^2}(\omega^1)', (\omega^2)') + \int (e^{\mu_1 x^2}(\omega^0)'', e^{\mu_2 x^2}(\omega^1)'', (\omega^2)'') \tag{2.8}$$

is a harmonic map into $G(\mu_1, \mu_2)$.

Conversely, if $\{(\omega_i)', (\omega_i)'' : i = 0, 1, 2\}$ is a solution to (2.4)-(2.7) and

$$\begin{aligned}
-(\omega^0)' \otimes (\omega^0)' + (\omega^1)' \otimes (\omega^1)' + (\omega^2)' \otimes (\omega^2)' &= 0,\\
-(\omega^0)'' \otimes (\omega^0)'' + (\omega^1)'' \otimes (\omega^1)'' + (\omega^2)'' \otimes (\omega^2)'' &= 0
\end{aligned} \tag{2.9}$$

on a simply connected domain $\mathfrak{D}$, then $\varphi(u, v)$ in (2.8) is a weakly conformal harmonic map into $G(\mu_1, \mu_2)$. In addition, if

$$-(\omega^0)' \otimes (\omega^0)'' + (\omega^1)' \otimes (\omega^1)'' + (\omega^2)' \otimes (\omega^2)'' \neq 0, \tag{2.10}$$

then $\varphi(u, v)$ in (2.8) is a minimal timelike surface in $G(\mu_1, \mu_2)$.

3 Normal Gauss maps

Let $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ be a Lorentz surface i.e. a conformal timelike surface. Take a unit normal vector field N along φ. Then by the left translation we obtain the smooth map

$$\varphi^{-1}N : \mathfrak{D} \longrightarrow \mathbb{S}^2_1(1),$$

where

$$\mathbb{S}^2_1(1) = \{u^0E_0 + u^1E_1 + u^2E_2 : -(u^0)^2 + (u^1)^2 + (u^2)^2 = 1\} \subset \mathfrak{g}(\mu_1, \mu_2)$$

is the de Sitter 2-space of constant Gaußian curvature 1. The Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$ is identified with Minkowski 3-space $\mathbb{E}^3_1(u^0, u^1, u^2)$ via the orthonormal basis $\{E_0, E_1, E_2\}$. Then smooth map $\varphi^{-1}N$ is called the normal Gauss map of φ. Let $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ be a minimal timelike surface determined by the data $((\omega^0)', (\omega^1)', (\omega^2)')$ and $((\omega^0)'', (\omega^1)'', (\omega^2)'')$. Write $(\omega^i)' = \xi^i du$ and $(\omega^i)'' = \eta^i dv$, $i = 0, 1, 2$. Then

$$\begin{aligned} I &= 2(-(\omega^0)' \otimes (\omega^0)'' + (\omega^1)' \otimes (\omega^1)'' + (\omega^2)' \otimes (\omega^2)'') \\ &= 2(-\xi^0\eta^0 + \xi^1\eta^1 + \xi^2\eta^2)dudv. \end{aligned} \tag{3.1}$$

The conformality condition (2.9) can be written as

$$\begin{aligned} -(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 = 0, \\ -(\eta^0)^2 + (\eta^1)^2 + (\eta^2)^2 = 0. \end{aligned} \tag{3.2}$$

It follows from (3.2) that one can introduce pairs of functions (q, f) and (r, g) such that

$$\begin{aligned} q &= \frac{-\xi^2}{\xi^0 - \xi^1},\ f = \xi^0 - \xi^1, \\ r &= \frac{\eta^2}{\eta^0 + \eta^1},\ g = -(\eta^0 + \eta^1). \end{aligned} \tag{3.3}$$

In terms of (q, f) and (r, g), $\varphi(u, v) = (x^0(u, v), x^1(u, v), x^2(u, v))$ is given by Weierstraß type representation formula

$$\begin{aligned} x^0(u, v) &= \frac{1}{2} e^{\mu_1 x^2(u,v)} \int (1 + q^2) f du - (1 + r^2) g dv, \\ x^1(u, v) &= -\frac{1}{2} e^{\mu_2 x^2(u,v)} \int (1 - q^2) f du + (1 - r^2) g dv, \\ x^2(u, v) &= -\int q f du + r g dv. \end{aligned} \tag{3.4}$$

with first fundamental form

$$I = (1 + qr)^2 f g du dv. \tag{3.5}$$

Remark 3.1. In the study of minimal timelike surfaces in Minkowski 3-space, one may assume that $f = g = 1$ so that (3.4) reduces to a simpler form called the *normalized Weierstraß formula*. This is possible as there are no restrictions on f and g other than f and g being Lorentz holomorphic and Lorentz anti-holomorphic respectively. (See [2] and [4].) However, this is not the case with minimal timelike surfaces in anti-de Sitter 3-space as we will see later.

It turns out that the pair (q, r) is the Normal Gauss map $\varphi^{-1}N$ projected into the Minkowski 2-pane $\mathbb{E}^2_1$. To see this, first the normal Gauss map is computed to be

$$\varphi^{-1}N = \frac{1}{qr + 1}[(q - r)E_0 + (q + r)E_1 + (qr - 1)E_2]. \tag{3.6}$$

Let $\wp_{\mathcal{N}} : \mathbb{S}^2_1(1) \setminus \{x^2 = 1\} \longrightarrow \mathbb{E}^2_1 \setminus \mathbb{H}^1_0$ be the stereographic projection from the north pole $\mathcal{N} = (0, 0, 1)$. Here, $\mathbb{H}^1_0$ is the hyperbola

$$\mathbb{H}^1_0 = \{x^0 E_0 + x^1 E_1 \in \mathbb{E}^2_1 : -(x^0)^2 + (x^1)^2 = -1\}.$$

Then

$$\wp_{\mathcal{N}}(x^0 E_0 + x^1 E_1 + x^2 E_2) = \frac{x^0}{1 - x^2} E^0 + \frac{x^1}{1 - x^2} E^1. \tag{3.7}$$

So, the normal Gauss map $\varphi^{-1}N$ is projected into the Minkowski plane $\mathbb{E}^2_1$ via $\wp_{\mathcal{N}}$ as

$$\wp_{\mathcal{N}} \circ \varphi^{-1} N = \frac{q - r}{2} E_0 + \frac{q + r}{2} E_1 \in \mathbb{E}^2_1(t, x). \tag{3.8}$$

In terms of null coordinates (u, v), (3.8) is written as

$$\wp_{\mathcal{N}} \circ \varphi^{-1} N = (q, r) \in \mathbb{E}^2_1(u, v). \tag{3.9}$$

The pair (q, r) is called the *projected normal Gauss map* of φ. It follows from (2.4) and (2.5) that

$$\begin{aligned} \frac{\partial \xi^i}{\partial v} &= \mu_{i+1} \eta^i \xi^2, \ i = 0, 1, \\ \frac{\partial \xi^2}{\partial v} &= \mu_1 \eta^0 \xi^0 - \mu_2 \eta^1 \xi^1. \end{aligned} \tag{3.10}$$

Using (3.10), we obtain

$$\begin{aligned} \frac{\partial f}{\partial v} &= \frac{\partial \xi^0}{\partial v} - \frac{\partial \xi^1}{\partial v} \\ &= \frac{1}{2} q[\mu_1(1 + r^2) - \mu_2(1 - r^2)] f g \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \frac{\partial q}{\partial v} &= -\frac{\frac{\partial \xi^2}{\partial v} f - \xi^2 \frac{\partial f}{\partial v}}{f^2} \\ &= \frac{1}{4}[\mu_1(1 - q^2)(1 + r^2) + \mu_2(1 + q^2)(1 - r^2)] g. \end{aligned} \tag{3.12}$$

It follows from (2.6) and (2.7) that

$$\begin{aligned} \frac{\partial \eta^i}{\partial u} &= \mu_{i+1} \xi^i \eta^2, \ i = 0, 1, \\ \frac{\partial \eta^2}{\partial u} &= \mu_1 \xi^0 \eta^0 - \mu_2 \xi^1 \eta^1. \end{aligned} \tag{3.13}$$

Using (3.13), we obtain

$$\begin{aligned} \frac{\partial g}{\partial u} &= -\frac{\partial \eta^0}{\partial u} - \frac{\partial \eta_1}{\partial u} \\ &= \frac{1}{2} r[\mu_1(1 + q^2) - \mu_2(1 - q^2)] f g \end{aligned} \tag{3.14}$$

and

$$\frac{\partial r}{\partial u} = -\frac{\frac{\partial \eta^2}{\partial u} g - \eta^2 \frac{\partial g}{\partial u}}{g^2} = \frac{1}{4}[\mu_1(1+q^2)(1-r^2) + \mu_2(1-q^2)(1+r^2)]f. \tag{3.15}$$

Remark 3.2. Setting $f = g = 1$, we obtain from (3.11), (3.12), (3.14), and (3.15)

$$0 = q[\mu_1(1+r^2) - \mu_2(1-r^2)], \tag{3.16}$$

$$\frac{\partial q}{\partial v} = \frac{1}{4}[\mu_1(1-q^2)(1+r^2) + \mu_2(1+q^2)(1-r^2)] \tag{3.17}$$

and

$$0 = r[\mu_1(1+q^2) - \mu_2(1-q^2)], \tag{3.18}$$

$$\frac{\partial r}{\partial u} = \frac{1}{4}[\mu_1(1+q^2)(1-r^2) + \mu_2(1-q^2)(1+r^2)]. \tag{3.19}$$

Let $\mu_1 = \mu_2 = c$. Then it follows from (3.16) that $qr^2 = 0$ i.e. we have $q = 0$ or $r = 0$. If $q = 0$ then (3.17) says $c = 0$. If $r = 0$ then (3.19) says $c = 0$ also. Hence, we cannot have $f = g = 1$ if $\mu_1 = \mu_2 = c \neq 0$.

Remark 3.3. For $G(0,0) = \mathbb{E}^3_1$,

$$\frac{\partial f}{\partial v} = \frac{\partial q}{\partial v} = 0,$$
$$\frac{\partial g}{\partial u} = \frac{\partial r}{\partial u} = 0.$$

That is, f, q are Lorentz holomorphic and g, r are Lorentz anti-holomorphic. From (3.4), we retrieve the Weierstraß representation formula ([2], [4]) for minimal timelike surface $\varphi(u,v) = (x^0(u,v), x^1(u,v), x^2(u,v))$ in $\mathbb{E}^3_1$ given by

$$\begin{aligned} x^0(u,v) &= \frac{1}{2}\int (1+q^2) f du - (1+r^2) g dv, \\ x^1(u,v) &= -\frac{1}{2}\int (1-q^2) f du + (1-r^2) g dv, \\ x^2(u,v) &= -\int q f du + r g dv. \end{aligned} \tag{3.20}$$

Remark 3.4. If $\mu_1 = \mu_2 = c \neq 0$, then (3.12) and (3.15) can be written respectively as

$$\frac{\partial q}{\partial v} = \frac{c}{2}(1-qr)(1+qr)g, \tag{3.21}$$

$$\frac{\partial r}{\partial u} = \frac{c}{2}(1-qr)(1+qr)f. \tag{3.22}$$

If $\frac{\partial q}{\partial v} = \frac{\partial r}{\partial u} = 0$, then $qr = 1$. Differentiating this with respect to u and v respectively, we obtain $\frac{\partial q}{\partial u} = \frac{\partial r}{\partial v} = 0$. So, q and r are constants such that $qr = 1$. This means that $\varphi^{-1}N$ is constant and $II = 0$. Hence, minimal timelike surface φ obtained by a Lorentz holomorphic map q and a Lorentz anti-holomorphic map r is totally geodesic.

From here on, we assume that $q^2 \neq 1$ and $r^2 \neq 1$. It follows from (3.11), (3.12), (3.14), and (3.15) that the projected normal Gauss map (q, r) satisfies the equations

$$\begin{aligned}(3.23)\quad &\frac{\partial^2 q}{\partial u \partial v} + \frac{2q[\mu_1(1+r^2) - \mu_2(1-r^2)]}{\mu_1(1-q^2)(1+r^2) + \mu_2(1+q^2)(1-r^2)} \frac{\partial q}{\partial u}\frac{\partial q}{\partial v} \\ &- \frac{4(\mu_1^2 - \mu_2^2)(1-q^4)r}{[\mu_1(1+q^2)(1-r^2) + \mu_2(1-q^2)(1+r^2)]} \\ &\frac{}{[\mu_1(1-q^2)(1+r^2) + \mu_2(1+q^2)(1-r^2)]}\frac{\partial r}{\partial u}\frac{\partial q}{\partial v} = 0\end{aligned}$$

and

$$\begin{aligned}(3.24)\quad &\frac{\partial^2 r}{\partial v \partial u} + \frac{2[\mu_1(1+q^2) - \mu_2(1-q^2)]r}{\mu_1(1+q^2)(1-r^2) + \mu_2(1-q^2)(1+r^2)} \frac{\partial r}{\partial u}\frac{\partial r}{\partial v} \\ &- \frac{4(\mu_1^2 - \mu_2^2)q(1-r^4)}{[\mu_1(1+q^2)(1-r^2) + \mu_2(1-q^2)(1+r^2)]} \\ &\frac{}{[\mu_1(1-q^2)(1+r^2) + \mu_2(1+q^2)(1-r^2)]}\frac{\partial r}{\partial u}\frac{\partial q}{\partial v} = 0.\end{aligned}$$

The equations (3.23) and (3.24) are not the harmonic map equations for the projected normal Gauss map (q, v) in general. The following theorem tells under what conditions they become the harmonic map equations for (q, v).

Theorem 3.1. *The projected normal Gauss map (q, r) is a harmonic map if and only if $\mu_1^2 = \mu_2^2$. If $\mu_1 = \mu_2 \neq 0$ then (3.23) and (3.24) reduce to*

$$(3.25)\quad \frac{\partial^2 q}{\partial u \partial v} + \frac{2qr^2}{(1+qr)(1-qr)}\frac{\partial q}{\partial u}\frac{\partial q}{\partial v} = 0,$$

$$(3.26)\quad \frac{\partial^2 r}{\partial v \partial u} + \frac{2q^2 r}{(1+qr)(1-qr)}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} = 0.$$

(3.25) and (3.26) are the harmonic map equations for the map $(q, r) : \mathfrak{D}(u, v) \longrightarrow \left(\mathbb{E}_1^2(\alpha, \beta), \frac{2d\alpha d\beta}{1-\alpha^2\beta^2}\right)$. If $\mu_1 = -\mu_2$ then (3.23) and (3.24) reduce to

$$(3.27)\quad \frac{\partial^2 q}{\partial u \partial v} - \frac{2q}{(q+r)(q-r)}\frac{\partial q}{\partial u}\frac{\partial q}{\partial v} = 0,$$

$$(3.28)\quad \frac{\partial^2 r}{\partial v \partial u} + \frac{2r}{(q+r)(q-r)}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} = 0.$$

(3.27) and (3.28) are the harmonic map equations for the map $(q, r) : \mathfrak{D}(u, v) \longrightarrow \left(\mathbb{E}_1^2(\alpha, \beta), \frac{2d\alpha d\beta}{\alpha^2-\beta^2}\right)$.

Proof. The tension field $\tau(q, r)$ of (q, r) is given by ([1], [8])

$$(3.29)\quad \tau(q, r) = 4\lambda^{-2}\left(\frac{\partial^2 q}{\partial u \partial v} + \Gamma_{\alpha\alpha}^{\alpha}\frac{\partial q}{\partial u}\frac{\partial q}{\partial v}, \frac{\partial^2 r}{\partial v \partial u} + \Gamma_{\beta\beta}^{\beta}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v}\right),$$

where λ is a conformal factor and $\Gamma_{\alpha\alpha}^{\alpha}, \Gamma_{\beta\beta}^{\beta}$ are the Christoffel symbols of $\mathbb{E}_1^2(\alpha, \beta)$. Comparing (3.23), (3.24) and $\tau(q, v) = 0$, we see that (3.23) and (3.24) are the harmonic map equations for (q, v) if and only if $\mu_1^2 = \mu_2^2$. In order to find a metric on $\mathbb{E}_1^2(\alpha, \beta)$ with which (3.23) and (3.24) are the harmonic map equations, one needs to solve the first-order partial differential equations

$$\begin{aligned} \Gamma^{\alpha}_{\alpha\alpha} &= g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial\alpha} \\ &= \begin{cases} \frac{2\alpha\beta^2}{1-\alpha^2\beta^2} & \text{if} \quad \mu_1 = \mu_2 \neq 0, \\ -\frac{2\alpha}{\alpha^2-\beta^2} & \text{if} \quad \mu_1 = -\mu_2, \end{cases} \\ \Gamma^{\beta}_{\beta\beta} &= g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial\beta} \\ &= \begin{cases} \frac{2\alpha^2\beta}{1-\alpha^2\beta^2} & \text{if} \quad \mu_1 = \mu_2 \neq 0, \\ \frac{2\beta}{\alpha^2-\beta^2} & \text{if} \quad \mu_1 = -\mu_2. \end{cases} \end{aligned} \tag{3.30}$$

The solutions are given by

$$(g_{\alpha\beta}) = \begin{cases} \begin{pmatrix} 0 & \frac{1}{1-\alpha^2\beta^2} \\ \frac{1}{1-\alpha^2\beta^2} & 0 \end{pmatrix} & \text{if} \quad \mu_1 = \mu_2 \neq 0, \\ \begin{pmatrix} 0 & \frac{1}{\alpha^2-\beta^2} \\ \frac{1}{\alpha^2-\beta^2} & 0 \end{pmatrix} & \text{if} \quad \mu_1 = -\mu_2. \end{cases} \tag{3.31}$$

Q.E.D.

Remark 3.5. Clearly, the projected normal Gauss map (q, r) of a minimal timelike surface in $G(0, 0) = \mathbb{E}_1^3$ satisfies the wave equation

$$\Box(q, r) = 0, \tag{3.32}$$

where $\Box$ denotes the d'Alembertian

$$\Box = \lambda^{-2}\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right) = 4\lambda^{-2}\frac{\partial^2}{\partial u \partial v}. \tag{3.33}$$

Remark 3.6. Theorem 3.1 tells that Minkowski 3-space $G(0, 0) = \mathbb{E}_1^3$, anti-de Sitter 3-space $G(c, c) = \mathbb{H}_1^3(-c^2)$, and $G(c, -c)$ are the only homogeneous Lorentzian 3-manifolds among $G(\mu_1, \mu_2)$ in which the projected normal Gauss map of a minimal timelike surface is harmonic.

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